NON-SINGULAR RECURRENT MARKOV PROCESSES HAVE STATIONARY MEASURES[1]

By Richard Isaac

Hunter College

1. Introduction. Let X_n be a discrete parameter Markov process on a measurable space (X, Σ) and let X_n have stationary transition probabilities $P^n(x, E)$. Σ is assumed separable [3]. Call the process singular with respect to a σ-finite measure m on Σ if for each x, except for an m-null set, there exists a set L_x, $m(L_x) = 0$, such that $P^n(x, L_x) = 1$ for all positive integers n. In the contrary case, call the process m-non-singular, or simply non-singular if there can be no confusion. In this paper we wish to continue work of Harris [3]. The methods and the notation of this paper rely heavily on [3] and all references to results in Harris refer to [3]. Our result is:

Theorem. *Let the X_n process be m-non-singular where m is a measure on Σ such that $m(E) > 0$ implies $P\{X_n \varepsilon E \text{ i.o.} \mid X_0 = x\} = 1$ for almost all (m) starting points x in X. Then there exists a σ-finite stationary measure Q for the process.* ("i.o." *means infinitely often.*)

This theorem is related to Theorem 1 of Harris. Our condition changes Condition C of [3] by relaxing the "everywhere" hypothesis to an "almost everywhere" assumption. However, we no longer obtain that $m(E) > 0$ implies $Q(E) > 0$. As an example, let us remark that all processes satisfying Doeblin's condition [1] satisfy the hypotheses of the theorem for some measure m. Indeed, if Y is a closed, indecomposable, ergodic subset of X, let m be the stationary measure vanishing outside Y with $m(Y) = 1$. That such an m exists is well known, as well as the fact that $m(E) > 0$ implies $P\{X_n \varepsilon E \text{ i.o.} \mid X_0 = x\} = 1$ for all $x \varepsilon Y$. The X_n process is thus m-non-singular and satisfies the recurrence condition.

We wish to emphasize that the exceptional null set of the theorem will depend, in general, upon the set E.

2. Notations. Following Harris, p. 115, we define the process on A with transition probability $P_A(x, E)$ to be:

$$P_A(x, E) = P(x, E) + \int_{X-A} P(x, dy)P(y, E) + \int_{X-A}\int_{X-A} P(x, dy)P(y, dz)P(z, E) + \cdots . \tag{1}$$

The transformation T_A takes measures into measures:

$$(T_A\mu)(\cdot) = \int_A P_A(x, \cdot)\mu(dx) \tag{2}$$

and represents the evolution of the process on A. If the process on A has a sta-

Received 21 October 1963.

[1] Work supported in part by NSF Grant GP-1542.

tionary probability measure Q_A, Harris proves on p. 116 that if, for every $E \varepsilon \Sigma$, we consider

$$Q(E) = \int_A P_A(x, E)\, Q_A(dx) \tag{3}$$

then this formula defines a measure Q, not necessarily finite, which is stationary on Σ for the original X_n process.

3. Proof of result. The theorem will be proved by means of several lemmas. Lemma 2 below follows very closely Lemma 2 of [3]. We repeat part of the argument to indicate the necessary changes.

LEMMA 1. *Let the X_n process be non-singular. Then there exists a set K, $m(K) > 0$, such that for each $x \varepsilon K$ and every $E \varepsilon \Sigma$ with $m(E) > 0$, there exists a positive integer $j = j(x, E)$ with $P^j(x, E) > 0$.*

PROOF. Since the process is non-singular, there exists a set K, $m(K) > 0$, such that for each $x \varepsilon K$ the Lebesgue decomposition:

$$P^n(x, \cdot) = \int_\cdot f^n(x, y)\, m(dy) + P_0^n(x, \cdot) \tag{4}$$

obtains, and for some positive integer k, $f^k(x, y) > 0$ for all y in a set Y, $m(Y) > 0$. k and Y depend, of course, upon x. Now let $m(E) > 0$. Almost all (m) points in Y enter E infinitely often with probability 1, so we may find a positive integer r, a subset $Y_0 \subseteq Y$, $m(Y_0) > 0$, and $\epsilon > 0$, satisfying $P^r(y, E) \geqq \epsilon$ for all $y \varepsilon Y_0$. Here r and Y_0 depend upon x and E. We now have:

$$\begin{aligned} P^{k+r}(x, E) &= \int P^k(x, dy) P^r(y, E) \\ &\geqq \int_{Y_0} P^k(x, dy) P^r(y, E) \geqq \epsilon P^k(x, Y_0) \geqq \epsilon \int_{Y_0} f^k(x, y) m(dy) > 0 . \end{aligned}$$

Thus the conclusion of the lemma holds for $j = k + r$.

LEMMA 2. *Let r be any real number, $0 < r < 1$. There exist a measurable set B, a positive number s, and a positive integer k, such that $0 < m(B) < \infty$, and for every $x \varepsilon B$:*

$$m\{y : y \varepsilon B, f^1(x, y) + \cdots + f^k(x, y) > s\} > rm(B). \tag{5}$$

Moreover $P\{X_n \varepsilon B \text{ i.o.} \mid X_0 = x\} = 1$ for every $x \varepsilon B$.

PROOF. Let K be the set of Lemma 1. Since each of the measures P_0^n, $n \geqq 1$, is singular, we can find, for each $x \varepsilon K$, a measurable set $S(x)$ with $m(S(x)) = 0$, such that

$$P_0^n\{X - S(x)\} = 0. \qquad n = 1, 2, \cdots . \tag{6}$$

For each $x \varepsilon K$, let $T(x)$ be the measurable x-set defined by

$$T(x) = \{y : f^n(x, y) = 0, n = 1, 2, \cdots\}. \tag{7}$$

Then if $X_0 = x$, the probability is 1 that there is no n such that $X_n \varepsilon T(x) - T(x)S(x)$. Therefore, by Lemma 1, $m(T(x)) = 0$.

Now let A_1 be any measurable subset of K such that $0 < m(A_1) < \infty$. For each $x \varepsilon A_1$, define the measurable set $A_{1i} = A_{1i}(x)$ for $i = 1, 2, \cdots$ by

$$A_{1i}(x) = \{y : y \varepsilon A_1, f^1(x, y) + \cdots + f^i(x, y) > i^{-1}\}. \tag{8}$$

Since $m(T(x)) = 0$ for $x \varepsilon K$, this implies that $m(A_1 - \bigcup_i A_{1i}) = 0$. Proceeding as Harris does in Lemma 2, we can construct the set A, $0 < m(A) < \infty$, satisfying (5). Thus, by hypothesis, $P\{X_n \varepsilon A \text{ i.o.} \mid X_0 = x\} = 1$ for almost each $x(m)$. Let $N = \{x : x \varepsilon A, P\{X_n \varepsilon A \text{ i.o.} \mid X_0 = x\} < 1\}$ and put $B = A - N$. Then $m(N) = 0$ and, for $x \varepsilon B$, $P^k(x, N) = 0$ for all positive integers k. This follows since points of B entering N would have positive probability of not returning infinitely often to A, which is impossible by definition of B. Moreover, it is clear that (5) holds, and the proof is complete.

LEMMA 3. *There is a stationary probability measure Q_B for the process on B.*

PROOF. Lemmas 3 and 4 of Harris hold without change and prove Lemma 3.

LEMMA 4. *Q_B may be extended to a σ-finite stationary measure Q for the X_n process.*

PROOF. Using the Definition (3) employed in Lemma 1 of Harris, we may extend Q_B to a measure Q which is stationary for the X_n process. We now show that Q is σ-finite. First of all, since

$$1 = Q(B) = \int P^n(x, B) Q(dx)$$

the set $V_n(x) = \{x : P^n(x, B) > 0\}$ is σ-finite for Q by Theorem F, p. 105 of [2] for each positive integer n, and so $\bigcup_n V_n$ is σ-finite for Q. By assumption, $X - \bigcup_n V_n = C$ has m-measure 0, and the proof will be complete if we show that C is σ-finite for Q. In fact, we shall show that $Q(C) = 0$. Now, by (3), we have $Q(C)$ defined by

$$Q(C) = \int_B P_B(x, C) Q_B(dx). \tag{9}$$

For each $x \varepsilon B$, however, $P^n(x, C) = 0$ for all positive integers n, otherwise x would have positive probability of escape from B, which is impossible. Therefore (1) shows that $P_B(x, C) = 0$ for each $x \varepsilon B$, and by (9), $Q(C) = 0$. Indeed if $D = \{x : P\{X_n \varepsilon B \text{ i.o.} \mid X_0 = x\} < 1\}$ then the same argument used above shows that $Q(D) = 0$ whence Q is concentrated on the set of points recurrent for B.

REFERENCES

[1] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.

[2] HALMOS, P. R. (1950). *Measure Theory*. Van Nostrand, New York.

[3] HARRIS, T. E. (1956). The existence of stationary measures for certain Markov processes. *Third Berkeley Symp. Math. Statist. Prob.* **2** 113–124.