RANKINGS FROM PAIRED COMPARISONS[1]

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1. Introduction. In many experimental contexts, preference relationships among objects can be obtained where numerical measurement is difficult or impossible. This situation occurs, for example, in regard to individual preferences for pieces of music. Frequently because of memory, fatigue or distance limitations the items will be presented to a subject only in pairs and he will be asked to state his preference. The authoritative statistical paper on this method of paired comparisons is that of Kendall and Smith [7]; they propose a criterion, based on the number of circular preference triads, for judging whether a given set of preferences can reasonably be considered as reflecting a single linear variable. Alternatively, Slater [6] proposes that the items should be ranked so as to minimize the number of violations of observed preference, and that this minimum should then replace Kendall and Smith's circular triad criterion. Thus, more generally than Slater's particular criterion, one reason for wishing to estimate rank order is to obtain a standard whose deviations from the observed set of paired comparisons can then be examined to determine whether or not ranking was justified at all. Other uses of rank order arise in psychological testing and market survey work. We may, for instance, wish to choose the three most preferred blends of coffee or place people into homogeneous groups according to their preferences.

Doehlert's master's thesis [5] was responsible for calling our attention to the subject of this paper. We study two theoretical aspects of the problem of obtaining rankings from paired comparisons. Section 2 treats the problem from the point of view of graph theory. Section 3 then introduces a mathematical model based on the concept of weak stochastic transitivity and uses the graph theoretic results of Section 2 to obtain a maximum likelihood ranking for this model. We show that the maximum likelihood weak stochastic ranking yields Slater's criterion when every pair of objects is compared exactly once.

2. Paired comparison graphs. Let $X = \{x_1, x_2, \cdots, x_m\}$ be a set of $m > 2$ distinct objects. A set of paired comparisons of X is a relation $\mathbf{R}$ in X which is anti-symmetric and anti-reflexive; that is, a subset of $X \times X$ such that $(x_i, x_i) \not\varepsilon \mathbf{R}$ and if $(x_i, x_j) \varepsilon \mathbf{R}$ then $(x_j, x_i) \not\varepsilon \mathbf{R}$. For brevity such a relation will be called a *comparison of* X. For definiteness the reader may interpret $(x_i, x_j) \varepsilon \mathbf{R}$ to mean "in the comparison $\mathbf{R}$, x_i is preferred to x_j"; however, the results of Section 2 in no way depend on this interpretation. A path $\mathbf{K}$ in $\mathbf{R}$ from

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y_1 to y_k and sometimes denoted by $(y_1, y_2, \cdots, y_k)$, is a finite collection of ordered pairs $(y_1, y_2) \varepsilon \mathbf{R}$, $(y_2, y_3) \varepsilon \mathbf{R}$, $\cdots$, $(y_{k-1}, y_k) \varepsilon \mathbf{R}$. If the $y_1, y_2, \cdots, y_k$ are distinct elements of X, the path is said to be *elementary*. If $y_1 = y_k$, then the path is called a *circuit*; elementary if all elements except y_1 and y_k are distinct. If $\mathbf{K}$ is a path in $\mathbf{R}$, with $y_1 \neq y_k$, it is easy to see that there is an elementary path $\mathbf{K}' \subset \mathbf{K}$ from y_1 to y_k. However, a circuit need not contain an elementary circuit coincident with a given pair of elements.

A full discussion of these concepts may be found in Berge [1]. An *arrangement* of X, denoted ordinarily by P or $(p_1, p_2, \cdots, p_m)$, is a reordering of X, i.e. an m-tuple whose elements are the elements of X. It is clear that a comparison on X may be regarded as a comparison on any arrangement of X; no distinction will be made between $\mathbf{R}$ on X and $\mathbf{R}$ on P. $\mathbf{R}$ is said to determine a *partial rank-order* (p.r.o.) on P if $(p_j, p_i) \not\varepsilon \mathbf{R}$ whenever $i < j$.

An $\mathbf{R}$-*bound* of X is an element y in X such that $(x, y) \not\varepsilon \mathbf{R}$ for all x in X; or more intuitively, an object y such that in the comparison $\mathbf{R}$ no object is preferred to y. By a familiar argument, involving contradiction, we may show that if $\mathbf{R}$ is a circuit-free comparison of X, then X has at least one $\mathbf{R}$-bound. Applying this comment stepwise m times yields

THEOREM 1. $\mathbf{R}$ *determines at least one* p.r.o. *if and only if it is circuit-free.*

A p.r.o. P determined by $\mathbf{R}$ is said to be a *semi rank-order* (s.r.o.) if there is a path in $\mathbf{R}$ from p_i to p_j whenever $i < j$. In particular if $(p_i, p_j) \varepsilon \mathbf{R}$ for all $i < j$ then P is called a *rank-order* (r.o.) determined by $\mathbf{R}$.

THEOREM 2. *P is a* s.r.o. *determined by* $\mathbf{R}$ *if and only if* $(p_i, p_{i+1}) \varepsilon \mathbf{R}$ *for* $i = 1, \cdots, m - 1$ *and* $(p_i, p_j) \not\varepsilon \mathbf{R}$ *for* $i > j$.

PROOF. Assume P is a s.r.o. and $(p_i, p_{i+1}) \not\varepsilon \mathbf{R}$. There must be a path $(p_i, p_{k_1}, p_{k_2}, \cdots, p_{k_r}, p_{i+1})$ in $\mathbf{R}$ from p_i to p_{i+1}. If, for example, $k_2 < k_1$ then the path from p_{k_2} to p_{k_1} combined with (p_{k_1}, p_{k_2}) would form a circuit, contrary to the fact that P is a s.r.o., hence the following sequence of inequalities must hold: $i < k_1 < k_2 < \cdots < k_r < i + 1$. This is clearly impossible so that $(p_i, p_{i+1}) \varepsilon \mathbf{R}$. The remainder of the proof is clear.

Theorem 2 gives a simple characterization of a s.r.o.; the next result indicates why it is worth-while to define such a concept.

THEOREM 3. *A circuit-free relation* $\mathbf{R}$ *determines a unique* p.r.o. *P if and only if P is a* s.r.o.

PROOF. If $\mathbf{R}$ determines the p.r.o. P, then $(p_i, p_j) \not\varepsilon \mathbf{R}$ for $i > j$. In particular $(p_{i+1}, p_i) \not\varepsilon \mathbf{R}$ for $i = 1, \cdots, m - 1$. If, in addition, P is not a s.r.o., then for some k, $(p_k, p_{k+1}) \not\varepsilon \mathbf{R}$ and $\mathbf{R}$ also determines the p.r.o. obtained from P by interchanging p_k and p_{k+1}. Hence $\mathbf{R}$ determines a unique p.r.o. P only if P is a s.r.o. On the other hand Theorem 2 makes it clear that if P is a s.r.o. determined by $\mathbf{R}$, then no other p.r.o. can be determined by $\mathbf{R}$.

The comparison $\mathbf{R}$ is said to be *complete* if for $1 \leqq i < j \leqq m$ then either $(x_i, x_j) \varepsilon \mathbf{R}$ or $(x_j, x_i) \varepsilon \mathbf{R}$. Let $\mathbf{C}$ be the generic symbol for a complete comparison on X, then $\mathbf{C}$ consists of $\binom{m}{2}$ pairs of the product space $X \times X$. If for every $i \neq j$ there is either a path in $\mathbf{R}$ from x_i to x_j or from x_j to x_i, then $\mathbf{R}$

will be called *semi-complete*. The following theorem is a straightforward consequence of Theorem 1 and the definitions of the various concepts involved.

THEOREM 4. *A comparison determines a unique* r.o. (s.r.o.) *if and only if it is complete (semi-complete) and circuit-free.*

R is a *maximal circuit-free subset* of a comparison **S** if and only if **R** is circuit-free but is not properly contained in any other circuit-free subset of **S**. It follows that if **R** is a circuit-free subset of **S**, there is at least one maximal circuit-free subset of **S** which contains **R**.

If $(p_1, p_2, \cdots, p_m)$ is an elementary path in **R**, then it will be called an *m-path* in **R**.

THEOREM 5. *Given a comparison* **S** *and an m-path P in* **S**, *then* $\mathbf{R} = \{(p_i, p_j) \varepsilon \mathbf{S}: i < j\}$ *is the unique maximal circuit-free subset of* **S** *which determines P as a* s.r.o.

PROOF. If **T** is a maximal circuit-free subset of **S** which determines P as a s.r.o. then, from Theorem 2, the m-path P is contained in **T**. If **T** contains (p_i, p_j) with $i > j$, then the path $(p_j, p_{j+1}, \cdots, p_{i-1}, p_i)$ taken with (p_i, p_j) forms a circuit in **T**. Hence $\mathbf{T} \subset \mathbf{R}$ but **R** is clearly circuit-free and **T** is maximal so that $\mathbf{T} = \mathbf{R}$.

If **C** is complete and **D** is the diagonal of $X \times X$ then $\mathbf{E} = X \times X - (\mathbf{D} \cup \mathbf{C})$ is likewise complete and if $r(x_i, x_j) \equiv (x_j, x_i)$ then r is a one-to-one mapping of **C** onto **E**. If $\mathbf{R} \subset \mathbf{C}$, then $\mathbf{F} = \mathbf{R} \cup r(\mathbf{C} - \mathbf{R})$ is clearly complete. We have in mind that **R** represents the comparisons remaining after "removing" those of $\mathbf{C} - \mathbf{R}$, while **F** represents the result of adding to **R** the "reversals" of $\mathbf{C} - \mathbf{R}$.

LEMMA. *If* **R** *is a maximal circuit-free subset of* **C** *and* $(x_j, x_i) \varepsilon \mathbf{C} - \mathbf{R}$, *then* $\mathbf{S} = \mathbf{R} \cup \{(x_i, x_j)\}$ *is a maximal circuit-free subset of* $\mathbf{C}' = [\mathbf{C} - \{(x_j, x_i)\}] \cup \{(x_i, x_j)\}$.

PROOF. We first show that **S** is circuit-free. By the maximality of **R**, there must be a path $\mathbf{K}_1$ in **R** from x_i to x_j completing with (x_j, x_i) a circuit. Now if **S** is not circuit-free, then any circuit in **S** must contain (x_i, x_j) and thus **R** would contain a chain $\mathbf{K}_2$ from x_j to x_i. But this yields a contradiction since $\mathbf{K}_2 \cup \mathbf{K}_1$ would then contain a circuit in **R**.

The lemma will be proved when we have shown that **S** is maximal in $\mathbf{C}'$. We see at once that $\mathbf{R} \subset \mathbf{S} \subset \mathbf{C}'$ and that $\mathbf{C}'$ is complete. If **S** is not a maximal circuit-free subset of $\mathbf{C}'$, then there is an element (x_p, x_q) of $\mathbf{C}' - \mathbf{S}$ which unites with **S** to form a circuit-free set **T**. Since $(x_p, x_q) \varepsilon \mathbf{C} - \mathbf{R}$, then $\mathbf{R} \cup (x_p, x_q)$ is not circuit-free because of the maximality of **R**. But $\mathbf{R} \cup (x_p, x_q) \subset \mathbf{T}$ so that **T** is not circuit-free. This contradiction proves the maximality of **S** in $\mathbf{C}'$ and concludes the proof of the lemma.

THEOREM 6. *If* **R** *is a maximal circuit-free subset of* **C**, *then* $\mathbf{F} = \mathbf{R} \cup r(\mathbf{C} - \mathbf{R})$ *is also circuit-free.*

The proof consists of applying the lemma iteratively a finite number of times.

THEOREM 7. *If* **R** *is a maximal circuit-free subset of the complete set* **C** *and* $\mathbf{F} = \mathbf{R} \cup r(\mathbf{C} - \mathbf{R})$ *then there is a unique arrangement P such that* 1. *P is a rank-order determined by* **F**; 2. *P is a semi rank-order determined by* **R**.

PROOF. Clearly **F** is complete; further, according to Theorem 6 **F** is circuit-free. Hence from Theorem 4 we have that there exists a unique rank order P determined by **F**. Now assume that P is not a semi rank-order determined by **R**. There must then exist elements p_i and p_j with $i < j$ such that there is no path in **R** from p_i to p_j. Therefore **R** $\cup$ (p_j, p_i) is circuit-free. But $(p_i, p_j) \varepsilon$ **F** $-$ **R** $=$ $r($**C** $-$ **R**$)$ and $(p_j, p_i) \varepsilon$ **C** $-$ **R**, so that **R** $\subset$ **R** $\cup$ $(p_j, p_i) \subset$ **C** contradicting the maximal nature of **R**. Thus P must be a semi rank-order determined by **R** and the theorem is proved.

COMMENT. This theorem shows that if we wish to convert a complete set of comparisons into a ranking, then we may do so by either deleting comparisons or reversing comparisons; the rankings obtained do not depend on which procedure we adopt.

THEOREM 8. *If* **C** *is a complete comparison, then there is a one-to-one correspondence between the maximal circuit-free subsets of* **C** *and the m-paths in* **C**.

PROOF. If **R** is maximal circuit-free, then Theorem 7 states the existence of a unique s.r.o. P determined by **R**. Theorem 2 then implies that P is an m-path in **R** but **R** $\subset$ **C** and hence P is an m-path in **C**. On the other hand, if P is an m-path in **C**, then Theorem 5 states the existence of a unique maximal circuit-free subset of **C** which determines P as a s.r.o.

3. Estimating a weak stochastic ranking. The following probabilistic model is introduced as one possible theoretic foundation for the method of paired comparisons. Let m items denoted individually by $x_1, x_2, \cdots, x_m$ and collectively by X be independently (in the probability sense) compared in pairs. Items x_i and x_j are compared on $n_{ij}(n_{ij} = 0, 1, 2, \cdots)$ independent trials, each trial having two possible outcomes denoted respectively by $x_i \rightarrow x_j$ and $x_j \rightarrow x_i$.

Let I denote the set of all subscripts pairs (ij) such that $1 \leqq j < i \leqq m$ and $n_{ij} > 0$ and let q be the number of pairs in I. We have $q \leqq \binom{m}{2}$ with equality holding in the frequent case where every pair of items is compared at least once. Define $P(x_i \rightarrow x_j) = \pi_{ij}$ for $i \neq j$. Since $\pi_{ij} + \pi_{ji} = 1$, exactly q functionally independent parameters enter into the model; we may as well restrict our consideration to those parameters with subscripts in I. The parameter space Ω, of all π_{ij} with subscripts in I is a q-dimensional unit cube with typical parameter point π. The probability that $x_i \rightarrow x_j$ occurs s_{ij} times $(s_{ij} + s_{ji} = n_{ij})$ for each and every pair of items is given by

$$\prod_I \binom{n_{ij}}{s_{ij}} \pi_{ij}^{s_{ij}} \pi_{ji}^{s_{ji}}. \tag{1}$$

Brunk [3] has reviewed much of the previous theoretical work on ranking from paired comparisons. The Bradley-Terry Model [2] is typical of what Brunk calls an intrinsic worth model. Bradley and Terry assume that $\pi_{ij} = \pi_i/(\pi_i + \pi_j)$ where π_i and π_j are the "worths" of x_i and x_j respectively. Here we investigate a method of determining rank-order without assuming the existence of an intrinsic worth. In order to motivate a more general definition, observe that in the Bradley-Terry model $\pi_i > \pi_j$ if and only if $\pi_{ij} > \frac{1}{2}$. In general we define the comparison $\mathbf{R}(\pi)$ as follows:

$$(x_i, x_j) \varepsilon \mathbf{R}(\pi) \quad \text{if and only if} \tag{2}$$
$$n_{ij} > 0 \quad \text{and} \quad \pi_{ij} = P_r(x_i \rightarrow x_j) > \tfrac{1}{2}.$$

Note that $x_i \rightarrow x_j$ and $\mathbf{R}(\pi)$ are sample and population orderings respectively, and that in some cases the two may easily be in opposite directions.

Following the related literature [4], if P is a rank-order (s.r.o. or p.r.o.) determined by $\mathbf{R}(\pi)$, then it will be called a *weak stochastic ranking*. Use of the modifier "weak" implies that there must also be a strong stochastic ranking. The difference between these two concepts can best be seen by an example. If $m = 3$, $\pi_{12} = \frac{3}{4}$, $\pi_{23} = \frac{3}{4}$ and $\pi_{13} = \frac{1}{2}$, then (x_1, x_2, x_3) is a weak stochastic s.r.o. determined by $\mathbf{R}(\pi)$. Some writers [3] would require $\pi_{12} \geqq \frac{1}{2}$, $\pi_{23} \geqq \frac{1}{2}$, $\pi_{13} \geqq \frac{1}{2}$, $\pi_{13} \geqq \pi_{12}$ and $\pi_{13} \geqq \pi_{23}$ before declaring (x_1, x_2, x_3) to be a stochastic ranking; such a ranking is called strong. We believe that there is a need for both weak and strong stochastic ranking theories. The rankings of the present paper will in general be weak.

We have now introduced the concepts necessary to state the main idea of this section; briefly it is as follows. Define ω to be that portion of the total parameter space Ω where $\mathbf{R}(\pi)$, as defined in (2), is a circuit-free comparison. A point in ω may be called a *circuit-free point*. Next maximize the likelihood function (1) over the closed region ω to obtain estimates $(\bar{\pi}_{ij}) = \bar{\pi}$ which would then immediately yield a p.r.o. determined by $\mathbf{R}(\bar{\pi})$. In general the likelihood function may assume its maximum at several points of ω, in which case the maximum likelihood ranking may not be unique. Let μ be the set of all circuit-free points at which the likelihood function assumes its maximum value.

From (1) we have that the log-likelihood is a constant plus $l(\pi)$, where

$$l(\pi) = \sum_I (s_{ij} \log \pi_{ij} + s_{ji} \log \pi_{ji}). \tag{3}$$

In maximizing the likelihood we may as well maximize l. In Ω, the unrestricted parameter cube, the maximum likelihood estimates of the π_{ij} are $\hat{\pi}_{ij} = s_{ij}/n_{ij}$ for $(ij) \varepsilon I$ and the maximum of l is

$$\max_\Omega l(\pi) = l(\hat{\pi}) = \sum_I n_{ij}(\hat{\pi}_{ij} \log \hat{\pi}_{ij} + \hat{\pi}_{ji} \log \hat{\pi}_{ji}).$$

In this notation we may rewrite (3) as

$$l(\pi) = \sum_I n_{ij}(\hat{\pi}_{ij} \log \pi_{ij} + \hat{\pi}_{ji} \log \pi_{ji}). \tag{3'}$$

The following theorem is fundamental to our objective.

THEOREM 9. *For all $(ij) \varepsilon I$ either $\bar{\pi}_{ij} = \frac{1}{2}$ or $\bar{\pi}_{ij} = \hat{\pi}_{ij}$.*

PROOF. Suppose that $(ab) \varepsilon I$ but $\bar{\pi}_{ab}$ equals neither $\frac{1}{2}$ nor $\hat{\pi}_{ab}$. Since $\bar{\pi}_{ab} + \bar{\pi}_{ba} = 1$ and $\hat{\pi}_{ab} + \hat{\pi}_{ba} = 1$, we may without loss of generality take $\bar{\pi}_{ab} < \frac{1}{2}$. Let $\tilde{\pi}$ and π' be the points obtained by substituting $\frac{1}{2}$ and $\hat{\pi}_{ab}$ respectively for $\bar{\pi}_{ab}$ in $\bar{\pi}$, the remaining $\bar{\pi}_{ij}$ being held constant. $\bar{\pi}$, $\tilde{\pi}$ and π' are colinear and on this line, l assumes its unique absolute maximum at π'. Further $\tilde{\pi}$ is circuit-free but π' can not be since $l(\tilde{\pi}) < l(\pi')$; hence we must have $\tilde{\pi}_{ab} < \frac{1}{2} < \hat{\pi}_{ab}$. From this it follows (by considering the derivative of l w.r.t. π_{ab}) that $l(\tilde{\pi}) < l(\bar{\pi}) < l(\pi')$. But $\bar{\pi}$ is circuit-free since $\tilde{\pi}$ is, and $l(\bar{\pi}) > l(\tilde{\pi})$ contradicting the definition of $\tilde{\pi}$.

Define $\beta = \{\pi: \pi_{ij} = \frac{1}{2} \text{ or } \pi_{ij} = \hat{\pi}_{ij} \text{ for all } (ij) \varepsilon I\}$, in view of Theorem 9, $\mu \subset \beta$. Note that β depends on the particular collection of comparisons which have been observed and contains at most 2 raised to the power $\binom{m}{2}$ distinct points, many of which will not be circuit-free.

COROLLARY. $\max_\omega l(\pi) = \max_{\omega \cap \beta} l(\pi)$.

Observe that $\omega \cap \beta$ is a finite set.

This method of estimating rank-order has an important information theoretic interpretation which adds to its intuitive appeal. The uncertainty of a single comparison of x_i and x_j is $-\pi_{ij} \log \pi_{ij} - \pi_{ji} \log \pi_{ji}$. Since information or uncertainty is additive over independent experiments, the uncertainty of all n_{ij} comparisons of x_i and x_j is

$$u_{ij} = u_{ij}(\pi_{ij}) = n_{ij}(-\pi_{ij} \log \pi_{ij} - \pi_{ji} \log \pi_{ji})$$

and the uncertainty of all comparisons is

$$U(\pi) = \sum_I u_{ij} = \sum_I n_{ij}(-\pi_{ij} \log \pi_{ij} - \pi_{ji} \log \pi_{ji}). \tag{4}$$

LEMMA. *If $\pi \varepsilon \beta$, then $l(\pi) = -U(\pi)$.*

PROOF. Either $\pi_{ij} = \frac{1}{2}$ or $\pi_{ij} = \hat{\pi}_{ij}$; in either case $\hat{\pi}_{ij} \log \pi_{ij} + \hat{\pi}_{ji} \log \pi_{ji} = \pi_{ij} \log \pi_{ij} + \pi_{ji} \log \pi_{ji}$.

THEOREM 10. *Maximizing the likelihood over ω is equivalent to minimizing the uncertainty over the set $\omega \cap \beta$.*

PROOF. From the previous lemma and the corollary to Theorem 9

$$\max_\omega l(\pi) = \max_{\omega \cap \beta} l(\pi) = -\min_{\omega \cap \beta} U(\pi).$$

Noting that $\max_{\pi_{ij}} u_{ij}(\pi_{ij}) = u_{ij}(\frac{1}{2})$ then the uncertainty increase in taking x_i and x_j to be equal in preference is seen to be

$$\Delta_{ij} = u_{ij}(\tfrac{1}{2}) - u_{ij}(\hat{\pi}_{ij})$$
$$= n_{ij}(1 - \hat{\pi}_{ij} \log \hat{\pi}_{ij} - \hat{\pi}_{ji} \log \hat{\pi}_{ji}) = \max u_{ij} - u_{ij}(\hat{\pi}_{ij}) \geqq 0.$$

From this relationship we see that non-transitive comparisons will in general be resolved by equating in preference items with Δ_{ij} small. Hence, if $n_{ij} = n$ for all i and j, then we have in particular that non-transitivities will be resolved by equating items with large estimated uncertainties.

As a first and simplest example consider that $m = 3$ and $n_{12} = n_{23} = n_{13} = n$. If $(\hat{\pi}_{12}, \hat{\pi}_{13}, \hat{\pi}_{23})$ is circuit-free, then it yields a p.r.o. Hence we need consider only the case where the estimated ranking is circular; without loss of generality we assume $\hat{\pi}_{12} > \frac{1}{2}$, $\hat{\pi}_{23} > \frac{1}{2}$ and $\hat{\pi}_{31} > \frac{1}{2}$. This circular preference relationship will be converted into a s.r.o. by deleting *any* single preference. Thus according to Theorem 10 and Equation (4) the maximum likelihood s.r.o. will be obtained by deleting that preference with the greatest estimated uncertainty, or since all items are compared an equal number of times, by deleting that preference for which $\hat{\pi}_{ij}$ is nearest to .5.

The rankings of this paper differ from those given by the method of rank sums, and in particular they differ from the Bradley-Terry rankings. For the example of the previous paragraph the rank sums are $s_{12} + s_{13}$, $s_{21} + s_{23}$ and $s_{31} + s_{32}$. We have

$$s_{12} + s_{13} = s_{21} + s_{23} + 2[s_{12} - \tfrac{1}{2}(s_{31} + s_{23})]$$

and

$$s_{21} + s_{23} = s_{31} + s_{32} + 2[s_{23} - \tfrac{1}{2}(s_{31} + s_{12})].$$

For $s_{12} = 8$, $s_{13} = 5$, $s_{23} = 11$ and $n = 12$ we have $\hat{\pi}_{12} = \frac{2}{3}$, $\hat{\pi}_{31} = \frac{7}{12}$ and $\hat{\pi}_{23} = \frac{11}{12}$. The weak stochastic ranking is (x_1, x_2, x_3); while that of the rank sum method is (x_2, x_1, x_3).

Before presenting an additional example we need one more result. Define $E = \{\pi: \mathbf{R}(\pi)$ is a maximal circuit-free subset of $\mathbf{R}(\hat{\pi})\}$ and notice that $E \subset \omega \cap \beta$. E is called the *estimation set* to emphasize the result of the following

THEOREM 11. *$\mu \subset E$, and if $\mathbf{R}(\hat{\pi})$ is complete then each $\bar{\pi} \varepsilon \mu$ determines a unique* s.r.o.

PROOF. The second part of the theorem follows from Theorem 7 as soon as we have proved that $\mu \subset E$. From Theorem 9 and its corollary we may assume that $\mathbf{R}(\bar{\pi})$ is circuit-free and $\mathbf{R}(\hat{\pi}) \supset \mathbf{R}(\bar{\pi})$. For purposes of contradiction we assume that $\mathbf{R}(\bar{\pi})$ is not maximal, i.e., there exists a point $\tilde{\pi} \varepsilon \beta$ such that $\mathbf{R}(\bar{\pi})$ is properly contained in $\mathbf{R}(\tilde{\pi})$ and $\mathbf{R}(\tilde{\pi}) \subset \mathbf{R}(\hat{\pi})$. But $\mathbf{R}(\bar{\pi})$ is properly contained in $\mathbf{R}(\tilde{\pi})$ if and only if for all $(ij) \varepsilon I$, $\bar{\pi}_{ij} = \frac{1}{2}$ whenever $\tilde{\pi}_{ij} = \frac{1}{2}$, but for some $(kl) \varepsilon I$, $\bar{\pi}_{kl} = \frac{1}{2}$ while $\tilde{\pi}_{kl} \neq \frac{1}{2}$. Hence, since $\bar{\pi}$ and $\tilde{\pi}$ are both in β, $l(\bar{\pi}) < l(\tilde{\pi})$, which contradicts the definition of $\bar{\pi}$. Therefore $\mathbf{R}(\bar{\pi})$ is a maximal circuit-free subset of $\mathbf{R}(\hat{\pi})$ and $\mu \subset E$.

The special case most frequently encountered in the applications is $n_{ij} = 1$ for all i and j. $\mathbf{R}(\hat{\pi})$ is complete since we have either $\hat{\pi}_{ij} = 0$ or $\hat{\pi}_{ij} = 1$, depending on whether $x_j \rightarrow x_i$ or $x_i \rightarrow x_j$ in the sample. For either alternative $u_{ij}(\hat{\pi}_{ij}) = 1(-0 \log 0 - 1 \log 1) = 0$ according to the usual information theory convention. On the other hand $u_{ij}(\frac{1}{2}) = \log 2 = 1$ provided the logarithm is taken to the base two. Thus for any $\pi \varepsilon E$, $u_{ij}(\pi_{ij}) = 1$ if the sample preference between x_i and x_j is violated by the s.r.o. determined by $\mathbf{R}(\pi)$, and $u_{ij}(\pi_{ij}) = 0$ otherwise. Hence finally, for $\pi \varepsilon E$,

$$U(\pi) = \sum_{i>j} u_{ij}(\pi_{ij}) = [\text{the total number of sample preferences violated by the s.r.o. determined by } \mathbf{R}(\pi)].$$

We may summarize as follows: In the special case $n_{ij} = 1$ for all $i \neq j$, the principle of maximum likelihood applied to the probabilistic model of this section yields Slater's criterion of choosing that ranking (or rankings) which minimizes the number of violations of observed preference.

4. Conclusion. In principle, for the case where $\mathrm{R}(\hat{\pi})$ is complete, Theorems 8 and 11 provide a full solution to the problem of calculating the maximum likelihood (or minimum uncertainty) weak stochastic ranking. The general procedure is as follows: Determine the m-paths in $\mathrm{R}(\hat{\pi})$ and from these, using Theorems 8 and 9, determine the estimation set E. By direct substitution in Equation (4) find the maximum likelihood estimates μ and their corresponding weak stochastic rankings.

When m is large, straightforward computation becomes tedious and shortened methods as well as computer implementation are necessary. However, as an example will show, for a small number of items the direct procedure outlined above may be carried out by hand. Suppose that $m = 4$, $n_{ij} = 4$ throughout and $\hat{\pi}_{12} = \frac{3}{4}$, $\hat{\pi}_{13} = \frac{1}{4}$, $\hat{\pi}_{14} = \frac{1}{4}$, $\hat{\pi}_{23} = \frac{3}{4}$, $\hat{\pi}_{24} = \frac{3}{4}$, $\hat{\pi}_{34} = \frac{3}{4}$. We have, using an obvious inequality notation,

$$x_1 > x_2, \qquad x_3 > \begin{cases} x_1 \\ x_4 \end{cases},$$

$$x_2 > \begin{cases} x_3 \\ x_4 \end{cases}, \qquad x_4 > x_1 .$$

From this layout, by exhausting all possibilities, we see that there are exactly five 4-paths. These maximal paths together with all points in the estimation set appear in Table 1. By examining the last column of the table it is clear that (x_2, x_3, x_4, x_1) is the unique maximum likelihood or minimum uncertainty weak stochastic rank order.

One final comment is in order. According to the definition (2), x_i and x_j may fail to be compared in two distinct ways. First, n_{ij} may be zero in which case the sample comparison never took place. Second, x_i and x_j may have been compared but the result was a tie. Thus, failure to obtain a ranking may be due to ties and missing comparisons as well as inconsistency in the data.

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TABLE 1

Points in the estimation set
(The tabled value under the subscript ij is $4\pi_{ij}$)

4-path	Subscript						Uncertainty
	12	13	14	23	24	34	
$x_1 > x_2 > x_3 > x_4$	3	2	2	3	3	3	$2a^* + 4b^* = 21.0$
$x_2 > x_3 > x_4 > x_1$	2	1	1	3	3	3	$a + 5b = 20.2$
$x_3 > x_4 > x_1 > x_2$	3	1	1	2	2	3	$2a + 4b = 21.0$
$x_4 > x_1 > x_2 > x_3$	3	2	1	3	2	2	$3a + 3b = 21.7$
$x_3 > x_1 > x_2 > x_4$	3	1	2	2	3	3	$2a + 4b = 21.0$

* $a = 4(-\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2}) = 4$, $b = 4(-\frac{3}{4}\log\frac{3}{4} - \frac{1}{4}\log\frac{1}{4}) \simeq 3.24$.

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