THE SPECTRAL THEOREM FOR FINITE MATRICES AND COCHRAN'S THEOREM[1]

By M. F. Smiley

University of California, Riverside

If $B_0, \cdots, B_k$ are real symmetric n by n matrices such that

$$\sum (B_i\,;\, i = 0, \cdots, k) = I \quad \text{and} \quad \sum (\text{rank } B_i\,;\, i = 0, \cdots, k) = n$$

where I is the n by n identity matrix, then there is a real orthogonal matrix P such that $P'B_iP$ is a diagonal matrix of zeros and ones $(i = 0, \cdots, k)$. This is W. G. Cochran's Theorem [2] which is of great importance in the analysis of variance ([5], p. 154). Our purpose here is to give a proof of this theorem which reveals it as a characterization of the spectral decomposition of finite real symmetric matrices and may suggest some useful generalizations.

Let us first agree on some notation. If D is a division ring, we let $V_n(D)$ denote the totality of row vectors $x = (\xi_1, \cdots, \xi_n)$ with ξ_g in D $(g = 1, \cdots, n)$. We also let D_n denote the totality of n by n matrices with elements in D.

Lemma 1. *Let D be a division ring and let A and B in D_n be such that $AB = BA$ and rank $(A + B) =$ rank $A +$ rank B. Then $AB = 0$.*

Proof. Let R_A denote the totality of vectors xA for x in $V_n(D)$. Define R_B and R_{A+B} in a similar manner. Then R_A is a subspace of $V_n(D)$ and rank $A = \dim R_A$. Now

$$\dim (R_A + R_B) + \dim (R_A \cap R_B) = \dim R_A + \dim R_B = \dim R_{A+B}.$$

Because $R_{A+B} \subseteq R_A + R_B$, this implies that $R_A \cap R_B = 0$. But then $xAB = xBA$ is in $R_A \cap R_B = 0$ so that $xAB = 0$ for all x in $V_n(D)$ and hence $AB = 0$.

Lemma 2. *Let R be a ring with identity element 1 for which $2x = 0$ implies that $x = 0$. If elements a_i $(i = 0, \cdots, k)$ of R satisfy, for $i \neq j$ and $i, j = 0, \cdots, k$,*

$$\sum (a_i\,;\, i = 0, \cdots, k) = 1, \tag{1}$$

$$a_i \sum (a_j\,;\, j \neq i, j = 0, \cdots, k) = 0, \tag{2}$$

$$(a_i + a_j) \sum (a_h\,;\, h \neq i, h \neq j, h = 0, \cdots, k) = 0, \tag{3}$$

then $a_i^2 = a_i$ and $a_ia_j = 0$ for $i \neq j$ $(i, j = 0, \cdots, k)$; that is, the elements a_i $(i = 0, \cdots, k)$ are orthogonal idempotents of R.

Proof. Multiply (1) by a_i on the left and use (2) to get $a_i^2 = a_i$ $(i = 0, \cdots, k)$. By (3) we have $(a_0 + a_1)(a_2 + \cdots + a_k) = 0$, and (2) gives $-a_0a_1 - a_1a_0 = 0$. By symmetry, we have $a_ia_j + a_ja_i = 0$ for $i \neq j$ $(i, j = 0, \cdots, k)$. But then $a_ia_j + a_ia_ja_i = 0 = a_ia_ja_i + a_ja_i$ so that $a_ia_j = a_ja_i$ and $2a_ia_j = 0$, and we have $a_ia_j = 0$, as desired.

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Proof of Cochran's Theorem. In Lemma 1, take $A = B_i$ and $B = \sum (B_j ; j \neq i, j = 0, \cdots, k)$. Then

$$n = \text{rank } (A + B) \leqq \text{rank } A + \text{rank } B = n$$

gives rank $(A + B) = \text{rank } A + \text{rank } B$, and $AB = BA$ is clear since $A + B = I$ This verifies (2) of Lemma 2. A similar argument with $A = B_i + B_j$ for $i \neq j$ and $B = \sum (B_h ; h \neq i, h \neq j, h = 0, \cdots, k)$ verifies (3) of Lemma 2. Thus the matrices B_i $(i = 0, \cdots, k)$ are orthogonal (hence commuting) idempotents and their simultaneous reduction to the desired diagonal forms is a consequence of a standard theorem on matrices.

Added at the suggestion of a referee: A. T. Craig [1] gives a beautifully direct and simple proof of Cochran's Theorem. Graybill and Marsaglia [3] and Hogg and A. T. Craig [4] discuss interesting modifications of Cochran's Theorem.

REFERENCES

[1] Craig, A. T. (1938). On the independence of certain estimates of variance. *Ann. Math. Statist.* **9** 48–55.

[2] Cochran, W. G. (1934). The distribution of quadratic forms in a normal system. *Proc. Cambridge Philos. Soc.* **30** 178–191.

[3] Graybill, F. A. and Marsaglia, G. (1957). Idempotent matrices and quadratic forms in the general linear hypothesis. *Ann. Math. Statist.* **28** 678–686.

[4] Hogg, R. V. and Craig, A. T. (1958). On the decomposition of certain χ^2 variables. *Ann. Math. Statist.* **29** 608–610.

[5] Tucker, H. G. (1962). *An Introduction to Probability and Mathematical Statistics.* Academic Press, New York.