A NOTE ON THE MAXIMIZATION OF A NON-CENTRAL CHI-SQUARE PROBABILITY

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If $\chi^2_{n;z}$ denotes a chi-square variate with n degrees of freedom and non-centrality parameter z then

$$\text{(1)} \qquad \text{Prob}\,(\chi^2_{n;z} \leqq r^2) = z^{1-\frac{1}{2}n} e^{-\frac{1}{2}z^2} \int_0^r t^{\frac{1}{2}n} e^{-\frac{1}{2}t^2} I_{\frac{1}{2}n-1}(zt)\, dt$$

where I_ν denotes the modified Bessel function of the first kind and order ν[1]. Designate the probability indicated in (1) by $g(z)$. Using the relationships $xI'_\nu(x) = \nu I_\nu(x) + xI_{\nu+1}(x)$, $(d/dx)\,[x^\nu I_\nu(x)] = x^\nu I_{\nu-1}(x)$ [2], integration by parts reveals that

$$\text{(2)} \qquad g'(z) = -r^{\frac{1}{2}n} z^{1-\frac{1}{2}n} \exp\,[-\tfrac{1}{2}(r^2 + z^2)]\, I_{\frac{1}{2}n}(rz).$$

Consider the function

$$\text{(3)} \qquad f(x) = g(|d + x|) + g(|d - x|)$$

where $d \geqq r$. Suppose two spherical targets of radii r have centers at $(-d, 0, \cdots, 0)$ and $(d, 0, \cdots, 0)$ in n-space. If a point bomb with impact density spherical normal, variance unity, were aimed at the point $(x, 0, \cdots, 0)$ then $f(x)$ represents the probability it intercepts either of the targets. It follows from (3) that $f'(0) = 0$.

Proposition 1. *The function $f(x)$ is maximized when $x = 0$ if $0 < r \leqq d \leqq 1$.*

Proof. Obviously, the maximum will occur for x between $-d$ and d. Using the series expansion of $I_{n/2}$ we have for $-d \leqq x \leqq d$ that

$$\text{(4)} \qquad f'(x) = 2^{-\frac{1}{2}n} r^n e^{-\frac{1}{2}r^2} \sum_{j=0}^{\infty} [r^{2j}/4^j j!\, \Gamma(\tfrac{1}{2}n + j + 1)]\, H_{2j+1}(x)$$

where

$$\text{(5)} \quad H_{2j+1}(x) = (d - x)^{2j+1} \exp[-\tfrac{1}{2}(d - x)^2] - (d + x)^{2j+1} \exp[-\tfrac{1}{2}(d + x)^2].$$

One can show that $H_{2j+1}(x) > 0$ for all j if $-d \leqq x < 0$, $d \leqq 1$. By the symmetry it follows that $f(x)$ attains a maximum when $x = 0$.

Proposition 2. *If $d > 1$ the maximum does not occur when $x = 0$ for sufficiently small r.*

Proof. Using $f''(0) = 2g''(d)$ and a recursion formula already stated it is easy to show that

$$\text{(6)} \quad f''(0) = 2\,(r/d)^{\frac{1}{2}n} \exp[-\tfrac{1}{2}(r^2 + d^2)]\,[(d^2 - 1)\, I_{\frac{1}{2}n}(rd) - rd\, I_{\frac{1}{2}n+1}(rd)].$$

Received 7 February 1963; revised 1 July 1963.

One can prove that $I_\nu(t) > I_{\nu+1}(t) > 0$ for $t > 0$, $\nu \geqq 0$. Thus, $f''(0) > 0$ for sufficiently small r if $d > 1$.

It is somewhat surprising that sufficient conditions exist under which the point of maximization is independent of both r and n.

REFERENCES

[1] RUBEN, HAROLD (1960). Probability content of regions under spherical normal distributions, I. *Ann. Math. Statist.* **31** 598–618.

[2] WATSON, G. N. (1922). *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press.