A PROPERTY OF SOME SYMMETRIC TWO-STAGE SEQUENTIAL PROCEDURES[1]

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1. Introduction. Some statistical problems concerning a one-parameter family of distributions may be formulated in such a way that they reduce to deciding whether some equivalent parameter θ is positive or negative. In this paper, the problem is assumed to be already in the reduced form and the family of distributions to be exponential with a real parameter θ. If $F_\theta(x)$ is the distribution function corresponding to θ and $F_{-\theta}(x) = 1 - F_\theta(-x^-)$, then the family and the problem have some degree of symmetry. If, in addition, the loss through wrong decision is an even function of θ which is zero for $\theta = 0$, then the problem is symmetric and it is reasonable that such symmetry be reflected in any procedure used. If Θ is the parameter space and a prior distribution on Θ symmetric about zero is selected, the corresponding Bayes procedure will certainly have the required symmetry and indeed any Bayes procedure which is symmetric can be shown to be also Bayes with respect to a symmetric prior distribution. Under certain conditions Wald [2] has shown that Bayes procedures and their limits form an essentially complete class and therefore, if attention is to be restricted to symmetric procedures, it may be further restricted to those procedures which are Bayes solutions for symmetric prior distributions. In a sequential solution to a problem of this type, Chernoff [1] obtained results which suggested that the expected number of observations required had a maximum at $\theta = 0$, precisely where the possible losses are least. The present paper shows that the same is true of two-stage sequential solutions in a somewhat more general context.

2. Assumptions and definitions. Suppose that there are available real-valued observations $\{X_i\}$ which are independently and identically distributed with common distribution belonging to a one-parameter exponential family. Suppose that the natural parameter set contains a neighborhood of the origin so that all members of the family are absolutely continuous with respect to the distribution given by $\theta = 0$. Let the latter probability measure be denoted by μ. Then the distribution corresponding to θ will have density $g(\theta) \exp\{\theta x\}$ with respect to μ.

Assumption 2.1. *The measure μ is symmetric about the origin and is either continuous or discrete. Also $\mu(\{0\}) \neq 1$.*

For the proof of Lemma 4 a further property is required of μ which will in

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point of fact tend to rule out measures which are mixtures of continuous and discrete measures. For any n, let H_n be the space of values of $S_n = \sum_{i=1}^n X_i$. For any real number s, let μ_s be the distribution of $X + s$ when μ is the distribution of X. Let $K_n = \{\mu_s : s \in H_n\}$. The distribution of $|X + s|$ will depend only on $|s|$.

ASSUMPTION 2.2. *If the distribution of a random variable Z belongs to K_n then the distribution of $|Z|$ is stochastically increasing in $|s|$ for $s \in H_n (n = 1, 2, \cdots)$.*

The preceding assumption will be verified if μ is continuous and its density is unimodal or, if μ is discrete and H_1 is a set of equally spaced points whose probabilities under μ are nonincreasing as a function of their distance from the origin.

REMARK. It is a simple matter to show by induction that, if μ satisfies Assumptions 2.1 and 2.2, then for every n the n-fold convolution of μ will also satisfy them.

Let there be a prior measure ν on the Borel sets of Θ.

ASSUMPTION 2.3. *ν is symmetric about zero and $\nu(\{0\}) \neq 1$.*

The statistical problem under consideration is to decide between the hypotheses $H^-: \theta \leqq 0$ and $H^+: \theta \geqq 0$ and it is assumed that if $\theta = 0$, then it is of no consequence which hypothesis is accepted. Let $L(\theta)$ be the loss made when θ is the value of the parameter and the wrong hypothesis is accepted.

ASSUMPTION 2.4. *$L(\theta)$ is a bounded Borel-measurable function of θ with $L(-\theta) = L(\theta) \geqq 0$, $L(0) = 0$ and $\nu(\{\theta: L(\theta) > 0\}) > 0$.*

Let the cost of each observation be c.

3. Symmetric procedures. By a symmetric two-stage procedure I shall mean a procedure which specifies a first-stage sample size n_1, a second-stage sample size function $n_2(X_1, X_2, \cdots, X_{n_1})$ and a final decision rule with the following properties.

(i) $n_2(-X_1, -X_2, \cdots, -X_{n_1}) = n_2(X_1, X_2, \cdots, X_{n_1})$ $(= N_2$, say$)$.

(ii) If $\phi^-(X_1, X_2, \cdots X_{n_1}, X_{n_1+1}, \cdots, X_{n_1+N_2})$ is the probability that H^- is accepted when $X_1, X_2, \cdots, X_{n_1}, X_{n_1+1}, \cdots, X_{n_1+N_2}$ have been observed and $\phi^+(X_1, X_2, \cdots X_{n_1}, X_{n_1+1}, \cdots, X_{n_1+N_2})$ is similarly defined with respect to H^+, then

$$\phi^+(-X_1, -X_2, \cdots, -X_{n_1}, -X_{n_1+1}, \cdots, -X_{n_1+N_2}) = \phi^-(X_1, X_2, \cdots, X_{n_1}, X_{n_1+1}, \cdots, X_{n_1+N_2}).$$

It is easy to see that for any such procedure the probability of error will be an even function of θ.

Since the admissible procedures are the ones of interest, there is no need to take into account procedures with arbitrary large values of n_1 or N_2. If Assumption 2.4 is satisfied and $L(\theta) \leqq M$, taking more than M/c observations would mean that the cost of observations would be larger than the largest loss that can be made.

DEFINITION. Let $\mathfrak{D}$ be the class of symmetric two-stage procedures with $n_1 \leqq M/c$ and $n_2(x_1 \cdots x_{n_1}) \leqq M/c$ for all $(x_1, \cdots, x_{n_1})$.

Assumptions 2.1 to 2.4 imply that Assumptions 3.1 to 3.6 of Wald's book [2] on statistical decision functions are satisfied and enable use of his Theorem 3.17 to show that the symmetric Bayes procedures obtained by using symmetric prior distributions ν and limits of such procedures form a class which is complete relative to $\mathfrak{D}$.

4. Preliminaries. If $X_1, \cdots, X_n$ are observed then $S_n = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ. For any two-stage procedure let $S^{(1)} = \sum_{i=1}^{n_1} X_i$ and $S^{(2)} = \sum_{j=1}^{N_2} X_{n_1+j}$. After n observations have been taken, whether in one group or more, the posterior distribution will be absolutely continuous with respect to the prior distribution, and if $\sum_{i=1}^{n} X_i = s$, there will be a Radon-Nikodym derivative for which the specific form

$$[g(\theta)]^n \exp\{\theta s\} / \int_\Theta [g(\theta)]^n \exp\{\theta s\} \nu(d\theta) \tag{4.1}$$

can be used.

The first two lemmas are well known or almost obvious and the proofs will be omitted.

LEMMA 1. *If $S^{(1)} = s_1$ is observed at the first stage and $S^{(2)} = s_2$ at the second, an optimal symmetric final decision rule is given by the following.*

If $(s_1 + s_2) <, =, > 0$ then accept H^- with probability 1, $\frac{1}{2}$, 0, *respectively.*

LEMMA 2. *If $X_1 \cdots X_{n_1}$ are observed at the first stage and $S^{(1)} = \sum_{i=1}^{n_1} X_i$, the Bayes second stage number $\hat{N}_2$ is a function of $S^{(1)}$.*

REMARK. Lemma 2 is true because $S^{(1)}$ is sufficient for θ. For the Bayes procedures discussed in this paper $\hat{N}_2$ will be also an even function of $S^{(1)}$.

NOTATION. Let $\phi^-(t, s) = 1, \frac{1}{2}, 0$, respectively, if $(t + s) <, =, > 0$. Let $\phi^+(t, s) = 1 - \phi^-(t, s)$.

If $S^{(1)} = s$, $S^{(2)} = t$ then $\phi^-(t, s)$ is the probability that H^- will be accepted when the optimal decision rule of Lemma 1 is used.

Let

$$\begin{aligned} p(\theta; n_2, s_1) &= E_\theta[\phi^+(S^{(2)}, S^{(1)}) \mid S^{(1)} = s_1, N_2(S^{(1)}) = n_2] && \text{if } \theta < 0, \\ &= 0, && \text{if } \theta = 0, \\ &= E_\theta[\phi^-(S^{(2)}, S^{(1)} \mid S^{(1)} = s_1, N_2(S^{(1)}) = n_2] && \text{if } \theta > 0. \end{aligned}$$

Thus $p(\theta; n_2, s_1)$ is the probability of error when θ is the true value of the parameter given that s_1 was the observed value of $S^{(1)}$ and the second stage is to consist of n_2 observations. Let

$$\begin{aligned} R(n_2 \mid s_1) &= cn_2 \\ &+ \int_\Theta L(\theta)p(\theta; n_2, s_1)|g(\theta)|^{n_1} \exp\{\theta s_1\}\nu(d\theta) / \int_\Theta |g(\theta)|^{n_1} \exp\{\theta s_1\}\nu(d\theta). \end{aligned} \tag{4.2}$$

Now $R(n_2 \mid s_1)$ is the conditional Bayes risk if the second stage consists of n_2 observations and $S^{(1)} = s_1$ was observed at the first stage (using (4.1)). The Bayes second stage number $\hat{n}_2(s_1)$ will be that value of n_2 which minimizes $R(n_2 \mid s_1)$.

5. The second-stage sample size. The main result of the paper is to show that for any first sample size n_1, the Bayes solution for the second sample size, $n_2(s_1)$, is a nonincreasing function of $|s_1|$. This is proved in Theorem 1 by considering $\Delta R(n_2 \mid s_1) = R(n_2 + 1 \mid s_1) - R(n_2 \mid s_1)$. Lemma 4 establishes a technical result which is needed in the proof of Lemma 5. Lemma 5 shows that $\Delta R(n_2 \mid s_1)$ is an increasing function of $|s_1|$ and is the key to the proof of Theorem 1. Theorem 2 establishes a general property of the admissible symmetric procedures. In Lemmas 4 and 5 and Theorem 1, numbers u_0 and $\bar{s}(n_2)$ are used and need to be defined.

Let μ^* be the n_2-fold convolution of μ if $n_2 \geqq 1$ and the distribution degenerate at 0 if $n_2 = 0$. Let $u_0 = \sup \{u: \mu_{-u}((0, \infty)) > 0\} > 0$ since $\mu(\{0\}) \neq 1$. Let $\bar{s}(n_2) = \sup \{s: \mu_s^*((-u_0, u_0)) > 0\}$.

Lemma 3. $\bar{s}(n_2) = (n_2 + 1)u_0$.

The proof is straightforward and will be omitted.

Lemma 4. *If Assumptions* 2.1 *to* 2.3 *are verified, then, for fixed* $\theta \neq 0$ *and* n_2 *the expression*

$$[-\exp\{\theta s_1\}\Delta p(\theta; n_2, s_1) - \exp\{-\theta s_1\}\Delta p(-\theta; n_2, s_1)] \tag{5.1}$$

is a positive decreasing function of $|s_1|$ *if* $|s_1| < \bar{s}(n_2)$. *If* $|s_1| \geqq \bar{s}(n_2)$ *the expression is zero.*

Proof. Suppose $\theta > 0$.

$$\begin{aligned}
-\Delta p(\theta; n_2, s_1) &= E_\theta[\phi^-(S^{(2)}, S^{(1)} \mid S^{(1)} = s_1, n_2(S^{(1)}) = n_2] \\
&\quad - E_\theta[\phi^-(S^{(2)}, S^{(1)}) \mid S^{(1)} = s_1, n_2(S^{(1)}) = n_2 + 1] \\
&= \textstyle\int_{-\infty}^{\infty} \phi^-(s, s_1)[\int_{-\infty}^{\infty} g(\theta) \exp\{\theta t\}\mu(dt)][(g(\theta)]^{n_2} \exp\{\theta s\}\mu^*(ds) \\
&\quad - \textstyle\int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} \phi^-(t, s + s_1) g(\theta) \exp\{\theta t\}\mu(dt)][g(\theta)]^{n_2} \exp\{\theta s\}\mu^*(ds) \\
&= \textstyle\int_{-\infty}^{\infty} \phi^-(s, s_1) \int_{-\infty}^{\infty} \phi^+(t, s + s_1) g(\theta) \exp\{\theta t\}\mu(dt)[g(\theta)]^{n_2} \exp\{\theta s\}\mu^*(ds) \\
&\quad - \textstyle\int_{-\infty}^{\infty} \phi^+(s, s_1) \int_{-\infty}^{\infty} \phi^-(t, s + s_1) g(\theta) \exp\{\theta t\}\mu(dt)[g(\theta)]^{n_2} \exp\{\theta s\}\mu^*(ds).
\end{aligned}$$

The first term represents the probability that one more observation will turn a wrong decision into a right one whilst the second term gives the probability that an extra observation will turn a correct decision into a wrong one.

$$\begin{aligned}
-\exp\{\theta s_1\}\Delta p(\theta; n_2, s_1)[g(\theta)]^{-(n_2+1)} \\
= \textstyle\int_{-\infty}^{\infty} \phi^-(s, s_1) \int_{-\infty}^{\infty} \phi^+(t, s + s_1) \exp\{\theta(t + s + s_1)\}\mu(dt)\mu^*(ds) \\
- \textstyle\int_{-\infty}^{\infty} \phi^+(s, s_1) \int_{-\infty}^{\infty} \phi^-(t, s + s_1) \exp\{\theta(t + s + s_1)\}\mu(dt)\mu^*(ds).
\end{aligned}$$

Let $u = s + s_1$, $v = t + s + s_1$.

$$\begin{aligned}
-\exp\{\theta s_1\}\Delta p(\theta; n_2, s_1)[g(\theta)]^{-(n_2+1)} \\
= \textstyle\int_{-\infty}^{\infty} \phi^-(u, 0) \int_{-\infty}^{\infty} \phi^+(v, 0) \exp\{\theta v\}\mu_u(dv) \cdot \mu_{s_1}^*(du) \\
- \textstyle\int_{-\infty}^{\infty} \phi^+(u, 0) \int_{-\infty}^{\infty} \phi^-(v, 0) \exp\{\theta v\}\mu_u(dv) \cdot \mu_{s_1}^*(du).
\end{aligned}$$

Similarly

$$-\exp\{-\theta s_1\}\Delta p(-\theta; n_2, s_1)[g(-\theta)]^{-(n_2+1)}$$
$$= \int_{-\infty}^{\infty} \phi^+(u, 0) \int_{-\infty}^{\infty} \phi^-(v, 0) \exp\{-\theta v\}\mu_u(dv)\mu_{s_1}^*(du)$$
$$- \int_{-\infty}^{\infty} \phi^-(u, 0) \int_{-\infty}^{\infty} \phi^+(v, 0) \exp\{-\theta v\}\mu_u(dv)\mu_{s_1}^*(du).$$

Since $g(-\theta) = g(\theta)$,

$$-[g(\theta)]^{-(n_2+1)}[\exp\{\theta s_1\}\Delta p(\theta; n_2, s_1) + \exp\{-\theta s_1\}\Delta p(-\theta; n_2, s_1)]$$
$$= 2\int_{-\infty}^{\infty} \phi^-(u, 0) \int_{-\infty}^{\infty} \phi^+(v, 0) \sinh\{\theta v\}\mu_u(dv)\mu_{s_1}^*(du)$$
$$+ 2\int_{-\infty}^{\infty} \phi^+(u, 0) \int_{-\infty}^{\infty} \phi^-(v, 0) \sinh\{-\theta v\}\mu_u(dv)\mu_{s_1}^*(du).$$

Now $\phi^+(v, 0) \sinh\{\theta v\} = 0$ if $v \leqq 0$ and is a positive increasing function if $v > 0$. Since the distributions $\{\mu_u : -\infty < u < \infty\}$ are stochastically increasing in u,

$$\int_{-\infty}^{\infty} \phi^+(v, 0) \sinh\{\theta v\}\mu_u(dv)$$

as a function of u is positive and increasing if $u > -u_0$ and, when u_0 is finite, it is 0 if $u \leqq -u_0$. Similarly

$$\int_{-\infty}^{\infty} \phi^-(v, 0) \sinh\{-\theta v\}\mu_u(dv)$$

is positive and decreasing if $u < u_0$ and, when u_0 is finite, it is 0 if $u \leqq u_0$. Let

$$h(u) = 2[g(\theta)]^{n_2+1} \int_{-\infty}^{\infty} \phi^+(v, 0) \sinh\{\theta v\}\mu_u(dv) \qquad \text{if } u \leqq 0,$$
$$= 2[g(\theta)]^{n_2+1} \int_{-\infty}^{\infty} \phi^-(v, 0) \sinh\{-\theta v\}\mu_u(dv) \qquad \text{if } u \geqq 0,$$

then $h(u)$ is an even function of u which is positive and decreasing if $0 \leqq u < u_0$ and if u_0 is finite, $h(u) = 0$ for $u \geqq u_0$.

Thus

$$[-\exp\{\theta s_1\}\Delta p(\theta; n_2, s_1) - \exp\{-\theta s_1\}\Delta p(-\theta; n_2, s_1)]$$
$$= \int_{-\infty}^{\infty} h(u)\mu_{s_1}^*(du) = E_{s_1}[h(|U|)]$$

where $U - s_1$ is a random variable with distribution μ^*. By Assumption 2.2 the distribution of $|U|$ is stochastically increasing in $|s_1|$ in the special sense defined. Therefore if $|s_1| < \bar{s}(n_2)$, $E_{s_1}[h(|U|)]$ is positive and decreasing as a function of $|s_1|$ so that the first result follows. If $|s_1| \geqq \bar{s}(n_2)$ then $E_{s_1}[h(|U|)] = 0$ and the second result is proved. By symmetry, the lemma holds for $\theta < 0$.

LEMMA 5. *If Assumptions 2.1, 2.3, and 2.4 are valid and the conclusion of Lemma 4 holds, for fixed* n_2, $\Delta R(n_2 \mid s_1)$ *is an increasing function of* $|s_1|$ *if* $|s_1| < \bar{s}(n_2)$ *and is equal to c if* $|s_1| \geqq \bar{s}(n_2)$.

PROOF. $\Delta R(n_2 \mid s_1) = c - B(s_1)/A(s_1)$ where

$$A(s_1) = \int_{\Theta} [g(\theta)]^{n_1} \exp\{\theta s_1\}\nu(d\theta),$$

and

$$B(s_1) = -\int_\Theta L(\theta)\Delta p(\theta; n_2, s_1)[g(\theta)]^{n_1} \exp\{\theta s_1\}\nu(d\theta),$$

using (4.2).

Let $\Theta^+ = \{\theta: \theta \varepsilon \Theta, \theta > 0\}$. Then

$$\begin{aligned} A(s_1) &= \int_{\Theta^+} [g(\theta)]^{n_1}(\exp\{\theta s_1\} + \exp\{-\theta s_1\})\nu(d\theta) + \nu(\{0\}) \\ &= 2\int_{\Theta^+} [g(\theta)]^{n_1} \cosh\{\theta s_1\}\nu(d\theta) + \nu(\{0\}) \end{aligned}$$

since g is even and ν is symmetric about zero.

But $\cosh\{\theta s_1\}$ is an increasing function of $|s_1|$ for all $\theta \neq 0$, $\nu(\{0\}) \neq 1$ and $A(0) > 0$ since $g(\theta) > 0$ for all $\theta \varepsilon \Theta$. Therefore, $A(s_1)$ is positive and is a strictly increasing function of $|s_1|$.

$$B(s_1) = \int_{\Theta^+} L(\theta)[-\exp\{\theta s_1\}\Delta p(\theta; n_2, s_1) - \exp\{-\theta s_1\}\Delta p(-\theta; n_2, s_1)] \cdot [g(\theta)]^{n_1}\nu(d\theta)$$

since L, g are even functions, $L(0) = 0$ and ν is symmetric. Since Assumption 2.4 and the conclusion of Lemma 4 hold, with positive probability the integrand is positive and decreasing as a function of $|s_1|$ if $|s_1| < \bar{s}(n_2)$.

Thus $B(s_1)/A(s_1)$ is a positive decreasing function of $|s_1|$ if $|s_1| < \bar{s}(n_2)$ and is zero if $|s_1| \geqq \bar{s}(n_2)$. Therefore, $\Delta R(n_2 \mid s_1) = c - B(s_1)/A(s_1)$ is an increasing function of $|s_1|$ if $|s_1| < \bar{s}(n_2)$. If $|s_1| \geqq \bar{s}(n_2)$ then $\Delta R(n_2 \mid s_1) = c$.

THEOREM 1. *If the conclusion of Lemma 5 holds, then for any first sample size, the Bayes solution for the second sample function is a nonincreasing function of* $|s_1|$.

PROOF. $\Delta R(n_2 \mid s_1) = c$ if $|s_1| \geqq \bar{s}(n_2)$, since for such values of s_1, $B(s_1) = 0$. But $\bar{s}(n_2)$ is an increasing function of n_2 by Lemma 3. Therefore, if $s_1 \geqq s(n_2)$ for some value of n_2, $B(s_1) = 0$ and $\Delta R(n_2 \mid s_1) = c > 0$ for all smaller values of n_2 and the Bayes solution $\hat{n}_2(s_1)$ must be either zero or $s_1 < \bar{s}(\hat{n}_2(s_1))$.

Now $\hat{n}_2(s_1)$ minimizes $R(n_2 \mid s_1)$ for each s_1 and since $R(n_2 \mid s_1) = R(n_2 \mid -s_1)$ it may be assumed that $\hat{n}_2(s_1) = \hat{n}_2(-s_1)$.

Suppose a, b are values of s_1 and $0 \leqq a < b$, $a < \bar{s}(\hat{n}_2(a))$. If $\hat{n}_2(a)$ is the Bayes solution for $s_1 = a$, for all $n_2 > \hat{n}_2(a)$, $0 \leqq R(n_2 \mid a) - R(\hat{n}_2(a) \mid a) = \sum_{k=\hat{n}_2(a)}^{n_2-1} \Delta R(k \mid a) < \sum_{k=\hat{n}_2(a)}^{n_2-1} \Delta R(k \mid b)$ (by Lemma 5) $= R(n_2 \mid b) - R(\hat{n}_2(a) \mid b)$, therefore, $R(n_2 \mid b) > R(\hat{n}_2(a) \mid b)$, therefore, $\hat{n}_2(b) \neq n_2 > \hat{n}_2(a)$, i.e., $\hat{n}_2(b) \leqq \hat{n}_2(a)$. If $a \geqq s(\hat{n}_2(a))$ and $\hat{n}_2(a) = 0$, a modified argument can be used to show that $\hat{n}_2(b) = 0$.

THEOREM 2. *If Assumptions* 2.1, 2.2, 2.4 *are valid and the conclusion of Lemma* 5 *holds, the expected number of observations for any admissible symmetric two stage procedure is a nonincreasing function of* $|\theta|$.

PROOF. Theorem 1 shows that, for every symmetric procedure with Bayes second stage, the second sample size function $\hat{n}_2(S^{(1)})$ is a nonincreasing function of $|S^{(1)}|$. Since the Bayes solution itself is such a procedure with some first sample size $\hat{n}_1$, the same property will hold. It is easy to see that the distribution of $|S^{(1)}|$ is stochastically increasing in $|\theta|$. Thus, $E_\theta[\hat{n}_2(|S^{(1)}|)]$ is a nonincreasing function of $|\theta|$.

Assumptions 2.1, 2.3, 2.4 are sufficient for Wald's theorem 3.17 [2] to hold. Using Wald's terminology, Bayes symmetric procedures in the wide sense form an essentially complete class of symmetric procedures. It is easy to see that any admissible procedure must have the required property by considering conditional Bayes risks with respect to appropriate prior distributions.

6. Some specific cases. The important conditions on the distribution of the observations are those of Assumptions 2.1, 2.2. If μ is continuous with a symmetric unimodal density, the conditions are satisfied. Thus, if μ is a normal distribution, Theorem 2 will be true and such a result compares nicely with the work of Chernoff on fully sequential solutions. If μ is discrete and its support is a set of equally spaced values, then Assumption 2.2 will be satisfied if the probabilities of the various possible values are nonincreasing as a function of their distance from zero. With some adjustments, the problem of testing whether a binomial parameter p is greater than or less than $\frac{1}{2}$ can be fitted into such a category. The required transformation will be to let $\theta = \ln (p/1 - p)$ and to let $\mu(\{-\frac{1}{2}\}) = \mu(\{\frac{1}{2}\}) = \frac{1}{2}$.

7. Remarks. The problem which led to the present results concerned two-stage solutions for the binomial case mentioned in the last section. Since the values of possible losses could be expected to be smallest for values of p near $\frac{1}{2}$, it was hoped that symmetric designs could be found with at least a relative minimum for the expected number of observations when $p = \frac{1}{2}$. However, Theorem 2 shows that such designs would be inadmissible. It is not known whether the same is true of fully sequential solutions to similar problems, but I would conjecture that it is, although it may not be possible to extend the methods of proof used here.

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REFERENCES

[1] CHERNOFF, H. (1960). Sequential tests for the mean of a normal distribution. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** 79–91. Univ. of California Press.

[2] WALD, A. (1950). *Statistical Decision Functions.* Wiley, New York.