AN EVALUATION OF A FUNCTIONAL ON INFINITELY DIVISIBLE STOCHASTIC PROCESSES[1]

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1. Introduction. We consider a separable, infinitely divisible stochastic process $\{x(t), 0 \leqq t < \infty\}$ with $x(0) = 0$ and $E\{e^{i\xi x(t)}\} = e^{-t\psi(\xi)}$. The result here concerns the evaluation of the functional

$$\varphi(\lambda, \alpha) = \int_0^\infty e^{-\lambda t} E\left\{\exp\left[\alpha \int_0^t \cos x(\tau)\, d\tau\right]\right\} dt$$

for $\lambda > \alpha \geqq 0$, and is obtained using the result of Nelson and Varberg [5] for the functional

$$\int_0^\infty e^{-st} E\left\{\exp\left[-\int_0^t V(r(\tau))\, d\tau\right]\right\} dt \tag{1}$$

on the collective risk process $r(t)$. The collective risk process is the sum of a Poisson distributed number of independent, Bernoulli variables each of which has distribution $P\{X = 1\} = P\{X = -1\} = \frac{1}{2}$. In [5] V is nonnegative real, but the result is still true if V is complex with nonnegative real part.

Our development parallels closely Baxter's derivation [1] concerning the evaluation of

$$\int_0^\infty e^{-st} E\left\{\exp\left[-u \int_0^t x^2(\tau)\, d\tau\right]\right\} dt,$$

using Kac's result [3] on the evaluation of

$$\int_0^\infty e^{-st} E\left\{\exp\left[-\int_0^t V(w(\tau))\, d\tau\right]\right\} dt$$

for the Wiener process $w(t)$.

The present result is

Theorem. *If $\{x(t), 0 \leqq t < \infty\}$ is a separable infinitely divisible process with $x(0) = 0$ and $E\{e^{i\xi x(t)}\} = e^{-t\psi(\xi)}$, then $\varphi(\lambda, \alpha)$ is given by*

$$\varphi(\lambda, \alpha) = \sum_{n=-\infty}^{\infty} \varphi_n(\lambda, \alpha)$$

$$\varphi_{n+1} - (2/\alpha)(\lambda + \psi(n))\varphi_n + \varphi_{n-1} = -(2/\alpha)\delta_{n,0} \tag{2}$$

$$\varphi_n(\lambda, \alpha) \to 0 \quad \text{as} \quad |n| \to \infty.$$

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2. Proof of theorem. In the derivation of this result we shall denote by E_r expectation with respect to the collective risk process $r(t)$ and by E_I expectation with respect to the process $x(t)$ of the theorem. Note that the collective risk process (with parameter α) is infinitely divisible with $E\{e^{i\xi r(t)}\} = e^{-\alpha t(1-\cos \xi)}$ and $r(0) = 0$. Then compute

$$E_I \left\{\exp\left(\sum_{k=0}^{n-1} \frac{\alpha t}{n}\left[\cos x\left(\frac{k}{n} t\right) - 1\right]\right)\right\}$$

$$= E_I\, E_r \left\{\exp\left(\sum_{k=0}^{n-1} ix\left(\frac{k}{n} t\right)\left[r\left(\frac{n-k}{n} t\right) - r\left(\frac{n-k-1}{n} t\right)\right]\right)\right\}$$

$$= E_r\, E_I \left\{\exp\left(i\sum_{k=1}^{n} r\left(\frac{n-k}{n} t\right)\left[x\left(\frac{k}{n} t\right) - x\left(\frac{k-1}{n} t\right)\right]\right)\right\}$$

$$= E_r \left\{\exp\left[-\sum_{k=1}^{n} \frac{t}{n} \psi\left(r\left(\frac{n-k}{n} t\right)\right)\right]\right\},$$

where we have used the boundedness of the exponentials to exchange expectations. Again using this boundedness we can take limits in the above and obtain

$$E_I \left\{\exp\left(\alpha \int_0^t [\cos x(\tau) - 1]\, d\tau\right)\right\} = E_r \left\{\exp\left[-\int_0^t \psi(r(\tau))\, d\tau\right]\right\}.$$

Then if $\lambda > \alpha \geqq 0$ we find

$$\varphi(\lambda, \alpha) = \int_0^\infty e^{-\lambda t} E_I \left\{\exp\left[\alpha \int_0^t \cos x(\tau)\, d\tau\right]\right\} dt$$

$$= \int_0^\infty e^{-(\lambda-\alpha)t} E_I \left\{\exp\left[\alpha \int_0^t (\cos x(\tau) - 1)\, d\tau\right]\right\} dt$$

$$= \int_0^\infty e^{-(\lambda-\alpha)t} E_r \left\{\exp\left[-\int_0^t \psi(r(\tau))\, d\tau\right]\right\} dt.$$

We now employ the result of Nelson and Varberg to get

$$\varphi(\lambda, \alpha) = \sum_{n=-\infty}^{\infty} \varphi_n(\lambda, \alpha)$$

where

$$\varphi_{n+1} - 2[(\lambda - \alpha)/\alpha + 1 + \alpha^{-1}\psi(n)]\varphi_n + \varphi_{n-1} = -(2/\alpha)\delta_{n,0}$$

$$\varphi_n \to 0 \quad \text{as} \quad |n| \to \infty.$$

3. Examples.

(a) Consider the Wiener process: $\psi(\xi) = \xi^2$. The solution of (2) in this case is given by Nelson and Varberg as the evaluation of (1) for the case $V(x) = x^2$. They find

$$\varphi(\lambda, \alpha) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} A_{2k,0}(\alpha) A_{2k,2n}(\alpha)/(\nu_k + 4\lambda)$$

where ν_k is an eigenvalue of the Mathieu equation (see [4], p. 46)

$$d^2y/dx^2 + (\nu - 8a \cos 2x)y = 0$$
$$y(x) = y(x + \pi) = y(-x)$$

and

$$-\nu_k A_{2k,0} + 4\alpha A_{2k,2} = 0$$
$$8\alpha A_{2k,0} + (4 - \nu_k)A_{2k,2} + 4\alpha A_{2k,4} = 0$$
$$4\alpha A_{2k,2n-2} + (4n^2 - \nu_k)A_{2k,2n} + 4\alpha A_{2k,2n+2} = 0, \qquad n > 1$$
$$2(A_{2k,0})^2 + (A_{2k,2})^2 + (A_{2k,4})^2 + \cdots = 1$$
$$A_{2k,2n} = A_{2k,-2n}, \qquad n < 0.$$

Inversion with respect to λ yields

$$E\left\{\exp\left[\alpha \int_0^t \cos w(\tau)\, d\tau\right]\right\} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} A_{2k,0}(\alpha)A_{2k,2n}(\alpha)e^{-(\alpha+\frac{1}{4}\nu_k)t}.$$

(b) As another example we consider the Cauchy process: $\psi(\xi) = |\xi|$. Then (2) becomes

$$\varphi(\lambda, \alpha) = \sum_{n=-\infty}^{\infty} \varphi_n(\lambda, \alpha)$$

$$\varphi_{n+1} - (2/\alpha)(\lambda + |n|)\varphi_n + \varphi_{n-1} = -(2/\alpha)\delta_{n,0} \tag{3}$$

$$\varphi_n \to 0 \quad \text{as} \quad |n| \to \infty.$$

The unique solution for φ_n is given by

$$\varphi_n = J_{\lambda+|n|}(\alpha)/\alpha J_\lambda'(\alpha)$$

where J is Bessel's function of the first kind, as can be easily verified using well-known properties of Bessel functions. To sum on n we use the formula

$$\int_0^x J_\lambda(\xi)\, d\xi = 2\sum_{\nu=0}^{\infty} J_{\lambda+2\nu+1}(x).$$

(See [2] p. 145) to obtain

$$\varphi(\lambda, \alpha) = \sum_{n=-\infty}^{\infty} J_{\lambda+|n|}(\alpha)/\alpha J_\lambda'(\alpha)$$

$$= \left[\int_0^\alpha (J_{\lambda-1}(\xi) + J_\lambda(\xi))\, d\xi - J_\lambda(\alpha)\right]\Big/ \alpha J_\lambda'(\alpha) \tag{4}$$

$$= \int_0^\alpha \frac{\xi + \lambda}{\xi} J_\lambda(\xi)\, d\xi/\alpha J_\lambda'(\alpha).$$

The inversion of (4) remains a problem.

(c) Consider the case of sums of independent random variables with identical distribution function $F(x)$, the number of summands being Poisson distributed with intensity β. Then we have $\psi(\xi) = \beta(1 - \int_{-\infty}^{\infty} e^{i\xi x}\, dF(x))$ and the difference equation is

$$\varphi_{n+1} - \frac{2}{\alpha}\left(\lambda + \beta - \beta \int_{-\infty}^{\infty} e^{inx}\, dF(x)\right)\varphi_n + \varphi_{n-1} = -\frac{2}{a}\delta_{n,0}. \tag{5}$$

Then let

$$\phi(t) = \sum_{n=-\infty}^{\infty} e^{int}\varphi_n$$

and note that $\phi(0) = \varphi(\lambda, \alpha)$. Then we obtain from (5) the relation

$$(\alpha \cos t - \lambda - \beta)\phi(t) + \beta \int_{-\infty}^{\infty} \phi(t + x)\, dF(x) = -1. \tag{6}$$

We are unable to solve (6) unless the distribution $F(x)$ is degenerate. Now if $m < n$ with m and n positive integers and if $gcd(m, n) = 1$, let

$$\begin{aligned} F(x) &= 0, x < (m/n)2\pi \\ &= 1, x \geqq (m/n)2\pi. \end{aligned}$$

Then (6) becomes $(\alpha \cos t - \lambda - \beta)\phi(t) + \beta\phi(t + 2\pi m/n) = -1$. Then set $t = 0, 2\pi m/n, 4\pi m/n, \cdots, 2\pi(n - 1)m/n$ and obtain n equations in n unknowns $\phi(0), \phi(2\pi m/n), \cdots, \phi(2\pi m(n - 1)/n)$. The solution for $\phi(0)$ is

$$\begin{aligned} \varphi(\lambda, \alpha) = \phi(0) = (\lambda + \beta - \alpha)^{-1}\{1 + [1 + \lambda/\beta - (\alpha/\beta) \cos (2\pi m/n)]^{-1} \\ + [1 + \lambda/\beta - \alpha/\beta \cos (2\pi m/n)]^{-1} \\ \cdot [1 + \lambda/\beta - \alpha/\beta \cos (4\pi m/n)]^{-1} + \cdots\}. \end{aligned}$$

Now let sequences $\{m_k\}$, $\{n_k\}$ be chosen with $2\pi(m_k/n_k) \to u$, and consider the corresponding processes with $\psi_k(\xi) = \beta(1 - \exp[2\pi i\xi m_k/n_k])$ and $\psi(\xi) = \beta(1 - e^{ui\xi})$. If the corresponding expectations are denoted by E_k and E, then it can be shown that

$$E\left\{\exp\left[\alpha \int_0^t \cos x(\tau)\, d\tau\right]\right\} = \lim_{k\to\infty} E_k\left\{\exp\left[\alpha \int_0^t \cos x(\tau)\, d\tau\right]\right\}.$$

Then

$$\begin{aligned} &\int_0^{\infty} e^{-\lambda t} E\left\{\exp\left[\alpha \int_0^t \cos x(\tau)\, d\tau\right]\right\} dt \\ &\quad = (\lambda + \beta - \alpha)^{-1}\{1 + (1 + \lambda/\beta - \alpha/\beta \cos u)^{-1} \\ &\quad + [(1 + \lambda/\beta - \alpha/\beta \cos u)(1 + \lambda/\beta - \alpha/\beta \cos 2u)]^{-1} + \cdots\}. \end{aligned}$$

Again the inversion of the transform remains a problem.

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