# THE LIMIT OF A RATIO OF CONVOLUTIONS[1]

By P. E. Ney

*Cornell University*

**1. Summary and Introduction.** Let $\{X_i\}$, $i = 1, 2, \cdots$ be a sequence of non-negative, independent, identically distributed random variables, and let $F_n(x) = P\{X_1 + \cdots + X_n < x\}$. The purpose of this note is to investigate

$$\lim_{n\to\infty} [F_n(a_1)/F_n(a_2)],$$

where $0 < a_1 < a_2 < \infty$. In order to do so, estimates are required of the extreme lower tail of the distribution $F_n(x)$, for large $n$. There are a number of known results on the probability in the tail of a convolution (see [1], [2], [3], [4]), but none are appropriate for the present problem. Let $X$ denote a typical $X_i$.

In the case when $P\{0 < X < x_0\} = 0$ for some $x_0 > 0$, the required limit is trivial to evaluate. This result is stated in Theorem 1. In the contrary case it is shown in Theorem 3 that a sufficient condition for $[F_n(a_1)/F_n(a_2)] \to 0$ is that there exist real $\gamma \geqq 0$ and $k > 0$ such that

$$(1) \qquad 0 < \lim_{x\to 0+} x^{-\gamma} P\{0 < X < x\} = k < \infty.$$

This result is achieved by means of an estimate of $F_n(x)$ which is given in Theorem 2.

The Condition (1) is satisfied by a wide class of random variables including those with density functions $f(x)$ such that for some

$$\alpha \geqq 0, \qquad 0 < \lim_{x\to 0+} x^{-\alpha} f(x) = k < \infty.$$

It is easy to verify that most "textbook" densities satisfy this condition, as do all densities with a positive right-continuous $k$th derivative $(0 \leqq k < \infty)$ at zero.

Although (1) is not necessary for $[F_n(a_1)/F_n(a_2)] \to 0$ (a class of examples will be given later to illustrate this point), one would expect that some regularity condition near the origin would be required. In attempting to weaken the sufficient Condition (1), one might conjecture that conditions such as the existence of a bounded density $f(x)$, such that $0 < f(x)$ on an open interval $(0, \epsilon)$, $0 < \epsilon$, is sufficient for $[F_n(a_1)/F_n(a_2)] \to 0$. This conjecture is at the present neither proved, nor disproved by a counterexample.

**2. Results.** First, for the sake of completeness, consider the simple case when

$$(2) \qquad P\{0 < X < x_0\} = 0 \qquad \text{for some } x_0 > 0.$$

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Let $P\{X = 0\} = p_0$. If $p_0 = 0$ then $F_n(x) = 0$ for $n > x/x_0$, and the problem degenerates. Define the conditional distribution

$$F_n^*(x) = P\{X_1 + \cdots + X_n < x \mid X_1 > 0, \cdots, X_n > 0\},$$

and let $m_i = \max\{k : F_k^*(a_i) > 0\}$, $i = 1, 2$.

THEOREM 1.

(i) *If $X$ satisfies* (2), $p_0 > 0$, *and* $m_1 = m_2 = m$, *then* $\lim_{n\to\infty} F_n(a_1)/F_n(a_2) = F_m^*(a_1)/F_m^*(a_2)$.

(ii) *If $X$ satisfies* (2), $p_0 > 0$, *and* $m_1 < m_2$, *then* $\lim_{n\to\infty} F_n(a_1)/F_n(a_2) = 0$.

PROOF. $\lim_{n\to\infty} F_n(a_1)/F_n(a_2)$

$$= \lim_{n\to\infty} \frac{\sum_{k=0}^{m_1} \binom{n}{k} (1 - p_0)^k p_0^{n-k} F_k^*(a_1)}{\sum_{k=0}^{m_2} \binom{n}{k} (1 - p_0)^k p_0^{n-k} F_k^*(a_2)} = \lim_{n\to\infty} \frac{\binom{n}{m_1} (1 - p_0)^{m_1} p_0^{-m_1} F_{m_1}^*(a)}{\binom{n}{m_2} (1 - p_0)^{m_2} p_0^{-m_2} F_{m_2}^*(a)},$$

which is $F_m^*(a_1)/F_m^*(a_2)$ if $m_1 = m_2 = m$, and 0 if $m_1 < m_2$.

REMARK 1. Theorem 1 disposes of the limit problem for discrete distributions on the integers.

REMARK 2. Part (i) of Theorem 1 shows that one can construct examples for which $[F_n(a_1)/F_n(a_2)]$ approaches any specified number $\leqq 1$.

THEOREM 2. *If for some* $\gamma \geqq 0$

$$(1^*) \qquad 0 < \lim_{x\to 0+} x^{-\gamma} P\{0 \leqq X < x\} = k < \infty$$

*then*

$$(3) \qquad \lim_{n\to\infty} [\Gamma(n\gamma + 1) F_n(y)]^{1/n} = k\Gamma(\gamma + 1) y^\gamma,$$

*where* $\Gamma(\cdot)$ *denotes the gamma function.*

PROOF. Choose and fix any $y > 0$. Define $G(x; b) = bx^\gamma$, $b > 0$, $G_0(x; b) = 1$, $G_n(x; b) = \int_0^x G_{n-1}(x - t; b)\, dG(t; b)$, $n = 1, 2, \cdots$. Choose $d$ so that $0 < d < y$, $bd^\gamma \leqq 1$, and define $G^-(x; b, d) = bd^\gamma$ when $d \leqq x < y$, $G^-(x; b, d) = G(x; b)$ otherwise, $G_0^-(x; b, d) = 1$,

$$G_n^-(x; b, d) = \int_0^x G_{n-1}^-(x - t; b, d)\, dG^-(t; b; d) n = 1, 2, \cdots.$$

Let $V_n(A)$ denote the measure of an $n$-dimensional set $A$ with respect to the $n$-dimensional product measure induced by $G$. For any $x > 0$, let $H_n^{(k)}(x)$ denote the half space $\{(x_1, \cdots, x_n) : x_\kappa \geqq x\}$, $C_n(x)$ denote the cube

$$\{(x_1, \cdots, x_n) : 0 \leqq x_i \leqq x, i = 1, \cdots, n\},$$

$S_n(x)$ denote the simplex

$$\{(x_1, \cdots, x_n) : x_1 + \cdots + x_n < x, 0 \leqq x_i, i = 1, \cdots, n\}.$$

Then $S_n(y) \subset [C_n(d) \cup H_n^{(1)}(d) \cup \cdots \cup H_n^{(n)}(d)]$, and hence

$$S_n(y) = [S_n(y) \cap C_n(d)] \cup [S_n(y) \cap H_n^{(1)}(d)] \cup \cdots \cup [S_n(y) \cap H_n^{(n)}(d)].$$

Thus

$$\begin{aligned} V_n\{S_n(y)\} &\leqq V_n\{S_n(y) \cap C_n(d)\} + V_n\{S_n(y) \cap H_n^{(1)}(d)\} + \cdots + V_n\{S_n(y) \cap H_n^{(n)}(d)\} \\ &= V_n\{S_n(y) \cap C_n(d)\} + nV_n\{S_n(y) \cap H_n^{(1)}(d)\}. \end{aligned}$$

But $V_n\{S_n(y)\} = G_n(y; b)$, $V_n\{S_n(y) \cap C_n(d)\} = G_n^-(y; b, d)$,

$$V_n\{S_n(y) \cap H_n^{(1)}(d)\} = \int_d^y dG(z; b) \int_0^{y-z} dG_{n-1}(v, b),$$

and therefore

$$G_n^-(y; b, d) \geqq G_n(y; b) - n\int_d^y dG(z; b)\int_0^{y-z} dG_{n-1}(v, b).$$

Computation of the right side of this inequality yields

$$(4) \qquad G_n^-(y; b, d) \geqq \frac{b^n\Gamma^n(\gamma + 1)}{\Gamma(n\gamma + 1)} y^{n\gamma} - \frac{\gamma b^n \Gamma^{n-1}(\gamma + 1)y^{\gamma-1}}{\Gamma(n\gamma - \gamma + 2)} (y - d)^{n\gamma-\gamma+1}.$$

Next define $G^+(x; b, d) = bx^\gamma$ when $0 \leqq x < d$, $G^+(x; b, d) = 1$ when $d \leqq x$, $G_0^+(x; b, d) = 1$, $G_n^+(x; b, d) = \int_0^x G_{n-1}^+(x - t; b, d)\, dG^+(t; b, d)$, and let $\beta = 1 - bd^\gamma$, $m_0 = \max\{n: n \leqq y/d\}$. Then

$$(5) \qquad \begin{aligned} G_n^+(y; b, d) &= \sum_{k=0}^{m_0} \binom{n}{k} \beta^k(1 - \beta)^{n-k} G_{n-k}^+(y; d^{-\gamma}, d) \\ &\leqq \sum_{k=0}^{m_0} \binom{n}{k} \beta^k(1 - \beta)^{n-k} G_{n-k}(y; d^{-\gamma}) \\ &= \sum_{k=0}^{m_0} \binom{n}{k} \beta^k(1 - \beta)^{n-k} \frac{(d^{-\gamma})^{n-k}\Gamma^{n-k}(\gamma + 1)}{\Gamma[(n - k)\gamma + 1]} y^{(n-k)\gamma} \\ &\leqq \binom{n}{m_0} n^C \frac{b^n\Gamma^n(\gamma + 1)}{\Gamma(n\gamma + 1)} y^{n\gamma}, \qquad \text{for } n > 2m_0, \end{aligned}$$

where $C$ is a constant depending on $(b, \gamma, d, m_0)$, but not on $n$.

Now by the hypothesis of the theorem, given any $\epsilon > 0$, there is a $\delta > 0$ such that for $0 < x < \delta$ one has $(1 - \epsilon)kx^\gamma < F(x) < (1 + \epsilon)kx^\gamma$. Hence by definition of $G_n^-$ and $G_n^+$, it follows that given any $\epsilon > 0$, there is a $\delta > 0$ such that for *any* $y > 0$ $G_n^-[y; (1 - \epsilon)k, \delta] \leqq F_n(y) \leqq G_n^+[y; (1 + \epsilon)k, \delta]$. Thus letting

$$B_n = \left\{\frac{\gamma k^n\Gamma^{n-1}(\gamma + 1)y^{\gamma-1}}{\Gamma(n\gamma - \gamma + 2)} (y - \delta)^{n\gamma-\gamma+1}\right\} \Big/ \left\{\frac{k^n\Gamma^n(\gamma + 1)y^{n\gamma}}{\Gamma(n\gamma + 1)}\right\}, \qquad C_n = n^C \binom{n}{m_0},$$

and applying (4) and (5), one obtains

$$(6) \qquad (1 - \epsilon)^n(1 - B_n) \leqq \frac{\Gamma(n\gamma + 1)F_n(y)}{[k\Gamma(\gamma + 1)y^\gamma]^n} \leqq (1 + \epsilon)^n C_n .$$

Since $B_n \to 0$ and $C_n^{1/n} \to 1$, the theorem follows from (6).

THEOREM 3. *If* (1) *is satisfied then* $\lim_{n\to\infty} [F_n(a_1)/F_n(a_2)] = 0$.

PROOF.

(i) If $p_0 = 0$ then (1) implies (1*), and hence by Theorem 2

$$[F_n(a_1)/F_n(a_2)]^{1/n} \to (a_1/a_2)^\gamma < 1.$$

This implies the desired result.

(ii) If $p_0 > 0$, then letting $b(j; n, p) = \binom{n}{j} p^k(1 - p)^{n-j}$, one has for any $0 < n_0 < n$,

$$\frac{F_n(a_1)}{F_n(a_2)} = \frac{\sum_{j=0}^{n} b(j; n, 1 - p_0)F_j^*(a_1)}{\sum_{j=0}^{n} b(j; n, 1 - p_0)F_j^*(a_2)}$$

$$\leqq \frac{\sum_{j=0}^{n_0} \frac{b(j; n, 1 - p_0)}{b(n_0; n, 1 - p_0)} F_j^*(a_1)}{\sum_{j=0}^{n_0} \frac{b(j; n, 1 - p_0)}{b(n_0; n, 1 - p_0)} F_j^*(a_2)} + \frac{\sum_{j=n_0+1}^{n} b(j; n, 1 - p_0)F_j^*(a_1)}{\sum_{j=n_0+1}^{n} b(j; n, 1 - p_0)F_j^*(a_2)}.$$

Letting $n \to \infty$, the first term on the right side of the inequality converges to $F_{n_0}^*(a_1)/F_{n_0}^*(a_2)$. Then letting $n_0 \to \infty$ the entire right side goes to zero by part (i) of the theorem. This proves the theorem.

REMARK 3. Condition (1) is not a necessary condition, as can be seen from the random variable $X$ with distribution $F$, and density $f(\cdot)$ satisfying $f(x) = \exp(-x^{-1})$, $0 < x < \epsilon$, for some $\epsilon > 0$. Approximate $f$ by functions $g_m(x)$ which in a neighborhood of the origin behave like

$$[1 + 1/x + \cdots + 1/m!x^m]^{-1}.$$

Let $G_{m,n}(x)$ be the associated $n$-fold convolution of the cumulative distribution associated with $g_m(\cdot)$. Then by Theorem 2, $[G_{m,n}(a_1)/G_{m,n}(a_2)] \to 0$ as $n \to \infty$. One can then show that the convergence is uniform in $m$, and hence conclude that $[F_n(a_1)/F_n(a_2)] \to 0$. The same result holds if for some $0 < x < \epsilon, f(x) = \exp(-x^{-\gamma})$ for any $\gamma \geqq 0$.

*Note added in proof.* The author would like to remark that R. Farrell has independently obtained a result very similar to Theorem 2.

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