ON TWO-SIDED TOLERANCE INTERVALS FOR A NORMAL DISTRIBUTION[1]

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0. Summary. In [6], Wald and Wolfowitz present a method for obtaining approximate two-sided tolerance intervals for a normal distribution. Some readers of [6] have been left with the impression that Wald and Wolfowitz proved that the confidence level attained by their approximation converges to the nominal confidence level with increasing sample size N, and that the difference is $O(1/N^2)$ (see, e.g., [3]). However, Wald and Wolfowitz did not consider that problem in their paper, nor does the problem seem to have been considered elsewhere in the literature. The principal result of Wald and Wolfowitz is given at the end of Section 1.

In Section 2 it is shown that the confidence level attained by the Wald-Wolfowitz approximation does converge to the nominal confidence level, that the difference is $O(1/N)$, and that $1/N$ is the exact order of the rate of convergence except for the confidence level .5. This is a corollary of a more general result obtained by considering the case in which $s^2 \sim \sigma^2\chi_n^2/n$, and independently $\bar{x} \sim$ normal $(\mu, \sigma^2/N)$, where n is not necessarily equal to $N - 1$.

It is found, perhaps surprisingly, that as $N \to \infty$, the confidence level attained by the obvious generalization of the Wald-Wolfowitz approximation converges to the nominal confidence level fastest when n is fixed, the difference being $O(1/N^2)$ in this case. Furthermore, if n increases "too rapidly" as $N \to \infty$, then the confidence level attained does not converge to the nominal confidence level. These results are consequences of the two theorems proved in this paper. Theorem 1 in Section 2 states that if $n/N^2 \to 0$, then the confidence level attained by the generalization of the Wald-Wolfowitz approximation converges to the nominal confidence level, that the difference is $O(n/N^2)$, and that, with certain unimportant exceptions, n/N^2 is the exact order of the rate of convergence. In Theorem 2, Section 5, it is shown that if $n/N^2 \to \infty$, then the confidence level attained by the generalization of the Wald-Wolfowitz approximation converges to a limit which does not depend on the nominal confidence level. A modification of the generalized Wald-Wolfowitz approximation which has the desired convergence property in this case is presented.

Certain facts used in Section 2 are verified in Sections 3 and 4. In Section 6, an heuristic explanation of the observed asymptotic behavior is given, and the possibility of improving the Wald-Wolfowitz approximation is discussed briefly.

The basic notation and method of determining error bounds in this paper are essentially the same as those employed by Wald and Wolfowitz in [6]. Sections

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2, 3 and 4 of this paper are counterparts, respectively, of Sections 4, 8 and 9 of the Wald-Wolfowitz paper. Equation (19) in the present paper plays the role that the basic Equation (4.1) did in the Wald and Wolfowitz paper.

1. Introduction. Let $\bar{x}$ be distributed as normal $(\mu, \sigma^2/N)$, and let s^2 be distributed as $\sigma^2\chi_n^2/n$ independently of $\bar{x}$. A two-sided tolerance interval is to be computed from $\bar{x}$ and s^2 such that the probability is equal to a preassigned value β that the tolerance interval includes at least a given proportion γ of the normal (μ, σ^2) population.

Let

$$A(\bar{x}, s, \lambda) = (1/(2\pi)^{\frac{1}{2}}\sigma) \int_{\bar{x}-\lambda s}^{\bar{x}+\lambda s} \exp[-(t-\mu)^2/2\sigma^2]\, dt \qquad \lambda > 0. \tag{1}$$

$A(\bar{x}, s, \lambda)$ is the proportion of the normal (μ, σ^2) population included between the limits $\bar{x} - \lambda s$ and $\bar{x} + \lambda s$. Since the distribution of $A(\bar{x}, s, \lambda)$ does not depend on the unknown parameters μ and σ, there is no loss of generality in assuming that $\mu = 0$ and $\sigma = 1$, and this assumption is made hereafter.

Let

$$P_n(\gamma, \lambda \mid \bar{x}) = \text{the conditional probability, given } \bar{x}, \text{ that } A(\bar{x}, s, \lambda) \geqq \gamma, \tag{2}$$

and let

$$P_{N,n}(\gamma, \lambda) = E[P_n(\gamma, \lambda \mid \bar{x})]. \tag{3}$$

Then $P_{N,n}(\gamma, \lambda)$ is the probability that the tolerance interval $[\bar{x} - \lambda s, \bar{x} + \lambda s]$ includes at least γ of the normal (μ, σ^2) population; i.e., $P_{N,n}(\gamma, \lambda)$ is the confidence level attained by the tolerance factor λ.

It is clear that for each N, n, β and γ there is a λ such that $P_{N,n}(\gamma, \lambda) = \beta$, so the problem of obtaining two-sided tolerance intervals for the normal distribution does have an exact solution. An approximation is used for computational reasons, since the equation $P_{N,n}(\gamma, \lambda) = \beta$ cannot be solved directly for λ.

Define $r(\bar{x}, \gamma)$ by

$$(2\pi)^{-\frac{1}{2}} \int_{\bar{x}-r(\bar{x},\gamma)}^{\bar{x}+r(\bar{x},\gamma)} \exp(-t^2/2)\, dt = \gamma. \tag{4}$$

Since $A(\bar{x}, s, \lambda)$ is a strictly increasing function of s, the inequality $A(\bar{x}, s, \lambda) \geqq \gamma$ is equivalent to the inequality $s \geqq r(\bar{x}, \gamma)/\lambda$. Hence (2) may be written as

$$P_n(\gamma, \lambda \mid \bar{x}) = P\{s \geqq r(\bar{x}, \gamma)/\lambda \mid \bar{x}\} = P\{(\chi_n^2/n)^{\frac{1}{2}} \geqq r(\bar{x}, \gamma)/\lambda \mid \bar{x}\}. \tag{5}$$

In this paper we consider approximate tolerance interval procedures for which the tolerance factor λ is given by

$$\lambda_{N,n} = r(N^{-\frac{1}{2}}x_0, \gamma)/c_{n,\beta} \quad \text{for some } x_0, \tag{6}$$

where

$$P\{s \geqq c_{n,\beta}\} = \beta. \tag{7}$$

The dependence of $\lambda_{N,n}$ on γ, β and x_0 is suppressed in the notation for typographical convenience.

The approximation presented by Wald and Wolfowitz in [6] for the case $n = N - 1$ has the above form with $x_0 = 1$. The tables of tolerance factors prepared by Bowker, which appear in [1], pp. 102–107, and elsewhere, are based upon the Wald-Wolfowitz approximation. In [7], Wallis considers the problem of tolerance intervals for linear regression. In that problem, Wallis' "effective number of observations," N', plays the role of our N, and $n \neq N' - 1$ (in particular, n may be much larger than N'). Wallis uses the generalization of the Wald-Wolfowitz approximation obtained by replacing N and $N - 1$ by N' and n, respectively. In [8], Weissberg and Beatty present tables of the generalized Wald-Wolfowitz approximation in which (in our notation) $r(N^{-\frac{1}{2}}, \gamma)$ and $c_{n,\beta}$ are tabulated separately. These tables are reproduced by Owen in [4], pp. 128–137.

Wald and Wolfowitz showed in [6] that for fixed λ and n, $P_{N,n}(\gamma, \lambda) = P_n(\gamma, \lambda \mid N^{-\frac{1}{2}}) + O(1/N^2)$. This result motivated their choice of the approximate tolerance factor as the solution in λ of $P_{N-1}(\gamma, \lambda \mid N^{-\frac{1}{2}}) = \beta$. Thus Wald and Wolfowitz assumed n fixed, $N \to \infty$, in their proof of asymptotic behavior, but they used $n = N - 1$ in their application of their result.

2. The case $n/N^2 \to 0$. In this section we investigate the asymptotic behavior of the difference between the confidence level attained and the nominal confidence level β, when the tolerance factor $\lambda_{N,n}$ is given by (6). The analysis in this section embraces the cases in which either or both of n, N tend to infinity. However, since bounds $O(n/N^2)$ and $O(n^{\frac{1}{2}}/N)$ are obtained, the results are useful only for the case $n/N^2 \to 0$. The case $n/N^2 \to \infty$ is treated in Section 5. The case in which n and N^2 increase at the same rate seems devoid of interest, and is not studied.

The difference between the confidence level attained and the nominal confidence level β, which we wish to study, is given by

$$P_{N,n}(\gamma, \lambda_{N,n}) - \beta = E[P_n(\gamma, \lambda_{N,n} \mid \bar{x}) - \beta]. \tag{8}$$

It will be convenient to have the notation

$$h_N(\bar{x}; x_0) = 1 - r(\bar{x}, \gamma)/r(N^{-\frac{1}{2}}x_0, \gamma), \tag{9}$$

where the dependence upon γ is suppressed for typographical convenience.

Since by (4), $r(\bar{x}, \gamma)$ is positive, an even function of $\bar{x}$, and increasing in $|\bar{x}|$, it follows that

$$\begin{aligned} h_N(\bar{x}; x_0) &> 0 \quad \text{for} \quad |N^{\frac{1}{2}}\bar{x}| < |x_0| \\ &= 0 \quad \text{for} \quad |N^{\frac{1}{2}}\bar{x}| = |x_0| \\ &< 0 \quad \text{for} \quad |N^{\frac{1}{2}}\bar{x}| > |x_0|. \end{aligned} \tag{10}$$

Starting from (5) and (6), we compute

$$P_n(\gamma, \lambda_{N,n} \mid \bar{x}) = P\{s \geqq r(\bar{x}, \gamma)c_{n,\beta}/r(N^{-\frac{1}{2}}x_0, \gamma)\}$$

$$= P\{s \geqq c_{n,\beta} - c_{n,\beta}[1 - r(\bar{x}, \gamma)/r(N^{-\frac{1}{2}}x_0, \gamma)]\}$$

$$(11) \qquad = P\{s \geqq c_{n,\beta} - c_{n,\beta}h_N(\bar{x}; x_0)\}$$

$$= \begin{cases} \beta + P\{c_{n,\beta} - c_{n,\beta}h_N(\bar{x}; x_0) \leqq s \leqq c_{n,\beta}\}, \\ \quad \text{for} \quad |N^{\frac{1}{2}}\bar{x}| \leqq |x_0| \\ \beta - P\{c_{n,\beta} \leqq s \leqq c_{n,\beta} - c_{n,\beta}h_N(\bar{x}; x_0)\}, \\ \quad \text{for} \quad |N^{\frac{1}{2}}\bar{x}| \geqq |x_0|. \end{cases}$$

Because s is asymptotically normal $(1, 1/2n)$ (see, e.g. [2], pp. 250–251), it will prove useful to introduce

$$(12) \qquad z = (2n)^{\frac{1}{2}}(s - 1)$$

since z is asymptotically normal $(0, 1)$. Similarly, let

$$(13) \qquad K_{n,\beta} = (2n)^{\frac{1}{2}}(c_{n,\beta} - 1).$$

Then

$$(14) \qquad P\{z \geqq K_{n,\beta}\} = \beta.$$

Using (12) and (13) we can rewrite (11) as

$$P_n(\gamma, \lambda_{N,n} \mid \bar{x}) - \beta$$

$$(15) \quad = \begin{cases} P\{K_{n,\beta} - [(2n)^{\frac{1}{2}} + K_{n,\beta}]h_N(\bar{x}; x_0) \leqq z \leqq K_{n,\beta}\}, \quad \text{for} \quad |N^{\frac{1}{2}}\bar{x}| \leqq |x_0| \\ -P\{K_{n,\beta} \leqq z \leqq K_{n,\beta} - [(2n)^{\frac{1}{2}} + K_{n,\beta}]h_N(\bar{x}; x_0)\}, \\ \qquad \text{for} \quad |N^{\frac{1}{2}}\bar{x}| \geqq |x_0|. \end{cases}$$

Let $g_n(\cdot)$ be the density function of the distribution of z. From the known density function for s (see, e.g., [2], p. 237), and from (12), it is easy to compute that

$$(16) \quad g_n(z) = (n/2)^{(n-1)/2}[1 + (2n)^{-\frac{1}{2}}z]^{n-1} \exp[(-n/2)(1 + (2n)^{-\frac{1}{2}}z)^2]/\Gamma(n/2).$$

For each n, the function $g_n(z)$ possesses derivatives of all orders at every z. For typographical convenience let

$$(17) \qquad t = [(2n)^{\frac{1}{2}} + K_{n,\beta}]h_N(\bar{x}; x_0).$$

Then (15) may be rewritten as

$$(18) \quad \begin{aligned} P_n(\gamma, \lambda_{N,n} \mid \bar{x}) - \beta &= \int_{K_{n,\beta}-t}^{K_{n,\beta}} g_n(z)\,dz, \quad \text{for} \quad |N^{\frac{1}{2}}\bar{x}| \leqq |x_0|, \quad \text{where} \quad t \geqq 0 \\ &= -\int_{K_{n,\beta}}^{K_{n,\beta}-t} g_n(z)\,dz, \quad \text{for} \quad |N^{\frac{1}{2}}\bar{x}| \geqq |x_0|, \quad \text{where} \quad t \leqq 0. \end{aligned}$$

The two functions of t appearing on the right-hand side of (18) have identical Taylor expansions in t about $t = 0$. Terminating the Taylor expansion at the second term, we obtain (19). The fact that $g_n(z)$ possesses derivatives of all orders at every z ensures the validity of (19).

$$P_n(\gamma, \lambda_{N,n} \mid \bar{x}) - \beta = tg_n(K_{n,\beta}) - \tfrac{1}{2}t^2 g_n'(K_{n,\beta} - t\theta(t)) \tag{19}$$

where $0 \leqq \theta(t) \leqq 1$. It is clear from (16) that for each n, the functions $g_n(\cdot)$ and $g_n'(\cdot)$ are bounded. Straightforward computations show that the sequences of functions $g_n(\cdot)$ and $g_n'(\cdot)$ are uniformly bounded ($g_n(\cdot)$ and $g_n'(\cdot)$ converge pointwise, respectively, to the normal (0, 1) density function and its derivative).

It follows that the error in confidence level, $P_{N,n}(\gamma, \lambda_{N,n}) - \beta = E[P_n(\gamma, \lambda_{N,n} \mid \bar{x}) - \beta]$, is bounded by a quantity whose order of magnitude is the larger of the orders of magnitude of $E[t]$ and $E[t^2]$. Therefore, in view of the definition, (17), of t, we need only obtain bounds on $E[h_N(\bar{x}; x_0)]$ and $E[h_N^2(\bar{x}; x_0)]$ in order to obtain a bound on $P_{N,n}(\gamma, \lambda_{N,n}) - \beta$. However, we wish to determine the best possible bound on $P_{N,n}(\gamma, \lambda_{N,n}) - \beta$ for sequences of N, n under which $P_{N,n}(\gamma, \lambda_{N,n}) - \beta \to 0$. This requires a finer analysis. The method employed here is to find a function of N, n, γ and β to which $P_{N,n}(\gamma, \lambda_{N,n}) - \beta$ is asymptotically equal in such cases.

It follows from the definition, (9), of $h_N(\bar{x}; x_0)$ and from the Taylor expansion for $r(\bar{x}, \gamma)$ given by Equation (34) in Section 3, that

$$\begin{aligned} h_N(\bar{x}; x_0) &= r(N^{-\frac{1}{2}}x_0, \gamma)^{-1} \\ &\cdot[(x_0^2 - N\bar{x}^2)r(0, \gamma)/2N + \{x_0^4 v(N^{-\frac{1}{2}}x_0, \gamma) - N^2\bar{x}^4 v(\bar{x}, \gamma)\}/N^2]. \end{aligned} \tag{20}$$

It is proved in Section 4 that the function $v(\cdot, \gamma)$ is bounded. Since $N^{\frac{1}{2}}\bar{x}$ has a normal (0, 1) distribution, its moments do not depend on N, and in particular

$$E[N\bar{x}^2] = 1, \qquad E[N^2\bar{x}^4] = 3$$
$$E[(x_0^2 - N\bar{x}^2)^2] = x_0^4 - 2x_0^2 + 3.$$

Let the symbol $\approx$ denote "is asymptotically equal to." Then it is clear that as $N \to \infty$

$$E[h_N(\bar{x}; x_0)] \approx (x_0^2 - 1)/2N \quad \text{for} \quad x_0^2 \neq 1, \tag{21}$$

and

$$E[h_N^2(\bar{x}; x_0)] \approx (x_0^4 - 2x_0^2 + 3)/4N^2 \quad \text{for all } x_0\,. \tag{22}$$

Further analysis is necessary to determine an asymptotic expression for $E[h_N(\bar{x}; x_0)]$ when $x_0^2 = 1$. The following result, which is obtained from Equations (32) and (33) of Section 3 by a straightforward computation, is needed.

$$v(0, \gamma) = [3r(0, \gamma) - 2r^3(0, \gamma)]/24. \tag{23}$$

Since all functions of x_0 in this paper depend upon x_0 only through $|x_0|$, we may suppose here that $x_0 = +1$.

$$\begin{aligned} E[h_N(\bar{x}; 1)] &= E[v(N^{-\frac{1}{2}}, \gamma) - N^2\bar{x}^4 v(\bar{x}, \gamma)]/N^2 r(N^{-\frac{1}{2}}, \gamma) \\ &\approx -2v(0, \gamma)/N^2 r(0, \gamma) = [2r^2(0, \gamma) - 3]/12N^2. \end{aligned} \tag{24}$$

The one exception to this result occurs when $2r^2(0, \gamma) - 3 = 0$; the γ for which this occurs is approximately .78. In this case $E[h_N(\bar{x}; 1)]$ converges to zero faster than $1/N^2$.

We continue the analysis for the case $x_0^2 = 1$. For the case $x_0^2 \neq 1$, only the results obtained from a similar analysis will be given. Since, by (17), $t = [(2n)^{\frac{1}{2}} + K_{n,\beta}]h_N(\bar{x}; x_0)$, it follows from (24) that for any sequence $n(N)$, $N = 1, 2, \cdots$,

$$E[t] \approx [(2n)^{\frac{1}{2}} + K_{n,\beta}][2r^2(0, \gamma) - 3]/12N^2 \tag{25}$$

(with one exception noted above), and it follows from (22) that

$$E[t^2] \approx [(2n)^{\frac{1}{2}} + K_{n,\beta}]^2/2N^2. \tag{26}$$

Consequently,

$$\mathrm{Var}[t] \approx [(2n)^{\frac{1}{2}} + K_{n,\beta}]^2/2N^2 \tag{27}$$

with no exception; and if $n/N^2 \to 0$, then the sequence of random variables t converges in probability to zero.

Consider first the case in which n is fixed as $n/N^2 \to 0$. The sequence of random variables $g_n'(K_{n,\beta} - t\theta(t))$ in Equation (19) converges in probability to $g_n'(K_{n,\beta})$, and (with exceptions noted below)

$$\begin{aligned} P_{N,n}(\gamma, \lambda_{N,n}) - \beta \approx {} & [(2n)^{\frac{1}{2}} + K_{n,\beta}][2r^2(0, \gamma) - 3]g_n(K_{n,\beta})/12N^2 \\ & - [(2n)^{\frac{1}{2}} + K_{n,\beta}]^2 g_n'(K_{n,\beta})/4N^2. \end{aligned} \tag{28}$$

Here the order of magnitude of the rate of convergence to zero is exactly $1/N^2$, unless the right-hand side is zero. This can occur only for special (γ, β, n) which one can expect never to encounter in practice.

Consider next the case in which $n \to \infty$ as $n/N^2 \to 0$. Denote the normal (0, 1) density function and its derivative, respectively, by $\phi(\cdot)$ and $\phi'(\cdot)$. Then

$$\begin{aligned} P_{N,n}(\gamma, \lambda_{N,n}) - \beta & \approx (2n)^{\frac{1}{2}}[2r^2(0, \gamma) - 3]\phi(K_\beta)/12N^2 - n\phi'(K_\beta)/2N^2 \\ & \approx -n\phi'(K_\beta)/2N^2. \end{aligned} \tag{29}$$

Here the order of magnitude of the rate of convergence to zero is exactly n/N^2 (except for $\beta = \frac{1}{2}$).

For $x_0^2 \neq 1$, a similar analysis shows that when n is fixed as $n/N^2 \to 0$,

$$P_{N,n}(\gamma, \lambda_{N,n}) - \beta \approx [(2n)^{\frac{1}{2}} + K_{n,}\beta](x_0^2 - 1)g_n(K_{n,\beta})/2N, \tag{30}$$

and when $n \to \infty$ as $n/N^2 \to 0$

$$P_{N,n}(\gamma, \lambda_{N,n}) - \beta \approx (2n)^{\frac{1}{2}}(x_0^2 - 1)\phi(K_\beta)/2N. \tag{31}$$

The order of magnitude of the rate of convergence to zero is exactly $1/N$ in the former case and exactly $n^{\frac{1}{2}}/N$ in the latter case, with no exceptions.

The principal results of the foregoing analysis are recorded in the following Theorem:

THEOREM 1. *If* $\lambda_{N,n} = r(N^{-\frac{1}{2}}x_0, \gamma)/c_{n,\beta}$ *and if* $n/N^2 \to 0$, *then*

$$P_{N,n}(\gamma, \lambda_{N,n}) - \beta = O(n^{\frac{1}{2}}/N) \quad \text{for} \quad x_0^2 \neq 1$$

and

$$P_{N,n}(\gamma, \lambda_{N,n}) - \beta = O(n/N^2) \quad \text{for} \quad x_0^2 = 1.$$

Furthermore, each bound gives the exact order of the rate of convergence, except for the special cases noted after Equations (28) *and* (29), *respectively.*

COROLLARY 1. *If* n *is fixed and* $x_0 = 1$ *in Theorem* 1, *then*

$$P_{N,n}(\gamma, \lambda_{N,n}) - \beta = O(1/N^2),$$

and $1/N^2$ *is the exact order of the rate of convergence except for the special cases noted after Equation* (28).

Corollary 1 is similar to the principal result of Wald and Wolfowitz in [6]. However, Wald and Wolfowitz held λ fixed as well as n, and then considered $P_n(\gamma, \lambda|N^{-\frac{1}{2}})$ as a computational approximation to the true confidence level $P_{N,n}(\gamma, \lambda)$. Their result was that the difference is $O(1/N^2)$.

COROLLARY 2. *If* $n = N - 1$ *and* $x_0 = 1$ *in Theorem* 1, *then the tolerance factor is that given by Wald and Wolfowitz, and the confidence level attained converges to the nominal confidence level* β, *and the difference is* $O(1/N)$, *and* $1/N$ *is the exact order of the rate of convergence except for* $\beta = \frac{1}{2}$.

Note that for fixed n, $\lim_{N\to\infty} \lambda_{N,n} = r(0, \gamma)/c_{n,\beta}$ for all x_0, and $\mu \pm [r(0, \gamma)/c_{n,\beta}]s$ are the well-known exact tolerance limits in the case where μ is known (see, e.g., [5]).

3. Validity of the Taylor expansion of $r(\bar{x}, \gamma)$**.** Denote the ith derivative of $r(\bar{x}, \gamma)$ by $r^{(i)}(\bar{x}, \gamma)$. A simple computation (given by Wald and Wolfowitz in [6]) shows that

$$r^{(1)}(\bar{x}, \gamma) = \tanh[\bar{x}r(\bar{x}, \gamma)]. \tag{32}$$

Hence $r(\bar{x}, \gamma)$ possesses derivatives of all orders for all real values of $\bar{x}$, and so $r(\bar{x}, \gamma)$ possesses a valid Taylor series expansion. Since $r(\bar{x}, \gamma)$ is an even function of $\bar{x}$, only even powers of $\bar{x}$ occur in the expansion. Terminating the Taylor expansion at the third term, we have

$$r(\bar{x}, \gamma) = r(0, \gamma) + (\bar{x}^2/2)r^{(2)}(0, \gamma) + (\bar{x}^4/4!)r^{(4)}(\xi(\bar{x}), \gamma),$$

where $0 \leqq \xi(\bar{x}) \leqq |\bar{x}|$. From (32) we have that

$$r^{(2)}(\bar{x}, \gamma) = \{\bar{x}r^{(1)}(\bar{x}, \gamma) + r(\bar{x}, \gamma)\} \operatorname{sech}^2[\bar{x}r(\bar{x}, \gamma)],$$

so $r^{(2)}(0, \gamma) = r(0, \gamma)$. Let

$$v(\bar{x}, \gamma) = (1/4!)r^{(4)}(\xi(\bar{x}), \gamma). \tag{33}$$

Then

$$r(\bar{x}, \gamma) = r(0, \gamma) + (\bar{x}^2/2)r(0, \gamma) + \bar{x}^4 v(\bar{x}, \gamma). \tag{34}$$

4. Proof that $v(\cdot, \gamma)$ is bounded. Fix $\delta > 0$. Since $r(\bar{x}, \gamma)$ possesses everywhere derivatives of all orders, the fourth derivative $r^{(4)}(\bar{x}, \gamma)$ is continuous and hence bounded for $|\bar{x}| \leqq \delta$. It follows from (33), in which $0 \leqq \xi(\bar{x}) \leqq |\bar{x}|$, that $v(\bar{x}, \gamma)$ is bounded for $|\bar{x}| \leqq \delta$.

Since $r(\bar{x}, \gamma)$ is minimum at $\bar{x} = 0$, it follows from (34) that $(\bar{x}^2/2)r(0, \gamma) + \bar{x}^4 v(\bar{x}, \gamma) \geqq 0$ and hence $v(\bar{x}, \gamma) \geqq -(1/2\bar{x}^2)r(0, \gamma)$. Consequently $v(\bar{x}, \gamma)$ is bounded below for $|\bar{x}| \geqq \delta$.

Next observe that

$$r(\bar{x}, \gamma) \leqq r(0, \gamma) + |\bar{x}|. \tag{35}$$

To see this suppose, without loss of generality, that $\bar{x} \geqq 0$. Then the interval

$$[\bar{x} - \{r(0, \gamma) + \bar{x}\}, \bar{x} + \{r(0, \gamma) + \bar{x}\}] = [-r(0, \gamma), r(0, \gamma) + 2\bar{x}]$$

has, under the normal (0, 1) distribution, probability content at least γ, since it contains the interval $[-r(0, \gamma), r(0, \gamma)]$ which, by (4), has probability content exactly γ. Since $r(\bar{x},\gamma)$ is an increasing function of γ for fixed $\bar{x}$, it follows that $r(0, \gamma) + \bar{x} \geqq r(\bar{x}, \gamma)$. Now from (34) and (35) we have that

$$(\bar{x}^2/2)r(0, \gamma) + \bar{x}^4 v(\bar{x}, \gamma) \leqq |\bar{x}|$$

and hence

$$v(\bar{x}, \gamma) \leqq |\bar{x}^3|^{-1} - r(0, \gamma)/2\bar{x}^2.$$

Consequently $v(\bar{x}, \gamma)$ is bounded above for $|\bar{x}| \geqq \delta$.

We have shown that $v(\cdot, \gamma)$ is bounded above and below for every real $\bar{x}$.

5. The case $n/N^2 \to \infty$. In this section we investigate the asymptotic behavior, as $n/N^2 \to \infty$, of the difference between the confidence level attained and the nominal confidence level β, when the tolerance factor $\lambda_{N,n}$ is given by (6). Here it is most convenient to work in terms of $y = N^{\frac{1}{2}}\bar{x}$, where y has a normal (0, 1) distribution for all N. If $P_n(\gamma, \lambda_{N,n} \mid \bar{x})$ written in terms of y is denoted by $P_{N,n}(\gamma, \lambda_{N,n} \mid y)$, then the confidence level attained is

$$P_{N,n}(\gamma, \lambda_{N,n}) = E[P_{N,n}(\gamma, \lambda_{N,n} \mid y)].$$

By (18) we have

$$\begin{aligned} P_{N,n}(\gamma, \lambda_{N,n} \mid y) &= \beta + \int_{K_{n,\beta}-t(y)}^{K_{n,\beta}} g_n(z)\, dz, && \text{for } |y| < |x_0|, \\ &= \beta + 0, && \text{for } |y| = |x_0|, \\ &= \beta + \int_{K_{n,\beta}}^{K_{n,\beta}-t(y)} g_n(z)\, dz, && \text{for } |y| > |x_0|, \end{aligned} \tag{36}$$

where $t(y)$ is the t defined by (17) written in terms of y. Explicitly,

$$t(y) = [(2n)^{\frac{1}{2}} + K_{n,\beta}]h_N(N^{-\frac{1}{2}}y; x_0) \tag{37}$$

where the first factor is always positive, since by (13), $(2n)^{\frac{1}{2}} + K_{n,\beta} = (2n)^{\frac{1}{2}} c_{n,\beta} > 0$. Hence, from (10),

$$t(y) > 0 \quad \text{for} \quad |y| < |x_0|$$
$$(38) \qquad = 0 \quad \text{for} \quad |y| = |x_0|$$
$$< 0 \quad \text{for} \quad |y| > |x_0|.$$

Clearly, if N is fixed (or bounded) as $n \to \infty$, then

$$t(y) \to +\infty \quad \text{for} \quad |y| < |x_0|$$
$$(39) \qquad \to 0 \quad \text{for} \quad |y| = |x_0|$$
$$\to -\infty \quad \text{for} \quad |y| > |x_0|.$$

We show next that (39) holds also in the case where $N \to \infty$ as $n/N^2 \to \infty$. Using (20), we can write (37) as

$$(40) \qquad \begin{aligned} t(y) &= [(2n)^{\frac{1}{2}} + K_{n,\beta}]/r(N^{-\frac{1}{2}}x_0, \gamma) \\ &\times[(x_0^2 - y^2)r(0, \gamma)/2N + \{x_0^4 v(N^{-\frac{1}{2}}x_0, \gamma) - y^4 v(N^{-\frac{1}{2}}y, \gamma)\}/N^2]. \end{aligned}$$

It was proved in Section 4 that $v(\cdot, \gamma)$ is bounded. Hence if $N \to \infty$ as $n/N^2 \to \infty$, then $t(y)$ is asymptotically equal to $(2n)^{\frac{1}{2}}(x_0^2 - y^2)/2N$, and (39) holds.

Since (39) holds when $n/N^2 \to \infty$, and since z is asymptotically normal (0, 1), it follows from (36) that as $n/N^2 \to \infty$,

$$P_{N,n}(\gamma, \lambda_{N,n} \mid y) \to \beta + (1 - \beta) = 1 \quad \text{for} \quad |y| < |x_0|$$
$$(41) \qquad \to \quad \beta + 0 = \beta \quad \text{for} \quad |y| = |x_0|$$
$$\to \quad \beta - \beta = 0 \quad \text{for} \quad |y| > |x_0|.$$

Let

$$(42) \qquad f_{N,n}(y) = P_{N,n}(\gamma, \lambda_{N,n} \mid y)(2\pi)^{-\frac{1}{2}} \exp[-y^2/2].$$

Then it follows from (41) that as $n/N^2 \to \infty$

$$f_{N,n}(y) \to f(y) = (2\pi)^{-\frac{1}{2}} \exp[-y^2/2] \quad \text{for} \quad |y| < |x_0|$$
$$(43) \qquad = \beta(2\pi)^{-\frac{1}{2}} \exp[-y^2/2] \quad \text{for} \quad |y| = |x_0|$$
$$= 0 \quad \text{for} \quad |y| > |x_0|.$$

Since the functions $f_{N,n}(y)$ are uniformly dominated by the integrable function $(2\pi)^{-\frac{1}{2}} \exp[-y^2/2]$, it follows that as $n/N^2 \to \infty$,

$$P_{N,n}(\gamma, \lambda_{N,n}) = E\,[P_{N,n}\,(\gamma,\lambda_{N,n} \mid y)] = \int_{-\infty}^{\infty} f_{N,n}(y)\,dy \to \int_{-\infty}^{\infty} f(y)\,dy$$
$$= P\{|y| < |x_0|\}.$$

This result is recorded in the following theorem:

THEOREM 2. *If* $\lambda_{N,n} = r(N^{-\frac{1}{2}}x_0, \gamma)/c_{n,\beta}$, *and if* $n/N^2 \to \infty$, *then* $P_{N,n}(\gamma, \lambda_{N,n}) \to P\{|y| < |x_0|\}$ *where* y *has a normal* $(0, 1)$ *distribution.*

COROLLARY 1. *If* $\lambda_{N,n} = r(N^{-\frac{1}{2}}x_0, \gamma)/c_{n,\beta}$, *and if* $n/N^2 \to \infty$, *then the confidence level attained converges to the nominal confidence level* β *if and only if* $|x_0| = r(0, \beta)$.

Note that for $|x_0| = r(0, \beta)$, and N fixed, $\lim_{n\to\infty}\lambda_{N,n} = r(N^{-\frac{1}{2}}r(0, \beta), \gamma)$, and $\bar{x} \pm r(N^{-\frac{1}{2}}r(0, \beta), \gamma)\sigma$ are the well-known exact tolerance limits in the case where σ is known (see, e.g., [5]).

6. Heuristic explanation of results and discussion. It is perhaps surprising that as $N \to \infty$ the confidence level attained by the generalization of the Wald-Wolfowitz approximation should converge fastest to the nominal confidence level when n is fixed, and that the confidence level attained should fail to converge to the nominal confidence level if n increases "too rapidly." This behavior is contrary to that observed in most statistical applications, where it is usually found that the larger any sample-size is, the better. An heuristic explanation of the behavior observed can be given as follows.

For any γ, the exact tolerance factors for various β are the corresponding percentage points of the distribution of $r(\bar{x}, \gamma)/s$. The approximate tolerance factors given by Wald and Wolfowitz are percentage points of the distribution of $r(N^{-\frac{1}{2}}, \gamma)/s$. Thus Wald and Wolfowitz replace the random variable $r(\bar{x}, \gamma)$ by the "typical value" $r(N^{-\frac{1}{2}}, \gamma)$. The variance of $r(\bar{x}, \gamma)$ is $O(1/N^2)$ and the variance of s is $O(1/n)$. Consequently, the substitution made by Wald and Wolfowitz does not matter much if N^2 is "large" in comparison with n, since then most of the variability of $r(\bar{x}, \gamma)/s$ is due to s. On the other hand, if N^2 is not "large" in comparison with n, then much of the variability of $r(\bar{x}, \gamma)/s$ is due to $r(\bar{x}, \gamma)$, and a serious error is introduced by replacing the latter random variable by the "typical value" $r(N^{-\frac{1}{2}}, \gamma)$.

These remarks suggest that a more adequate approximation for general n, N could be obtained by replacing $r(\bar{x}, \gamma)$ by some other random variable, rather than by a constant. A normal random variable would be a natural candidate, since then the approximate tolerance factors would be given by the non-central t-distribution (as are the exact tolerance factors in the case of one-sided tolerance intervals). Furthermore, it is possible to choose the mean and variance of the normal random variable in such a way that as $N \to \infty$ with n fixed, or as $n \to \infty$ with N fixed, the tolerance factors tend to the exact tolerance factors for the respective cases of μ known and σ known. The asymptotic behavior of the difference between the confidence level attained and the nominal confidence level could be determined by using the methods employed in this paper.

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