A NOTE ON IDEMPOTENT MATRICES

By K. S. Banerjee

Cornell University

1. Introduction. The importance of idempotent matrices (abbreviated in this note as idices or, as idix for the singular number, meaning a real symmetric matrix A such that $A = A^2$) is being increasingly recognized in the field of distribution of quadratic forms and tests of linear hypothesis. In this context, Graybill and Marsaglia [1] and Hogg and Craig [2] (which reference was pointed out to the author by a referee) have proved some highly useful theorems on idices which play an important role in the decomposition of χ^2 variables. The aim of the present note is to present a more direct proof of these theorems. Most of the demonstration has been based on the use of "trace" alone. (Incidentally, the attention of mathematical statisticians may be drawn to the necessity for a systematic account of the theory on idices).

2. The theorems. Graybill and Marsaglia [1] have proved the following theorem:

Given a collection of $n \times n$ *symmetric matrices* $A_i(i = 1, 2, \cdots, m)$, *where the rank of* A_i *is* p_i, $A = \sum A_i$, *the rank of* A *is* p, *and four conditions*: (a) *each* A_i *is an idix*, (b) $A_iA_j = 0$ (*null matrix*) *for all* $i \neq j; i, j = 1, 2, \cdots, m$, (c) A *is an idix, and* (d) $p = \sum p_i$: *then*, (i): (a) *and* (c) *would imply* (b); (ii): (b) *and* (c) *would imply* (a); (iii): (a) *and* (b) *would imply* (c); (iv): *any two of* (a), (b) *and* (c) *would imply all the four*; (v): (c) *and* (d) *would imply* (a) *and* (*b*).

Subsequently, Hogg and Craig [2] proved a stronger result which may be summarized as: (vi): *If* A, $A_i(i = 1, 2, \cdots, m - 1)$ *are idices and* A_m *positive semi-definite, then* A_m *is also an idix.*

3. Proofs. We shall prove here parts (i), (ii), (iv) and (vi). Part (iii) as given in [1] is quite simple and straightforward, and is, therefore, omitted. Although it is felt that the proof of part (v) as given in [1] could perhaps be slightly simplified, that demonstration is not reproduced here. Line of reasoning for proving (vi) as adopted here is different from that taken by Professors Hogg and Craig [2] who have shown the result for $m = 2$, and have stated that "for $m > 2$, the proof of the theorem is easily completed by induction".

Part (i): We have

$$A = A^2 = \left(\sum_{i=1}^{m} A_i\right)^2 = \sum_{i=1}^{m} A_i^2 + \sum_{\substack{ij \\ i \neq j}}\sum A_i A_j$$
$$= \sum_{i=1}^{m} A_i + \sum_{\substack{ij \\ i \neq j}}\sum A_i A_j = A + \sum_{\substack{ij \\ i \neq j}}\sum A_i A_j .$$

Received 4 April 1963; revised 5 August 1963.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.* www.jstor.org

Hence,

$$\mathrm{Tr}\,(A) = \mathrm{Tr}\,(A) + 2\,\mathrm{Tr} \sum_{\substack{ij \\ i<j}}\sum A_i\, A_j\,.$$

Therefore,

$$\mathrm{Tr} \sum_{\substack{ij \\ i<j}}\sum A_i\, A_j = 0. \tag{1}$$

A typical term of (1) is $\mathrm{Tr}\,(A_iA_j) = \mathrm{Tr}\,(A_i^2A_j^2) = \mathrm{Tr}\,(A_jA_i^2A_j) = \mathrm{Tr}\,(A_iA_j)'(A_iA_j)$. As sum of such terms is zero, we shall have $A_iA_j = 0$ for all $i \neq j$.

Part (ii): Since $A_iA_j = 0$ for all $i \neq j$, we shall have

$$0 = \mathrm{Tr}\,A_iA_j = \mathrm{Tr}\,A_i^2A_j^2 = \mathrm{Tr}\,A_iA_j^2 = \mathrm{Tr}\,A_i^2A_j\,; \tag{2}$$

also,

$$\sum_{i=1}^{m} A_i^2 = \sum_{i=1}^{m} A_i\,.$$

Let

$$A_i^2 - A_i = -\sum_{\substack{j=1 \\ j\neq i}}^{m} (A_j^2 - A_j) = -B, \quad \text{where} \quad B = \sum_{\substack{j=1 \\ j\neq i}}^{m} (A_j^2 - A_j).$$

Now, $\mathrm{Tr}\,(A_i^2 - A_i) = -\mathrm{Tr}\,B$. Since A_i's are symmetric,

$$\mathrm{Tr}\,(A_i^2 - A_i)'(A_i^2 - A_i) = -\mathrm{Tr}\,(A_i^2 - A_i)B = 0, \qquad \text{by (2)}.$$

Hence, $A_i^2 = A_i$ for any i. That is, A_i's are idices.

Part (iv): This part of the theorem will be proved, if it can be shown that (a), (b), and (c) imply (d). Since the rank of an idix is equal to its trace, we have

$$\begin{aligned}\text{Rank } A = p = \mathrm{Tr}\,(A) &= \mathrm{Tr}\,(A_1) + \mathrm{Tr}\,(A_2) + \cdots + \mathrm{Tr}\,(A_m) \\ &= \text{Rank } A_1 + \text{Rank } A_2 + \cdots + \text{Rank } A_m = \sum_{i=1}^{m} p_i\,.\end{aligned}$$

Part (vi): When $E + F = I$, where I is the identity matrix and E, an idix, then F is also an idix, since $F^2 = (I - E)^2 = I - E = F$. Hence, if $I = \sum_{i=1}^{k-1} H_i + H_k$, where any matrix H_i is an idix, then the remainder of the right hand side taken together, by the above result, is also an idix.

Let

$$A = \sum_{i=1}^{m-1} A_i + A_m\,, \tag{3}$$

where A, $A_i (i = 1, 2, \cdots, m - 1)$ are idices, and A_m is positive semi-definite. Let P and Q be orthogonal matrices, such that $P'A_mP$ and $Q'(P'AP)Q$ are diagonal. We then pre-multiply and post-multiply (3) by orthogonal matrices

(P', P) and (Q', Q) at two successive stages, and we recall that such operations do not alter the character of idices and the semi-definiteness of A_m and that the diagonal elements of an idix and those of a diagonalized positive semi-definite matrix are non-negative. We also note that the elements in the row and column corresponding to a zero in the diagonal of an idix are zeros. Making operations as above and recalling these facts, (3) can be reduced to the form

$$\begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_2 & 0 \\ 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} B_{m-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_m & 0 \\ 0 & 0 \end{pmatrix},$$

where $B_i = (i = 1, 2, \cdots, m - 1)$ are idices, B_m is positive semi-definite and p represents the relevant dimension of the identity matrix. Hence, we have $I_p = \sum_{i=1}^{m-1} B_i + B_m$. Therefore, $\sum_{i=2}^{m-1} B_i + B_m = B$ (say) is also an idix. We thus revert to the original situation. By repeated application of the above operations, we shall have, eventually, the identity $I_q = C_{m-1} + C_m$, where q is the relevant dimension of the identity matrix and C_{m-1} is an idix. C_{m-1} and C_m emerge from A_{m-1} and A_m as a result of the above mentioned operations. Hence, C_m and, therefore, A_m is an idix.

4. Acknowledgment. My grateful thanks are due to Professor D. S. Robson of the Biometrics Unit and Professor R. P. Murphy of the Plant Breeding Department, Cornell University, for affording me facilities for research in this connection.

REFERENCES

[1] Graybill, F. A. and Marsaglia, G. (1957). Idempotent matrices and quadratic forms in the general linear hypothesis. *Ann. Math. Statist.* **28** 678–686.

[2] Hogg, Robert V. and Craig, Allen T. (1958). On the decomposition of certain variables. *Ann. Math. Statist.* **29** 608–610.