AN UPPER BOUND FOR THE NUMBER OF DISJOINT BLOCKS IN CERTAIN PBIB DESIGNS

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1. Introduction and summary. Majinder [3] obtained an upper bound for the number of disjoint blocks in BIB designs. In this paper we give an upper bound for the number of disjoint blocks in (i) Semi-regular GD designs, (ii) PBIB designs with two associate classes having triangular association scheme, (iii) PBIB designs with two associate classes having L_2 association scheme, and (iv) PBIB designs with three associate classes having rectangular association scheme. The main tools used to establish the results of this paper are the theorems proved by (i) Bose and Connor [1], (ii) Raghavarao [4] and (iii) Vartak [6].

2. An upper bound for the number of disjoint blocks in semi-regular GD designs. An incomplete block design with v treatments each replicated r times in b blocks of size k is said to be group divisible (GD) [2], if the treatments $v = mn$ can be divided into m groups, each with n treatments, so that the treatments belonging to the same group occur together in λ_1 blocks and treatments belonging to different groups occur together in λ_2 blocks ($\lambda_1 \neq \lambda_2$). The parameters of such a design are $v = mn$, b, r, k, λ_1, λ_2, $n_1 = n - 1$, $n_2 = n(m - 1)$. They obviously satisfy the relations $bk = vr$, $(n - 1)\lambda_1 + n(m - 1)\cdot\lambda_2 = r(k - 1)$, $r \geqq \lambda_1$, $r \geqq \lambda_2$.

Semi-regular GD designs [1] are characterised by $r - \lambda_1 \geqq 0$ and $rk - v\cdot\lambda_2 = 0$. Bose and Connor [1] proved the following theorem for semi-regular GD designs.

Theorem 2.A. *For a semi-regular* GD *design, k is divisible by m. If $k = cm$, then every block must contain c treatments from every group.*

We use Theorem 2.A to obtain an upper bound for the number of disjoint blocks which have no treatment common with a given block of semi-regular GD designs. The result is given in Theorem 2.1.

Theorem 2.1. *A given block of the semi-regular* GD *design cannot have more than*

$$b - 1 - \frac{v(v - m)(r - 1)^2}{(v - k)(b - r) - (v - rk)(v - m)}$$

disjoint blocks with it and if some block has that many disjoint blocks, then

$$k[(v - k)(b - r) - (v - rk)(v - m)]/(v - m)(r - 1)$$

is an integer and each non-disjoint block has

$$k[(v - k)(b - r) - (v - rk)(v - m)]/(v - m)(r - 1)$$

treatments common with that given block.

Received 13 March 1963.

PROOF. Let the given block have d disjoint blocks. Let it have x_i treatments common with the ith of the remaining $(b - d - 1)$ blocks. Considering the treatments of the given block singly, we obtain

$$\sum_{i=1}^{b-d-1} x_i = k(r - 1). \tag{2.1}$$

The given block, by virtue of Theorem 2.A contains k/m treatments from each group which form pairs of first associates. Hence considering the treatments of the given block pairwise, we get

$$\sum_{i=1}^{b-d-1} x_i(x_i - 1) = k[(k - m)\lambda_1 + k(m - 1)\lambda_2 - m(k - 1)]/m. \tag{2.2}$$

Now for semi-regular GD designs, $\lambda_2 = rk/v$. Then from $n_1 \cdot \lambda_1 + n_2 \cdot \lambda_2 = r(k - 1)$, we get $\lambda_1 = r(k - m)/(v - m)$.
Let $\bar{x} = \sum_{i=1}^{b-d-1} x_i/(b - d - 1)$. It follows from (2.1) and (2.2) that

$$\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = \frac{k^2[(v - k)(b - r) - (v - rk)(v - m)}{v(v - m)} - \frac{k^2(r - 1)^2}{(b - d - 1)}. \tag{2.3}$$

As $\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 \geqq 0$, it follows from (2.3) that

$$d \leqq b - 1 - \frac{v(v - m)(r - 1)^2}{(v - k)(b - r) - (v - rk)(v - m)}, \tag{2.4}$$

which proves the first part of Theorem 2.1. If, however,

$$d = b - 1 - \frac{v(v - m)(r - 1)^2}{(v - k)(b - r) - (v - rk)(v - m)}, \tag{2.5}$$

then $\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = 0$, giving that

$$x_i = \frac{k[(v - k)(b - r) - (v - rk)(v - m)]}{v(v - m)(r - 1)}. \tag{2.6}$$

is an integer, and the given block has $k[(v - k)(b - r) - (v - rk)(v - m)]/v(v - m) \cdot (r - 1)$ treatments common with each of the non-disjoint blocks.

The following are the companion theorems to Theorem 2.1.

THEOREM 2.2. *The necessary and sufficient condition that a block of semi-regular* GD *design has the same number of treatments common with each of the remaining blocks is that* (i) $b = v - m + 1$ *and* (ii) $k(r - 1)/(v - m)$ *is an integer.*

PROOF. Let a block have x_i treatments common with the ith of the remaining $(b - 1)$ blocks. Then from (2.3), noting that $d = 0$, we obtain

$$\sum_{i=1}^{b-1} (x_i - \bar{x})^2 = \frac{k^2(v - k)(b - r)(b - v + m - 1)}{v(v - m)(b - 1)}. \tag{2.7}$$

All factors of the r.h.s. of (2.7) except $(b - v + m - 1)$ are positive. Hence the result immediately follows.

THEOREM 2.3. *If a block of the semi-regular* GD *design with parameters* $v = mn = tk$, $b = tr$, *(*t *an integer greater than* 1*), has* $(t - 1)$ *blocks disjoint with it, then the necessary and sufficient condition that it has the same number of treatments common with each of the non-disjoint blocks is that* (i) $b = v - m + r$ *and* (ii) k/t is an integer.

PROOF. Let a block have x_i treatments common with the ith of the remaining $(b - t) = t(r - 1)$ non-disjoint blocks. Then, we have, from (2.3), noting that $d = t - 1$,

$$\sum_{i=1}^{b-t} (x_i - \bar{x})^2 = \frac{k^2(t-1)(b-v+m-r)}{t(v-m)}, \tag{2.8}$$

where $\bar{x} = k/t$. The result, then, immediately follows from (2.8).

We get the following two corollaries from this theorem.

COROLLARY 2.3A. *For a resolvable semi-regular* GD *design*, $b \geqq v - m + r$. This is also proved in [1].

COROLLARY 2.3B. *The necessary and sufficient condition that a resolvable semi-regular* GD *design be affine resolvable is that it has a block which has the same number of treatments common with each block not belonging to its own replication.*

3. An upper bound for the number of disjoint blocks in PBIB designs with two associate classes having a triangular association scheme. A PBIB design with two associate classes is said to have a triangular association scheme [2], if the number of treatments is $v = n(n - 1)/2$ and the association scheme is an array of n rows and n columns with the following properties:

(a) the positions in the principal diagonal are blank,

(b) the $n(n - 1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \cdots, n(n - 1)/2$, corresponding to the treatments,

(c) the array is symmetric about the principal diagonal,

(d) for any treatment θ, the first associates are exactly those treatments which lie in the same row and the same column as θ.

The primary parameters of this design are $v = n(n - 1)/2$, b, r, k, λ_1, λ_2, $n_1 = 2n - 4$, $n_2 = (n - 3) \cdot (n - 2)/2$. The following Theorem is proved by Raghavarao [4].

THEOREM 3.A. *If in a* PBIB *design with two associate classes having a triangular association scheme* $rk - v\lambda_1 = n(r - \lambda_1)/2$, *then* $2k$ *is divisible by* n. *Further every block of this design contains* $2k/n$ *treatments from each of the* n *rows of the association scheme.*

We use Theorem 3.A and obtain an upper bound for the number of disjoint blocks which have no treatments common with a given block of this design. The result is contained in Theorem 3.1.

THEOREM 3.1. *A given block of the* PBIB *with two associate classes having triangular association scheme and* $rk - v\lambda_1 = n(r - \lambda_1)/2$ *cannot have more than*

$$b - 1 - \frac{n(v - n)(r - 1)^2}{n(b + 1 - 2r) - (v - rk)(n - 2)}$$

disjoint blocks with it and if some block has that many disjoint blocks, then $k[n(b + 1 - 2r) - (v - rk)\cdot(n - 2)]/n(v - n)\cdot(r - 1)$ *is an integer and each nondisjoint block has* $k[n\cdot(b + 1 - 2r) - (v - rk)\cdot(n - 2)]/n(v - n)\cdot(r - 1)$ *treatments common with that given block.*

PROOF. Let the given block have d disjoint blocks. Let it have x_i treatments common with the ith of the remaining $(b - d - 1)$ non-disjoint blocks. Then, considering the treatments of the given block singly, we have,

$$\sum_{i=1}^{b-d-1} x_i = k(r - 1). \tag{3.1}$$

Considering treatments of the given block pairwise and using Theorem 3.A, we have

$$\sum_{i=1}^{b-d-1} x_i(x_i - 1) = n\cdot(2k/n)(2k/n - 1)(\lambda_1 - 1) + [k(k - 1) - n\cdot(2k/n)\cdot(2k/n - 1)]\cdot(\lambda_2 - 1). \tag{3.2}$$

Let $v = v_1\cdot v_2$, where $v_1 = n/2$, $v_2 = n - 1 = 2v_1 - 1$. From $rk - v\cdot\lambda_1 = n(r - \lambda_1)/2$, we get $\lambda_1 = r(k - v_1)/2\, v_1(v_1 - 1)$. Also, we see that $n_1 = 4(v_1 - 1)$, $n_2 = (v_1 - 1)\,(v_2 - 2)$ and $\lambda_2 = r(kv_1 + v_1 - 2k)/v_1\cdot(v_1 - 1)\,(v_2 - 2)$. Putting $n = 2\,v_1$ and substituting the values of λ_1 and λ_2 in (3.2), we obtain

$$\sum_{i=1}^{b-d-1} x_i(x_i - 1) = k^2[v_1(b + 1 - 2r) - (v - rk)(v_1 - 1)]/v_1(v - 2v_1) - k(r - 1). \tag{3.3}$$

Let $\bar{x} = k(r - 1)/(b - d - 1)$. Then from (3.1) and (3.3), we have

$$\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = \frac{k^2[n(b + 1 - 2r) - (v - rk)(n - 2)]}{n(v - n)} - \frac{k^2(r - 1)^2}{(b - d - 1)} \geqq 0. \tag{3.4}$$

It follows from (3.4) that

$$d \leqq b - 1 - \frac{n(v - n)(r - 1)^2}{n(b + 1 - 2r) - (v - rk)(n - 2)}. \tag{3.5}$$

If, however, $d = b - 1 - n(v - n)\cdot(r - 1)^2/[n(b + 1 - 2r) - (v - rk)(n - 2)]$, then, $\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = 0$, giving

$$x_i = k[n(b + 1 - 2r) - (v - rk)\,(n - 2)]/n(v - n)\,(r - 1). \tag{3.6}$$

Theorem 3.1, then, follows immediately from (3.5) and (3.6).

The following are the companion Theorems to Theorem 3.1.

THEOREM 3.2. *The necessary and sufficient condition that a block of a* PBIB *design with two associate classes having a triangular association scheme and* $rk - v\cdot\lambda_1 = n(r - \lambda_1)/2$ *has the same number of treatments common with each of the remaining blocks is that* (i) $b = v - n + 1$ *and* (ii) $k(r - 1)/(v - n)$ *is an integer.*

PROOF. Let a block have x_i treatments common with the ith of the remaining $(b - 1)$ blocks. Then from (3.4), noting that $d = 0$ we get

$$\sum_{i=1}^{b-1} (x_i - \bar{x})^2 = k^2(b - r)\,(v - k)\,(b + n - 1 - v)/v(v - n)\,(b - 1). \tag{3.7}$$

Then, the Theorem 3.2 follows immediately from (3.7).

THEOREM 3.3. *If a block of a* PBIB *design with two associate classes having a triangular association scheme with parameters* $v = n(n - 1)/2 = tk$, $b = tr$, (t *an integer greater than* 1), *and* $rk - v\cdot\lambda_1 = n(r - \lambda_1)/2$ *has* $(t - 1)$ *blocks disjoint with it, then the necessary and sufficient condition that it has the same number of treatments common with each of the remaining non-disjoint blocks is that* (i) $b = v + r - n$ *and* (ii) k/t *is an integer.*

PROOF. Let a block have x_i treatments common with the ith of the remaining $b - t = t(r - 1)$ non-disjoint blocks. Then we have from (3.1) and (3.3),

$$\sum_{i=1}^{b-t} (x_i - \bar{x})^2 = k^2[v_1(b + 1 - 2r) - (v - rk)(v_1 - 1)]/v_1(v - 2v_1) - k^2(r - 1)/t. \tag{3.8}$$

Now $v = tk = v_1\cdot v_2 = v_1\cdot(2\cdot v_1 - 1)$, hence $t = v_1(2\cdot v_1 - 1)/k$. Therefore, from (3.8), we get

$$\sum_{i=1}^{b-t} (x_i - \bar{x})^2 = k^2(v - k)\,(b + n - v - r)/v(v - n). \tag{3.9}$$

Then, Theorem 3.3 follows from (3.9). We get the following corollaries from this Theorem.

COROLLARY 3.3A. *For a resolvable* PBIB *design with two associate classes having a triangular association scheme and* $rk - v\cdot\lambda_1 = n(r - \lambda_1)/2$, $b \geqq v - n + 1$.

COROLLARY 3.3B. *The necessary and sufficient condition that a resolvable* PBIB *design with two associate classes having a triangular association scheme and* $rk - v\cdot\lambda_1 = n(r - \lambda_1)/2$ *be affine resolvable is that it has a block which has the same number of treatments common with each block not belonging to its own replication.*

4. An upper bound for the number of disjoint blocks in PBIB designs with two associate classes having L_2 association scheme. A PBIB design with two associate classes is said to have a L_2 association scheme [2] if the number of treatments is $v = s^2$, where s is a positive integer and the treatments can be arranged in an $s \times s$ square such that treatments in the same row or the same column are first associates; while others are second associates. The primary parameters of this design are $v = s^2$, b, r, k, $n_1 = 2(s-1)$, $n_2 = (s-1)^2$, λ_1 and λ_2.

The following theorem has been proved by Raghavarao [4].

THEOREM 4.A. *If in a* PBIB *design with two associate classes having a* L_2 *association scheme* $rk - v\cdot\lambda_1 = s(r - \lambda_1)$, *then* k *is divisible by* s. *Further every block of this design contains* k/s *treatments from each of the* s *rows (or columns) of the association scheme.*

We use Theorem 4.A to obtain an upper bound for the number of disjoint blocks which have no treatments common with a given block of this design for which $rk - v\cdot\lambda_1 = s\cdot(r - \lambda_1)$. The result is given in Theorem 4.1.

THEOREM 4.1. *A given block of the* PBIB *design with two associate classes having a* L_2 *association scheme and* $rk - v\cdot\lambda_1 = s(r - \lambda_1)$ *cannot have more than*

$$b - 1 - \frac{v(r-1)^2\cdot(s-1)^2}{(b-r)(v-k) - (s-1)^2\cdot(v-rk)}$$

disjoint blocks with it and if some block has that many, then $k\cdot[(b-r)\,(v-k) - (s-1)^2\cdot(v-rk)]/v\cdot(r-1)\,(s-1)^2$ *is an integer and each non-disjoint block has* $k[(b-r)\cdot(v-k) - (s-1)^2\cdot(v-rk)]/v\cdot(r-1)\,(s-1)^2$ *treatments common with that given block.*

PROOF. Let the given block have d disjoint blocks with it. Let it have x_i treatments common with the ith of the remaining $(b - d - 1)$ non-disjoint blocks. Then considering the treatments of the given block singly, we have

$$\sum_{i=1}^{b-d-1} x_i = k(r-1). \tag{4.1}$$

Considering the treatments of the given block pairwise and using Theorem 4.A, we have

$$\sum_{i=1}^{b-d-1} x_i(x_i - 1) = k[2(k-s)\lambda_1 + (sk + s - 2k)\lambda_2 - s(k-1)]/s. \tag{4.2}$$

Now $rk - v\cdot\lambda_1 = s\,(r - \lambda_1)$ gives $\lambda_1 = r(k-s)/s(s-1)$. Also $n_1\cdot\lambda_1 + n_2\cdot\lambda_2 = r\,(k-1)$ gives $\lambda_2 = r\,(sk + s - 2k)/s(s-1)^2$. Hence (4.2) becomes

$$\sum_{i=1}^{b-d-1} x_i(x_i - 1) = \frac{k[2r(k-s)^2\cdot(s-1) + r)sk + s - 2k)^2 + v(s-1)^2(r-k)]}{v\cdot(s-1)^2} - k(r-1). \tag{4.3}$$

From (4.1) and (4.3), we get

$$\sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = k^2\cdot[(b-r)(v-k) - (s-1)^2(v-rk)]/v\cdot(s-1)^2 - k^2(r-1)^2/(b-d-1) \geqq 0. \tag{4.4}$$

From (4.4), we have

$$(4.5)\quad d \leqq b - 1 - v\cdot(s-1)^2(r-1)^2/[(b-r)(v-k) - (s-1)^2\cdot(v-rk)].$$

If, however, $d = b - 1 - v\cdot(s-1)^2\,(r-1)^2/[(b-r)\,(v-k) - (s-1)^2\,(v-rk)]$, then, $x_i = k[(b-r)\,(v-k) - (s-1)^2\cdot(v-rk)]/v\cdot(s-1)^2\,(r-1)$. Hence the theorem is proved.

The following are the companion theorems to Theorem 4.1.

THEOREM 4.2. *The necessary and sufficient condition that a block of a* PBIB *design with two associate classes having a L_2 association scheme and $rk - v\cdot\lambda_1 = s(r-\lambda_1)$ has the same number of treatments common with each of the remaining blocks is that* (i) $b = v - 2s + 2$ *and* (ii) $k(r-1)/(s-1)^2$ *is an integer.*

PROOF. Let a block have x_i treatments common with the ith of the remaining $(b-1)$ blocks. Then from (4.4), noting that $d = 0$, we get

$$(4.6)\quad \sum_{i=1}^{b-1}(x_i - \bar{x})^2 = k^2(b-r)\,(v-k)\,(b-v+2s-2)/v\cdot(s-1)^2\,(b-1).$$

The result follows from (4.6).

THEOREM 4.3. *If a block of a* PBIB *design with two associate classes having a L_2 association scheme with parameters $v = s^2 = tk$, $b = tr$, (t an integer greater than* 1), *and $rk - v\cdot\lambda_1 = s\,(r-\lambda_1)$, has $(t-1)$ blocks disjoint with it, then the necessary and sufficient condition that it has a block which has the same number of treatments common with each of the remaining non-disjoint blocks is that* (i) $b = v + r - 2s + 1$ *and* (ii) k/t *is an integer.*

PROOF. Let a block have x_i treatments common with the ith of the remaining $(b-t) = t\,(r-1)$ non-disjoint blocks. Then from (4.4), noting that $d = t-1$, we have

$$(4.7)\quad \sum_{i=1}^{b-t}(x_i - \bar{x})^2 = k^2(t-1)\,(b-v-r+2s-1)/t(s-1)^2.$$

The required result follows from (4.7). We get the following corollaries from this theorem.

COROLLARY 4.3.A. *For a resolvable* PBIB *design with two associate classes having a L_2 association scheme and $rk - v\cdot\lambda_1 = s(r-\lambda_1)$, $b \geqq v + r - 2s + 1$.*

COROLLARY 4.3.B. *The necessary and sufficient condition that a resolvable* PBIB *design with two associate classes having a L_2 association scheme and $rk - v\cdot\lambda_1 = s(r-\lambda_1)$ be affine resolvable is that it has a block which has the same number of treatments common with each block not belonging to its own replication.*

5. An upper bound for the number of disjoint blocks in PBIB designs with three associate classes having a rectangular association scheme. A PBIB design with three associate classes is said to have a rectangular association scheme [5], if the number of treatments $v = v_1\cdot v_2$ can be arranged in the form of a rectangle of v_1 rows and v_2 columns, so that the first associates of any treatment are the other $(v_2 - 1)$ treatments of the same row, the second associates are the other $(v_1 - 1)$ treatments of the same column; while the remaining $(v_1 - 1)\cdot(v_2 - 1)$ treatments are the third associates. The primary parameters of this design are $v = v_1 \cdot v_2$, b, r, k, $n_1 = v_2 - 1$, $n_2 = v_1 - 1$, $n_3 = n_1 \cdot n_2$, λ_1, λ_2 and λ_3. Vartak [5] has proved that the characteristic roots of NN' of this design are $\theta_0 = rk$, $\theta_1 = r - \lambda_1 + (v_1 - 1)\cdot(\lambda_2 - \lambda_3)$, $\theta_2 = r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3)$, $\theta_3 = r - \lambda_1 - \lambda_2 + \lambda_3$.

In this paper, we consider this design with $\theta_1 = 0$ and $\theta_2 = 0$. The following theorems were proved by Vartak [6].

THEOREM 5.A. *If in a* PBIB *design with three associate classes having a rectangular association scheme $\theta_1 = 0$, then k is divisible by v_2 and every block of this design contains k/v_2 treatments from every column of the association scheme.*

THEOREM 5.B. *If in a* PBIB *design with three associate classes having a rectangular association scheme $\theta_2 = 0$, then k is divisible by v_1 and every block of this design contains k/v_1 treatments from every row of the association scheme.*

We use Theorems 5.A and 5.B to obtain an upper bound for the number of disjoint blocks which have no treatments common with a given block of this design in which $\theta_1 = 0$ and $\theta_2 = 0$. The result is given in Theorem 5.1.

THEOREM 5.1. *A given block of a* PBIB *design with three associate classes having a rectangular association scheme and $\theta_1 = 0 = \theta_2$ cannot have more than*

$$b - 1 - k(r-1)^2 vp/[r(v-k)^2 - k\cdot p\cdot(v - rk)]$$

disjoint blocks and if some block has that many disjoint blocks, then $[r\cdot(v-k)^2 - kp(v-rk)]/v\cdot p\cdot(r-1)$ is an integer and each non-disjoint block has $[r\cdot(v-k)^2 - kp(v-rk)]/v\cdot p\cdot(r-1)$ treatments common with that given block, where $p = (v_1 - 1)\cdot(v_2 - 1)$.

PROOF. Let a block have d disjoint blocks and have x_i treatments common with the ith of the remaining $(b - d - 1)$ non-disjoint blocks. Then considering the treatments of the given block singly, we have

$$\sum_{i=1}^{b-d-1} x_i = k(r-1). \tag{5.1}$$

Considering the treatments of the given block pairwise, and using Theorems 5.A and 5.B, we have

$$\sum_{i=1}^{b-d-1} x_i(x_i - 1) \tag{5.2}$$
$$= k\cdot[v_2(k - v_1)(\lambda_1 - \lambda_3) + v_1(k - v_2)(\lambda_2 - \lambda_3) + v(k-1)(\lambda_3 - 1)]/v.$$

Next we have

$$\theta_1 = r - \lambda_1 + (v_1 - 1)(\lambda_2 - \lambda_3) = 0 \tag{5.3}$$

$$\theta_2 = r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3) = 0 \tag{5.4}$$

$$r(k-1) = \lambda_1(v_2 - 1) + \lambda_2(v_1 - 1) + \lambda_3\cdot p, \quad \text{where} \tag{5.5}$$
$$p = (v_1 - 1)(v_2 - 1).$$

Solving Equations (5.3), (5.4) and (5.5) for λ_1, λ_2 and λ_3, we obtain $\lambda_1 = r\cdot v_2 (k - v_1)\cdot(v_1 - 1)/v\cdot p$, $\lambda_2 = rv_1\cdot(k - v_2)\cdot(v_2 - 1)/v\cdot p$, and $\lambda_3 = r(v + kv - k\cdot v_1 - k\cdot v_2)/v\cdot p$. Substituting the values of λ_1, λ_2 and λ_3 in (5.2), we obtain

$$(5.6)\quad \sum_{i=1}^{b-d-1} x_i(x_i - 1) = k[r(v - k)^2 - k\cdot p(v - rk) - vp(r - 1)]/v\cdot p.$$

From (5.1) and (5.6), we have

$$(5.7)\quad \sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = k[r(v - k)^2 - kp(v - rk)]/v\cdot p - k^2(r - 1)^2/(b - d - 1).$$

From (5.7), it follows that

$$(5.8)\quad d \leqq b - 1 - k(r - 1)^2\cdot v\cdot p/[r(v - k)^2 - k\cdot p(v - rk)].$$

If, however,

$$d = b - 1 - k(r - 1)^2\cdot v\cdot p/[r(v - k)^2 - k\cdot p\cdot(v - rk)],$$

then, $x_i = [r(v - k)^2 - kp\cdot(v - rk)]/(r - 1)\cdot v\cdot p$. Hence the result.

The following are the companion theorems to Theorem 5.1.

THEOREM 5.2. *The necessary and sufficient condition that a block of a* PBIB *design with three associate classes having a rectangular association scheme and* $\theta_1 = 0 = \theta_2$ *has the same number of treatments common with each of the remaining blocks is that* (i) $b = p + 1$, *and* (ii) $k(r - 1)/p$ *is an integer.*

PROOF. Let a block have x_i treatments common with the ith of remaining $(b - 1)$ blocks. Then from (5.7), noting that $d = 0$ we get

$$(5.9)\quad \sum_{i=1}^{b-1} (x_i - \bar{x})^2 = k^2[(v - k)\cdot(b - r)\ (b - p - 1)]/v\cdot p\cdot(b - 1),$$

which establishes the required result.

THEOREM 5.3. *If a block of a* PBIB *design with three associate classes having a rectangular association scheme and* $\theta_1 = 0 = \theta_2$ *and parameters* $v = v_1\cdot v_2 = tk$, $b = tr$, (t *an integer greater than* 1), *has* $(t - 1)$ *blocks disjoint with it, then the necessary and sufficient condition that it has the same number of treatments common with each of the non-disjoint blocks is that* (i) $b = p + r$ *and* (ii) k/t *is an integer.*

PROOF. Let a block have x_i treatments common with each of the remaining $b - t = t\ (r - 1)$ non-disjoint blocks. Then from (5.7), noting that $d = t - 1$, we obtain

$$(5.10)\quad \sum_{i=1}^{b-t} (x_i - \bar{x})^2 = k\cdot(v - k)\ (b - r - p)/pt,$$

which proves the required result. We get the following corollaries from this theorem.

COROLLARY 5.3A. *For a resolvable* PBIB *design with three associate classes having a rectangular association scheme and* $\theta_1 = 0 = \theta_2$, $b \geqq r + p$.

COROLLARY 5.3B. *The necessary and sufficient condition that a resolvable* PBIB *design with three associate classes having a rectangular association scheme and* $\theta_1 = 0 = \theta_2$ *be affine resolvable is that it has a block which has the same number of treatments common with each block not belonging to its own replication.*

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