ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Western Regional Meeting, Monterey, California, January 29–31, 1964.)

1. Variances of Estimates of Variance Components in a Three-Way Classification.

WALLACE R. BLISCHKE, TRW Space Technology Laboratories, Redondo Beach, California.

Methods of estimating the fixed effects and the variances of the random effects in a three-way classification components of variance model including a fixed factor, two random factors and a random interaction have been given by Henderson for the case of unbalanced data (Henderson, C. R., Estimation of variance and covariance components. *Biometrics* **9** (1953) 226–252). The purpose of this paper is to compute the variances of the estimates of the variance components obtained by Henderson's "Method II," under the assumption that all random effects in the model are normally distributed. The technique used is that which was developed by Searle and which was earlier applied in a two-way classification (Searle, S. R., Sampling variances of estimates of components of variance. *Ann. Math. Statist.* **29** (1958) 167–178). The results given here include as special cases some of Searle's results as well as the case of a two-way mixed model without interaction.

2. Non-Parametric Maximum Likelihood Estimation.

G. B. CRAWFORD and S. C. SAUNDERS, Boeing Scientific Research Laboratories, Seattle, Washington. (Invited)

The asymptotically optimum properties of the maximum likelihood estimates of parameters which lie in finite dimensional Euclidian space are classical results and the connection of these estimates with the concepts of sufficiency and efficiency are well known. It is possible to formulate a definition of maximum likelihood which agrees with the usual definition on finite dimensional parameter spaces such that in several non-parametric cases (infinite dimensional parameter spaces) the maximum likelihood estimate exists and is consistent. The possible extension of certain other properties such as asymptotic normality, the construction and calculation of several of these estimates, and their advantages and application is discussed.

3. Linear Regression of Adjacent Order Statistics.

THOMAS S. FERGUSON, University of California, Los Angeles.

Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics of a sample of size n from some continuous distribution. If $E(X_{(m)} \mid X_{(m+1)})$ is linear, then the parent distribution function is, except for change of location and scale, either $F(x) = e^x$ for $x < 0$, or $F(x) = (-x)^\alpha$ for $x < -1$ and $\alpha < 0$, or $F(x) = x^\alpha$ for $0 < x < 1$ and $\alpha > 0$. This generalizes a result of G. S. Rogers (*Amer. Math. Monthly*, (1963) 857–858).

4. Max-m Distributions in the Theory of the Type II Particle Counter and the Infinitely Many Server Queue.

JOSEPH L. GASTWIRTH, Stanford University.

The max-m distribution is the distribution of m i.i.d. exponential random variables. Assume that particles arrive at a Type II counter according to a recurrent process and produce i.i.d. dead times. We determine the mean time between consecutive registrations when the dead times follow a max-m law, using the techniques developed by L. Takács (*Acta. Math. Acad. Sci. Hungar.* 1958, *Ann. Math. Statist.* 1961). The related queue size problem is discussed when the service times obey a max-m law or an Erlang law. The results of the paper indicate that, mathematically speaking, it is better to assume that the dead time distribution is a max-m law rather than an Erlang one.

5. Tolerance Limits for a Class of Distributions. D. L. Hanson and L. H. Koopmans, University of Missouri and Sandia Laboratory, Albuquerque, New Mexico. (Invited)

Let $X_1, X_2, \cdots, X_N$ be the order statistics of a sample from a population with a continuous distribution function F. Upper tolerance limits of the form $U_{N,k,j,b} = X_{N-k-j} = b(X_{N-k} - X_{N-k-j})$ are derived which are distribution free for the class $\mathfrak{F}$ of distribution functions F for which $-\log(1 - F)$ is convex. $\mathfrak{F}$ coincides with the class of distributions with increasing hazard rate (studied by Barlow, Marshall, and Proschan in the *Ann. Math. Statist.* **34** (1963) 375–389) and contains most of the common distribution functions. Suppose $0 < P, \gamma < 1$. In the non-parametric case, making the statement "$1 - F(X_{N-k}) \leqq P$ with probability at least $1 - \gamma$ for all continuous F" imposes a condition on N of the form $N \geqq N(k, \gamma, P)$. If we restrict ourselves to F's in $\mathfrak{F}$ we can obtain tolerance limits of the above form (with $U_{N,k,j,b}$ replacing X_{N-k}) for all $N \geqq 2$. Thus, what amounts to imposing an exponential rate of decrease on $1 - F$ makes it possible to obtain tolerance limits for all sample sizes (excluding one).

6. Fiducial Expectation Identities for Distributions With Group Structure (Preliminary report). R. B. Hora and R. J. Buehler, University of Minnesota.

Let P^ω be a family of distributions satisfying Fraser's (*Biometrika* **48** (1961) 261) assumptions for applicability of fiducial theory. Essentially: (i) $G = \{g\}$ is a continuous group defined on both the sample space $\mathfrak{X} = \{x\}$ and the parameter space $\Omega = \{\omega\}$; (ii) G is exactly transitive on Ω; (iii) a Haar measure exists on the space G; (iv) the identity $(*) P^{g\omega}(gX) = P^\omega(X)$ holds. This formulation includes the case of an ancillary statistic which labels the orbit of any x under G. Let E_f denote fiducial expectation and E_R denote conditional expectation given the ancillary. It is shown that $(**) H(gx, g\omega) = H(x, \omega)$ is a sufficient (but not necessary) condition for validity of Pitman's (*Biometrika* **30** (1939) 391) identity $E_f H = E_R H$. The identity has been applied for example to give "best" invariant estimators in the case of families of distributions closed under rotation. In a more general model $x = (y, z)$ where y and z are "past" and "future" observations. Here it is shown that some assumptions including $(*)$ and $(**)$ imply an expectation identity in which E_f relates to an appropriately defined joint fiducial distribution of z and ω and E_R is conditional expectation given the y-related ancillary. This generalizes results of Ramsey and Buehler (*Ann. Math. Statist.* **34** (1963) 1114) on "best" invariant predictors. For both identities special cases include multiple location and scale parameter families.

7. Pseudo Inverses in the Analysis of Variance. Peter W. M. John, University of California, Davis. (Invited)

We consider a linear model $\mathcal{E}(\mathbf{Y}) = \mathbf{X}\boldsymbol{\theta}$ where $\mathbf{Y}$ is a vector of n observations, $\boldsymbol{\theta}$ is of a vector of p unknown constant parameters and $\mathbf{X}$ is the design matrix. $\mathbf{X}$ is of rank $p - m$. There exists a matrix $\mathbf{D}$ of order $p \times m$ with rank m such that $\mathbf{XD} = \mathbf{0}$. The normal equations

$X'X\hat{\theta} = X'Y$ are consistent. Let $\mathbf{P}$ be any matrix such that $\mathbf{PX'XP} = \mathbf{P}$ and $\mathbf{X'XPX'X} = \mathbf{X'X}$; then $\mathbf{P}$ is a pseudo inverse of $\mathbf{X'X}$. Under the set of linear constraints $\mathbf{H\theta} = \mathbf{0}$, where $\mathbf{H}$ is a matrix of order $m \times p$ such that $|\mathbf{HD}| \neq \mathbf{0}$, the estimates $\hat{\theta}$ are unique. The pseudo inverses are not unique, and the relationships between them are discussed. In particular, the variance covariance matrix $\mathbf{C}$ and the matrix $(\mathbf{X'X} + \mathbf{H'H})^{-1}$ are pseudo inverses and it is shown that $\mathbf{C} = (\mathbf{X'X} + \mathbf{H'H}) - \mathbf{D}(\mathbf{D'H'HD})^{-1}\mathbf{D'}$. For the reduced intrablock equations $\mathbf{A}\hat{\tau} = \mathbf{Q}$, $\mathbf{H}$ is a row vector (h_i) and $(\mathbf{A} + \mathbf{H'H})^{-1} = \mathbf{C} + \mathbf{11'}/(\sum_i h_i)^2$.

8. Differentiable Distribution Function Processes. Charles H. Kraft, University of Minnesota.

Processes, F, in the unit interval, which have, with probability one, distribution functions as sample functions can be constructed by various interpolation procedures. If the interpolations are made with independent, identically distributed random variables, Dubins and Friedman have shown that such processes generally produce sample functions which are singular with probability one. We show that if interpolations, at the nth stage, are made at the points $k/2^n$, k odd, with independent random variables $Z(k/2^n)$, sufficient conditions for the sample function of F to be absolutely continuous are (i) $E\{Z(k/2^n)\} = \frac{1}{2}$, and (ii) $\sum_n [\sup_{0<k<2^n, k \text{ odd}} \sigma^2\{Z(k/2^n)\}]$ converges.

9. On Optimum Estimates of Parameters of Continuous Distributions. Nancy R. Mann, Rocketdyne (Division of North American Aviation, Inc.), Canoga Park, California.

Consider any specified linear function ψ of a location parameter μ and a scale parameter σ from any absolutely continuous distribution. It is shown that among estimates which are linear in the observations and which have location-invariant expected squared error, the uniformly best is a linear function of the best linear unbiased estimates of μ and σ. Furthermore, it is proved that among location-invariant-risk estimates which are linear in a complete sufficient statistic, the uniformly best $(\tilde{\psi})$ can be expressed as a function of the best unbiased parameter estimates $\hat{\mu}$ and $\hat{\sigma}$, the variance of $\hat{\sigma}$, and the covariance between $\hat{\mu}$ and $\hat{\sigma}$. Whenever the best unbiased estimate of ψ is efficient (in the Cramér-Rao sense), $\tilde{\psi}$ is the unique admissible minimax estimate of ψ. Next we consider any parameter $\phi > 0$ with uniformly minimum variance unbiased estimate $\hat{\phi}$ having variance of the form $C\phi^2$ (where C is independent of observed sample values). It is shown that $\tilde{\phi}$, the uniformly best among estimates which are linear functions of a complete sufficient statistic and which have location-invariant risk, is a function of C and $\hat{\phi}$. If $\hat{\phi}$ is efficient, $\tilde{\phi}$ is the unique admissible minimax estimate of ϕ.

10. Reversal of Lyapunov's Inequality and Other Inequalities for Means. Albert W. Marshall and Ingram Olkin, Boeing Scientific Laboratories, Seattle, Washington and Stanford University.

Lyapunov's inequality asserts that if X is a non-negative random variable with $EX^m \equiv \mu_m$, then $\mu_v^{w-u} \leqq \mu_u^{w-v}\mu_w^{v-u}$, $0 \leqq u \leqq v \leqq w$. Under the restriction $P\{m \leqq X \leqq M\} = 1$, $m > 0$, we find the largest constant c for which $\mu_v^{w-u} \geqq c\mu_u^{w-v}\mu_w^{v-u}$ (the case $u = 0$ was obtained for discrete random variables by Cargo and Shisha, *J. Res. Nat. Bur. Standards*, **66B** 169–170). Our result is obtained from more general results, which specialize also to yield generalizations of Diaz and Metcalf's extension and improvement (*Bull. Amer. Math. Soc.*, **69** 415–418) of the Kantorovich inequality. As a special case of a more general inequality, we obtain an upper bound for $Eg(X)$, (g convex) in terms of EX, which may be regarded as a reversal of Jensen's inequality.

11. **Explicit Results for the Probability Density of the First-Passage Time for Two Classes of Gaussian Processes.** C. B. MEHR and J. A. MCFADDEN, I.B.M. Research Laboratory, San Jose, California; Purdue and Stanford Universities.

Let $\{X(t):t \geqq 0\}$ be a real Gaussian Markov process, with $E[X(t)] = 0$ and $R_X(t_1, t_2) \equiv E[X(t_1)X(t_2)]$, $0 \leqq t_1 \leqq t_2$. Then $R_X(t_1, t_2) = f(t_1)g(t_2)$, $0 \leqq t_1 \leqq t_2$. Let $a(t) = f(t)/g(t)$. Let $\{W(t):t \geqq 0\}$ be the Wiener process; then $X(t) = g(t)W[a(t)]$, $0 \leqq t$. [Doob, *Ann. Math. Statist.* **20** (1949) 393]. Let $T_X(t_0, x_0; t_0 + t, 0) = \sup \{t:X(s) > 0, t_0 \leqq s \leqq t_0 + t;\ X(t_0) = x_0 > 0\}$. Using the above transformation, the density function f_{T_X} of T_X is obtained for all G. M. processes $\{X(t)\}$ from the known density of $T_W(t_0, x_0; t_0 + t, 0)$ [Fortet, *J. Math. Pures Appl.* **22** (1943) 177]. Let $\{Y(t):t \geqq 0\}$ be a stationary Gaussian process, with $E[Y(t)] = 0$ and $R_Y(t_2 - t_1) = E[Y(t_1)Y(t_2)]$, $0 \leqq t_1 \leqq t_2$. Let $\{Z(t):t \geqq 0\}$ be the conditional process obtained by taking $Y(0) = y_0$. A functional equation is solved for R_Y, giving the whole class for which $\{Z(t)\}$ is Markovian. The density function of T_Y is obtained for $\{Y(t)\}$. Included is the solution given by Slepian, *Ann. Math. Statist.* **32** (1961) 610.

12. **Admissibility of Monotone Decision Functions.** JOSEPH M. MOSER, San Diego State College.

In curve fitting one sometimes tests the hypothesis $H_0 : F = U$, where U is the uniform distribution on the unit interval, against the alternative hypothesis H_1: $F < U$. This paper considers the case where F belongs to a class of cpfs., $\mathfrak{F}$, which has the following properties; (i) $F = 0$, $x < 0$; $F \leqq U$, $0 \leqq x \leqq 1$; $F = 1$, $x > 1$. (ii) F is continuous and has a positive second derivative everywhere. The class, $\mathfrak{D}$, of decision functions considered is the class of all monotone decision functions, i.e. $\delta \in \mathfrak{D}$ if $\delta(x_1, \cdots, x_n) \leqq \delta(x_1^*, \cdots, x_n^*)$ whenever $x_i \leqq x_i^*$, $i = 1, \cdots, n$. It is then shown that a sub-class, $\mathfrak{M}$, viz. the class of non-randomized monotone decision functions, is admissible for the given hypothesis.

13. **Estimation of Bivariate Probability Density.** V. K. MURTHY, Douglas Aircraft, Inc., Santa Monica, California.

Given a random sample of size n from a distribution $F(x)$, the author has earlier established the consistency and asymptotic normality of a class of estimators for estimating the density at every point of continuity of the distribution $F(x)$. In this paper the problem of estimating the density of a bivariate distribution $F(x, y)$ at every point of continuity is considered. Based on a random sample $\{(X_1, Y_1), (X_2, Y_2), \cdots (X_n, Y_n)\}$ of size n from the bivariate distribution $F(x, y)$, a consistent class of estimators is obtained for estimating the bivariate density at every point of continuity of the distribution $F(x, y)$ and they are shown to be asymptotically normally distributed. The results for the bivariate case essentially imply those of the multivariate situation, in that, the procedure reveals in an obvious manner how one obtains the corrseponding results for higher dimensional distributions.

14. **Nonparametric Upper Confidence Bounds for** $\Pr\{Y < X\}$ **and Confidence Limits for** $\Pr\{Y < X\}$ **When** X **and** Y **Are Normal.** D. B. OWEN, K. J.

CRASWELL and D. L. HANSON, Sandia Corporation, Albuquerque, New Mexico. (By title)

Birnbaum and McCarty give a distribution-free upper confidence bound on Pr $\{Y < X\}$ when X and Y are independent and have continuous cumulative distribution functions. In this paper the restriction to a continuous distribution function is removed. The same problem is then considered where X and Y have a joint bivariate normal distribution function. The distribution of the estimator of Pr $\{Y < X\}$ is noncentral t. Upper confidence limits on Pr $\{Y < X\}$ are examined for sample sizes $n = 10(10)100$ in two cases—where X and Y are independent and where observations are taken in pairs and X and Y may be dependent.

15. **An Application of a Generalized Gamma Distribution.** GERALD S. ROGERS, University of Arizona.

Let $\{x_{i1}, \cdots, x_{in_i}\}, i = 1, 2, \cdots, k, k \geqq 2, n_i \geqq 2$, be random samples from stochastically independent Gaussian populations with unknown means μ_i and unknown variances σ_i^2. To test the hypothesis $H_0: \sigma_i^2 = \cdots = \sigma_k^2 = \sigma^2$, unknown, one may use a likelihood ratio test wherein H_0 is rejected if $\lambda < \lambda_0$, $P\{\Lambda < \lambda_0 \mid H_0\} = \alpha$ is the significance level and $\Lambda = \prod_{i=1}^k [N \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2/n_i \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2]^{n_i/2}$, $N = \sum_{i=1}^k n_i$, $\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i$. The distribution of Λ under H_0 is obtained via the convolution theorem for Mellin transforms. The details are presented for the case $k = 3$ beginning with the generalized gamma distribution introduced by E. W. Stacy [A generalization of the gamma distribution, *Ann. Math. Statist.* **33** (1962) p. 1187–1192]. The general case then follows by induction.

16. **Some Aspects of Density Fluctuation Under Diffusion Equilibrium.** H. RUBEN, University of Minnesota. (Invited)

Smoluchowski's classical analysis of the temporal fluctuation, under diffusion equilibrium, of the number of particles in a fixed geometrically well-defined region, R, of space is generalized to a set of disjoint regions: specifically, the single Smoluchowski region is divided into a finite set of nonintersecting subregions. This generalization allows a more rigorous test of some of the consequences of the classical Einstein-Smoluchowski theory of Brownian motion to be carried out, and at the same time enables the fundamental Avogadro constant to be estimated with greater precision than is possible with the single region. In particular, the reversibility paradox of Loschmidt and the recurrence paradox of Zermelo are reexamined from the point of view of the fluctuation of configurations (a configuration being defined as the complex of occupation numbers for the component subregions), rather than that of total concentration for the single region.

A fundamental notion in Smoluchowski's original analysis was that of "probability after-effect", $1 - p_R(t)$, $p_R(t)$ being (in modern stochastic process theory terminology) the autocorrelation function of the scalar stochastic process represented by the total concentration in R. The corresponding notion in the present generalized analysis is that of the covariance matrix of the vector stochastic process represented by the configuration relative to the specified subregions.

The mean speed of pedestrians has been estimated by Fürth with the aid of an empirically observed probability after-effect. The precision of the latter estimate can (as for the Avogadro constant) be increased with the aid of an empirically observed covariance matrix of the process.

17. Approximations to the Distribution of Quadratic Forms. M. M. SIDDIQUI, Boulder Laboratories National Bureau of Standards, Boulder, Colorado. (Invited)

Let $Q = \frac{1}{2} \sum_{j=1}^{2k} a_j X_j^2$, where $0 < a_1 \leqq a_2 \leqq \cdots \leqq a_{2k}$ and $X_1, \cdots, X_{2k}$ are independent $N(0, 1)$ variates. Let $F(x) = \Pr(Q > x)$. Let $Q_1 = \frac{1}{2} \sum_1^k a_{2j-1}(X_{2j-1}^2 + X_{2j}^2)$, $Q_2 = \frac{1}{2} \sum_1^k a_{2j}(X_{2j-1}^2 + X_{2j}^2)$ and $F_i(x) = \Pr(Q_i > x)$, $i = 1, 2$. $F_i(x)$ can be exactly evaluated as a linear combination of gamma distribution functions. Noting that $Q_1 < Q < Q_2$ almost surely, we have $F_1(x) < F(x) < F_2(x)$ if $x > 0$. An approximation to $F(x)$ is obtained by minimizing $d(F, \hat{F})$ where $\hat{F}$ is a linear combination of F_1 and F_2 and $d(\cdot, \cdot)$ is the distance function of the metric space $L^2(0, \infty)$. Other approximation techniques are also briefly discussed and some numerical examples have been worked out.

18. Optimum Classification Rules. M. S. SRIVASTAVA, University of Toronto.

Let x_i, $i = 0, 1, \cdots k$ be a p dimensional random vector which is distributed independently in π_i as multivariate normal with mean μ_i and covariance matrix Δ. Let $\bar{x}_i$'s be the sample mean vectors from random samples of sizes n, and let S independent of x_i's be distributed as Wishart with n' degrees of freedom and mean $n'\Delta$. The problem is to select one of the k decisions $D_i : \mu_0 = \mu_i (i = 1, 2, \cdots k)$. For simple loss function, it is proved that the rule, to take the decision D_i if i is the smallest integer for which the minimum of the statistics $(\bar{x}_i - \bar{x}_0)'(A^{-1} + W)^{-1}(\bar{x}_i - \bar{x}_0)$, where $W = S + (2\beta + 1)^{-1} \sum_{j=1}^k (\bar{x}_i - \bar{x}_0)(\bar{x}_i - \bar{x}_0)' - (2\beta + 1)^{-1}(k + 1)^{-1} (\sum_{j=1}^k (\bar{x}_j - \bar{x}_0)) (\sum_{j=1}^k (\bar{x}_j - \bar{x}_0)')$, $\beta \neq -\frac{1}{2}$ and A a known positive definite matrix, is attained, is admissible in the class of translation invariant procedures.

19. Asymptotic Independence of Certain Statistics Connected With Extreme Order Statistics in a Bivariate Distribution. O. P. SRIVASTAVA, Seton Hall University. (By title)

Let (X_i, Y_i) $i = 1, 2, \cdots, n$ be n independent observations from a continuous distribution with distribution function $F(x, y)$ and marginals $F_1(x)$ and $F_2(y)$. Let $Z_1^{(n)} \leqq Z_2^{(n)} \leqq \cdots \leqq Z_n^{(n)}$ and $W_1^{(n)} \leqq W_2^{(n)} \leqq \cdots \leqq W_n^{(n)}$ be the ordered values of X's and Y's respectively. Further let a_n, b_n, c_n, and d_n be siutably chosen constants which ensure the convergence of $U_n(x) = nF_1(a_n x + b_n)$ and $V_n(y) = nF_2(c_n y + d_n)$ to $U(x)$ and $V(y)$ respectively where $U(x)$ and $V(y)$ are non-negative, non-decreasing functions of x and y respectively satisfying the conditions $U(-\infty) = V(-\infty) = 0$; $U(\infty) = V(\infty) = \infty$. In this paper it has been proved that a necessary and sufficient condition for the asymptotic independence of $Z_1^{(n)}$ and $W_1^{(n)}$ is $F(a_n x + b_n, c_n y + d_n) = o(1/n)$. Under the same condition with the help of a recurrence relation it is proved that $Z_k^{(n)}$ and $W_l^{(n)}$ are asymptotically independent for any $k > 1$, $l > 1$, $\lim_{n\to\infty} k/n = \lim_{n\to\infty} l/n = 0$. Further the distances $Z_k^{(n)} - Z_1^{(n)}$ and $W_l^{(n)} - W_1^{(n)}$ between two order statistics in both components of the distribution, are also asymptotically independent under the same condition mentioned above.

(Abstract of a paper to be presented at the Annual Meeting, Amherst, Massachusetts, August 30 to September 4, 1964. An additional abstract appeared in the December, 1963, issue, and others will appear in the June and September, 1964, issues.)

2. A Symmetric Stopping Rule and Its Applications to Probability Sampling. P. K. PATHAK, University of Illinois. (Introduced by J. L. Doob)

Let (Ω, S, P) be a probability measure space. Let $X_1(\omega), X_2(\omega), \cdots, X_n(\omega), \cdots$ be interchangeable random variables defined on (Ω, S, P). Let $M(\omega)$ be an inter-valued random variable defined on (Ω, S, P) with essential infimum at $k > 0$ such that $P[\omega: M(\omega) = m \mid X_1(\omega), \cdots, X_m(\omega)]$ is a symmetric function of $X_1(\omega), X_2(\omega), \cdots, X_m(\omega)$. $M(\omega)$ is called a symmetric stopping rule. Then the following assertions hold: (1) $X_1(\omega), X_2(\omega), \cdots, X_m(\omega)$ given $M(\omega) = m$ are interchangeable; (2) If $f(x_i, \cdots, x_p)(0 < p \leqq k)$ is a Baire function on R^p such that $E[|f(X_1(\omega), \cdots, X_p(\omega))|] < \infty$ then

$$E\left[\binom{M(\omega)}{p}^{-1} \sum{}' f(X_{i1}(\omega), \cdots, X_{ip}(\omega))\right] = E[f(X_1(\omega), \cdots, X_p(\omega))]$$

where $\sum'$ extends over all combinations of $X_{i1}(\omega), \cdots, X_{ip}(\omega)$ from $X_1(\omega), \cdots, X_n(\omega)$. Applications of the above results to fixed cost sampling schemes and inverse sampling with unequal probabilities are given.

(Abstracts not connected with any meeting of the Institute)

1. **The Martin Boundary for Pólya's Urn Scheme.** DAVID BLACKWELL and DAVID G. KENDALL, University of California, Berkeley, and Churchill College, Cambridge.

Let Pólya's urn initially contain k differently colored balls, and let each ball drawn from the urn be returned to it together with a fresh ball of the same color. If we consider the contents of the urn after the 1st, 2nd, $\cdots$, nth, $\cdots$ replacement we have a discrete-parameter Markov chain whose state-space consists of all ordered $k$ads of positive integers. The Martin boundary is calculated and is shown to be homeomorphic with a standard Euclidean $(k-1)$-cell U_{k-1} (the portion of the hyperplane $\sum \tau_j = 1$ cut off by the closed positive cone in R_k). If the fractional composition of the urn after the nth replacement is denoted by $f^n = (f_1^n, \cdots, f_k^n)$, then the martingale $\{f^n : n = 1, 2, \cdots\}$ converges with probability one to $\tau = (\tau_1, \cdots, \tau_k)$ which can be identified with the Doob limit on the Martin boundary M; moreover, τ is uniformly distributed over the simplex $M = U_{k-1}$. Applications to moment problems and to microbiology are sketched, and a connection is established with a classical argument due to Bayes and Crofton.

2. **The Estimation of Variance Components I: One-Way Models.** BRUCE M. HILL, University of Michigan.

Estimation of variance components in the one-way random model is considered from a subjective Bayesian point of view: the situation in which the classical unbiased estimate of the between variance component is negative is explored in some detail, and exact and approximate posterior distributions for the unbalanced case are obtained.

3. **Recurrent Events and Completely Monotonic Sequences.** J. F. C. KINGMAN, Pembroke College, Cambridge.

Let $\{u_n; n = 0, 1, 2, \cdots\}$ be any completely monotonic sequence with $u_0 = 1$. Then there exists a recurrent event $\mathcal{E}$ such that u_n is the probability that $\mathcal{E}$ occurs at time n. Let $h(t)$ be any function continuous and completely monotonic in $t \geqq 0$. Then there exists a renewal process with renewal density $h(t)$.

4. **Sharp Bounds for Two Measures of Skewness.** C. L. MALLOWS and DONALD RICHTER, Bell Telephone Laboratories, Inc., Murray Hill, New Jersey.

Suppose X is a random variable with distribution function $F(x)$, mean μ, and variance σ^2. We distinguish two conventions regarding the definition of the median of X, as follows: m_1 is any number satisfying $F(m_1 - 0) \leqq \frac{1}{2} \leqq F(m_1)$, while $m_2 = \frac{1}{2} \sup \{x:F(x) < \frac{1}{2}\} + \frac{1}{2} \inf \{x:F(x) > \frac{1}{2}\}$. The quantities $s_j = (\mu - m_j)/\sigma (j = 1, 2)$ are measures of skewness. Hotelling and Solomons (*Ann. Math. Statist.* **3** 141–142) showed that $|s_1| \leqq 1$, and Majindar (*Ann. Math. Statist.* **33** 1192–1194) has obtained the bounds $|s_2| < 2(pq)^{\frac{1}{2}}(p + q)^{-\frac{1}{2}}$ where $p = F(\mu - 0)$, $q = 1 - F(\mu)$. We prove the following *Theorem*. Sharp bounds are: $s_1 = s_2 = 0$ if $0 < p, q < \frac{1}{2}$; $0 \leqq s_1 \leqq 2q^{\frac{1}{2}}(1 + 2q)^{-\frac{1}{2}}$ if $0 < q \leqq \frac{1}{2} \leqq p$; $0 < s_2 < 2q^{\frac{1}{2}}(1 + 2q)^{-\frac{1}{2}}$ if $0 < q < \frac{1}{2} < p$; $0 \leqq s_2 \leqq q^{\frac{1}{2}}(1 + 2q)^{-\frac{1}{2}}$ if $0 < q < \frac{1}{2} = p$; and $|s_1| \leqq 1$, $|s_2| < \frac{1}{2}$ if $p = q = \frac{1}{2}$. The equalities are attainable if and only if $p = \frac{1}{2}$.

5. Exact Distributions of Extremes, Ranges and Mid-Ranges in Samples From a Multivariate Population With Applications to Normal and Pareto Type 1 Populations (Preliminary report). K. V. MARDIA, University of Rajasthan, Jaipur (Raj.) India. (Introduced by B. D. Tikkiwal)

Let $(x_{1r}, \cdots, x_{kr})$, $r = 1, 2, \cdots, n$, be a random sample of size n from a k-variate continuous population having the probability density function (p.d.f.) $f(x_1, \cdots, x_k)$. Let the ith range be $R_i = Y_i - X_i$ and ith mid-range be $V_i = (Y_i + X_i)/2$, where Y_i and X_i are the maximum and minimum observations of the ith variate, $i = 1, 2, \cdots, k$. The p.d.f. of $(X_1, \cdots, X_k, Y_1, \cdots, Y_k)$ is obtained and utilized to derive the exact distributions of $(R_1, \cdots, R_k)$ and $(V_1, \cdots, V_k)$. In particular, the joint distribution function of the ranges is given by

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(\frac{(-\partial)^k}{\partial X_1 \cdots \partial X_k} \left\{ \int_{X_1}^{C_1} \cdots \int_{X_k}^{C_k} f(x_1, \cdots, x_k)\, dx_1 \cdots dx_k \right\}^n \right) dX_1, \cdots, dX_k,$$

$C_i = X_i + R_i$, $i = 1, \cdots, k$. These distributions simplify considerably for $n = 2$ and $n = 3$. They further simplify for the normal and the Pareto Type 1 populations (Mardia, *Ann. Math. Statist.* **33** (1962) 1008–1015).

6. Some Distribution-Free k-Sample Rank Tests of Homogeneity Against Ordered Alternatives. MADAN L. PURI, New York University.

For testing homogeneity against ordered alternatives on the basis of k independent random samples of sizes m_i; $i = 1, \cdots, k$; the test statistics of the form $S = \sum \sum_{i<j} m_i m_j h_{ij}$ are considered. Here $h_{ij} = m_i^{-1} \sum_{\lambda=1}^{m_i} E_\psi[V(s_{ij,i\lambda})] - m_j^{-1} \sum_{\lambda=1}^{m_j} E_\psi[V(s_{ij,j\lambda})$ where $s_{ij,K1}, \cdots, s_{ij,Km_K}$, $(k = i, j)$ denote the ranks of $X_{K1}, \cdots, X_{Km_K}$, $(K = i, j)$ in the combined (i, j)th sample, and where $V(1) < \cdots < V(m_i + m_j)$ denote an ordered sample of size $(m_i + m_j)$ from a distribution ψ. Under suitable regularity conditions, the asymptotic normality of the class of S statistics is established. These results are specialized to the S_N statistic and the S_R statistic obtained by taking for ψ the standard normal distribution and the rectangular distribution on (0, 1) respectively. The asymptotic relative efficiencies of the S_N and the S_R tests for translation alternatives are shown to be the same as those of the two-sample normal scores test and the Wilcoxon text (cf. Hodges and Lehmann, *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** (1961) 307–318). Under the assumptions of normality and for some k's the asymptotic power comparisons are made between the S tests and some of the existing tests in literature.

7. On a Problem of Servicing a Poisson Flow of Demands (Preliminary report). D. N. SHANBHAG, Karnatak University, Dharwar, India. (Introduced by B. R. Bhat)

Downton (*J. Roy. Statist. Soc.* (1962) 107–111) has considered the problem of servicing a flow of demands with (i) the inter-arrival time distribution $\lambda \exp(-\lambda t)\, dt (0 < t < \infty)$; (ii) the service-time distribution $dB(t) (0 < t < \infty)$; and (iii) an unbounded number of servers. For an infinite sequence of ordered random variables $W_1 > W_2 > \cdots$, where W_r is the rth largest unexpired service time at an arbitrary instant during the steady state, he proved that $\psi(z, x) = \sum_{r=1}^{\infty} \Pr(W_r \leqq x) z^r (0 < x < \infty, |z| < 1)$ satisfies the equation $\partial\psi(z, x)/\partial x = (1 - z)\{1 - B(x)\}\lambda\psi(z, x)$ and is given by $\psi(z, x) = z(1 - z)^{-1} \exp[-\lambda(1 - z) \int_{y=x}^{\infty} \{1 - B(y)\}\, dy]$. Now it is proved that if $W_r(t)$ is the rth largest unexpired service time at t, $\psi(z, x, t) = \sum_{r=1}^{\infty} \Pr(W_r(t) \leqq x) z^r (0 < x < \infty, |z| < 1)$ satisfies the partial differential equation $\partial\psi(z, x, t)/\partial t - \partial\psi(z, x, t)/\partial x = -(1 - z)\{1 - B(x)\}\lambda\psi(z, x, t)$. In fact $\psi(z, x, t)$ equals $z^{r+1}(1 - z)^{-1} \exp[-\lambda(1 - z) \int_{y=x}^{t+x} \{1 - B(y)\}\, dy]$ for $W_{r+1}(0) < t + x < W_r(0)$ and equals $z(1 - z)^{-1} \exp[-\lambda(1 - z) \int_{y=x}^{t+x} \{1 - B(y)\}\, dy]$ for $t + x > W_1(0)$. Further $P_n(t)$, the probability that exactly n servers are busy at t, equals $\{(n - r)!\}^{-1} \exp[-\lambda \int_{y=0}^{t} \{1 - B(y)\}\, dy][\lambda \int_{y=0}^{t} \{1 - B(y)\}\, dy]^{n-r}$ for $W_{r+1}(0) < t < W_r(0)$, and equals $(n!)^{-1} \exp[-\lambda \int_{y=0}^{t} \{1 - B(y)\}\, dy][\lambda \int_{y=0}^{t} \{1 - B(y)\}\, dy]^n$ for $t > W_1(0)$. $\lim_{t\to\infty} \Pr(W_r(t) \leqq x)$ and $\lim_{t\to\infty} P_n(t)$ coincide with those given by Downton.

8. On a General Class of Contagious Distributions and Pascal-Poisson Distribution. K. Subrahmaniam, The Johns Hopkins University.

In this paper we discuss the method of fitting the Pascal-Poisson distribution of which the p.g.f. is given by $H(z) = [1 - (\mu/\alpha c)\{\exp c(z - 1) - 1\}]^{-\alpha}$. This distribution arises as a limiting form of a very general class of contagious distributions defined by the p.g.f. ${}_1F_1[\alpha, \alpha + \beta, \{m_1 \exp \lambda(z - 1) - m_1\}]$, where ${}_1F_1$ stands for Kummer's form of the confluent hypergeometric series [Whittaker and Watson, *A Course of Modern Analysis*, (1962)]. This general class of contagious distributions is the outcome of a modified approach to the Neyman type A distribution. The classical approach is discussed by Gurland (*Biometrics*, 1958). Following Gurland's notation (*Biometrika*, 1957), we can summarize this approach as $P(m) \vee [B(n; p) \wedge_n P(\lambda)] \approx$ Neyman type A $[\exp m \{\exp p(z - 1) - 1\}]$. Upon changing the order of operations, $[B(n; p) \vee P(\lambda)] \wedge_n P(m) \approx$ Neyman type A $[\exp mp \cdot \{\exp(z - 1) - 1\}]$. By a process of compounding this resulting Newman type A on a Beta distribution for p, we obtain the general class of contagious distributions defined above. We have studied the various limiting forms of this distribution. A method for fitting the Pascal-Poisson is devised by using a reduction formula for the derivatives of $H(z)$ at $z = 0$.