EXTENDED GROUP DIVISIBLE PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS[1]

By Klaus Hinkelmann[2]

Iowa State University

1. Introduction and summary. The class of partially balanced incomplete block designs (PBIB) with more than two associate classes has not yet been explored to a great extent. In fact, only a few m-associate class PBIB's ($m > 2$) are known explicitly. One way to obtain such designs is certainly by generalizing the well-known PBIB's with two associate classes. Among these particularly the Group Divisible PBIB's lend themselves rather obviously to a generalization in this direction. Roy [8] and Raghavarao [7] have generalized the Group Divisible design of Bose and Connor [1] to m-associate class designs. The idea of another type of Group Divisible PBIB's with three associate classes, given by Vartak [11], was extended to an m-associate class design by Hinkelmann and Kempthorne [5] which they called an Extended Group Divisible PBIB (EGD/m-PBIB).

In this paper we shall investigate the EGD/m-PBIB in some detail. The definition and parameters of this design are given in Section 2. In Section 3 we shall prove the uniqueness of its association scheme. For a design given by its incidence matrix $\mathbf{N}$, the properties of the matrix $\mathbf{NN}'$ will be explored in Section 4. The eigenvalues of $\mathbf{NN}'$, its determinant and its Hasse-Minkowski invariants c_p are obtained, and non-existence theorems are given. These theorems are illustrated by examples. An example of an existent EGD/m-PBIB plan is given.

2. Definition of the EGD/m-PBIB. It is convenient to characterize the associate classes of this design by the ordered ν-plet $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_\nu)$ where γ_i is either zero or one ($i = 1, 2, \cdots, \nu$). The ν-plet $(0, 0, \cdots, 0)$ corresponds to what is sometimes called the 0th associate. Thus we have in this way $m = 2^\nu - 1$ associate classes. Denote the collection of all ν-plets γ by Γ_0 and the collection of all such ν-plets except $(0, 0, \cdots, 0)$ by Γ. We shall refer to the components in a ν-plet which are equal to one as the unity components of this ν-plet, and similarly to the components which are zero as the zero components. Then we define the EGD/m-PBIB as follows.

Definition 2.1. An incomplete block design is said to be an EGD/$(2^\nu - 1)$-PBIB if it satisfies the following conditions:

(i) The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.

Received 20 May 1963; revised 13 December 1963.

[1] This work was supported by National Science Foundation Grant NSF-G16491. Journal paper No. J-4754 of the Iowa Agricultural and Home Economics Experiment Station, Ames, Iowa. Project No. 1508.

[2] Now at Federal Research Station for Forestry and Forest Products, Schmalenbeck, Germany.

(ii) There are $N = N_1 N_2 \cdots N_\nu$ treatments denoted by $(i_1, i_2, \cdots, i_\nu)$ $(i_1 = 1, 2, \cdots, N_1; i_2 = 1, 2, \cdots, N_2; \cdots; i_\nu = 1, 2, \cdots, N_\nu)$; i_k being called the kth component of the treatment $(k = 1, 2, \cdots, \nu)$. Each treatment occurs in r blocks.

(iii) There can be established a relation of association between any two treatments satisfying the following requirements: (a) Two treatments are γth associates if they differ only in the components that correspond to the unity components of $\gamma(\gamma \varepsilon \Gamma_0)$. (b) Each treatment has exactly $n(\gamma)$ γth associates. (c) Given any two treatments that are γth associates, the number of treatments common to the γ'th associates of the first and the γ''th associates of the second is $p_\gamma(\gamma'; \gamma'')$ and is independent of the pair of treatments with which we start. Also

$$p_\gamma(\gamma'; \gamma'') = p_\gamma(\gamma''; \gamma')(\gamma, \gamma', \gamma'' \varepsilon \Gamma_0).$$

(iv) Two treatments which are γth associates occur together in exactly $\lambda(\gamma)$ blocks, with $\lambda(0\,0 \cdots 0) = r$.

One can see immediately that for any $\gamma \varepsilon \Gamma$

$$n(\gamma) = \prod_{i \varepsilon I(\gamma)} (N_i - 1) \tag{2.1}$$

where $I(\gamma)$ is the set of all $i(1 \leqq i \leqq \nu)$ for which $\gamma_i = 1$ in γ, and $n(0\,0 \cdots 0) = 1$.

The parameters of the second kind can be exhibited in 2^ν symmetric **P**-matrices of order $2^\nu \times 2^\nu$ in the following way. A balanced incomplete block design (BIB) can be considered as a special case of a PBIB with one associate class. For N treatments its parameters of the second kind according to the zeroth and first associates, respectively, are the elements of the two matrices

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & N-1 \end{bmatrix}, \mathbf{P}_1 = \begin{bmatrix} 0 & 1 \\ 1 & N-2 \end{bmatrix}. \tag{2.2}$$

To express the dependence of these matrices on the number of treatments, N, we shall write these matrices more specifically as $\mathbf{P}_0^{(N)}$ and $\mathbf{P}_1^{(N)}$. Since for $\nu = 2$, i.e., $N = N_1 N_2$, the EGD/3-PBIB is the Kronecker product of two BIB designs with N_1 and N_2 treatments, respectively, (Vartak [9]) its **P**-matrices can also be written as the Kronecker product of the corresponding **P**-matrices of the BIB designs as follows (it is convenient to write the associate classes in the standard order in which one writes down the main effects and interactions of a 2^ν factorial plan):

$$\mathbf{P}_{00} = \mathbf{P}_0^{(N_2)} \times \mathbf{P}_0^{(N_1)}, \qquad \mathbf{P}_{10} = \mathbf{P}_0^{(N_2)} \times \mathbf{P}_1^{(N_1)}$$

$$\mathbf{P}_{01} = \mathbf{P}_1^{(N_2)} \times \mathbf{P}_0^{(N_1)}, \qquad \mathbf{P}_{11} = \mathbf{P}_1^{(N_2)} \times \mathbf{P}_1^{(N_1)}$$

where $\mathbf{P}_0^{(N_1)}$, $\mathbf{P}_1^{(N_1)}$, $\mathbf{P}_0^{(N_2)}$, $\mathbf{P}_1^{(N_2)}$ are obtained from (2.2) with $N = N_1, N_2$, respectively. In general, this procedure can be used to obtain the **P**-matrices for ν factors from the **P**-matrices for $\nu - 1$ factors. We then have

$$\mathbf{P}_{\gamma_1\gamma_2\cdots\gamma_{\nu-1}0} = \mathbf{P}_0^{(N_\nu)} \times \mathbf{P}_{\gamma_1\gamma_2\cdots\gamma_{\nu-1}}$$
$$\mathbf{P}_{\gamma_1\gamma_2\cdots\gamma_{\nu-1}1} = \mathbf{P}_1^{(N_\nu)} \times \mathbf{P}_{\gamma_1\gamma_2\cdots\gamma_{\nu-1}} \tag{2.3}$$

where $\mathbf{P}_0^{(N_\nu)}$ and $\mathbf{P}_1^{(N_\nu)}$ are as defined by (2.2) with $N = N_\nu$.

The parameters of the PBIB so defined satisfy the relations

$$Nr = bk, \qquad \sum_{\gamma \varepsilon \Gamma_0} n(\gamma) = N$$
$$\sum_{\gamma \varepsilon \Gamma_0} n(\gamma)\lambda(\gamma) = rk, \qquad \sum_{\gamma' \varepsilon \Gamma_0} p_\gamma(\gamma'; \gamma'') = n(\gamma'') \tag{2.4}$$
$$n(\gamma)p_\gamma(\gamma'; \gamma'') = n(\gamma')p_{\gamma'}(\gamma; \gamma'') = n(\gamma'')p_{\gamma''}(\gamma; \gamma').$$

To illustrate the notation introduced in Definition 2.1, we consider the following example,

EXAMPLE 2.1. Suppose $N = 24$, $N_1 = 2$, $N_2 = 3$, $N_3 = 4$. The treatments are denoted by (1, 1, 1), (2, 1, 1), (1, 2, 1), (2, 2, 1), (1, 3, 1), (2, 3, 1), (1, 1, 2), $\cdots$, (2, 3, 4). The (1, 0, 1)th associates of treatment (2, 2, 1), for example, are (1, 2, 2), (1, 2, 3) and (1, 2, 4). From (2.1) we obtain $n(100) = N_1 - 1 = 1$, $n(010) = N_2 - 1 = 2$, $n(110) = (N_1 - 1)(N_2 - 1) = 2$, $n(001) = N_3 - 1 = 3$, $n(101) = (N_1 - 1)(N_3 - 1) = 3$, $n(011) = (N_2 - 1)(N_3 - 1) = 6$, $n(111) = (N_1 - 1)(N_2 - 1)(N_3 - 1) = 6$.

3. Uniqueness of the association scheme. In this section we shall prove a theorem on the uniqueness of the association scheme of the EGD/$(2^\nu - 1)$-PBIB.

THEOREM 3.1. *If the parameters of an* EGD/$(2^\nu - 1)$-PBIB *satisfy the Conditions* (2.1) *and* (2.3) *in connection with* (2.2), *then the association scheme for its treatments is uniquely determined and is given by* iii(a) *of Definition* 2.1.

PROOF. We shall prove this theorem by induction. First we observe that the association scheme is unique for $\nu = 2$, i.e. if $N = N_1N_2$. This has been shown by Vartak [11] (if we take the second associates of his design to be our (1, 0)th associates, his first associates to be our (0, 1)th associates, and his third associates to be our (1, 1)th associates, we obtain the association scheme iii(a) for $\nu = 2$). Now suppose the uniqueness has been shown for $(\nu - 1)$ factors; i.e., for $N^* = N_1N_2 \cdots N_{\nu-1}$. Then we have to show that it holds also for ν factors; i. e., for $N = N_1 N_2 \cdots N_{\nu-1} N_\nu$.

Because the proof is somewhat long, we shall first describe the steps to be followed:

1. The N treatments can be divided into N^* groups of N_ν treatments each on the basis of the $(0, \cdots, 0, 1)$th association, where there are $\nu - 1$ zeros.

2. Taking one treatment from each group, the N treatments can be divided into N_ν groups, $G_1, G_2, \cdots, G_{N_\nu}$, of N^* treatments each on the basis of the $(\gamma_1, \gamma_2, \cdots, \gamma_{\nu-1}, 0)$th association for all possible $(\gamma_1, \gamma_2, \cdots, \gamma_{\nu-1})$.

3. We show that the treatments in each $G_k (k = 1, 2, \cdots, N_\nu)$ form an EGD/$(2^{\nu-1} - 1)$-PBIB. Using the induction hypothesis, the treatments in each such group can be indexed by $(\nu - 1)$-plets according to the requirements of an EGD/$(2^{\nu-1} - 1)$-PBIB.

4. Each treatment is therefore indexed by a $(\nu - 1)$-plet and a (1)-plet, the latter indicating the group G_k it belongs to. These indices are combined into a ν-plet by adjoining the (1)-plet to the $(\nu - 1)$-plet.

5. We show that the resulting ν-plets give an indexing that satisfies the EGD/$(2^\nu - 1)$-PBIB requirements.

We are given symmetry of the association scheme. Let θ and ϕ be any two treatments which are $(0, \cdots, 0, 1)$th associates. Denote the $n(0 \cdots 0\, 1)$ $(0, \cdots, 0, 1)$th associates of θ by $\theta_{11}, \theta_{12}, \cdots, \theta_{1,n(0\cdots 0\, 1)}$ and those of ϕ by $\phi_{11}, \phi_{12}, \cdots, \phi_{1,n(0\cdots 0\, 1)}$. Then θ is one of the ϕ_{1i}'s and ϕ one of the θ_{1i}'s $(i = 1, 2, \cdots, n(0 \cdots 0\, 1))$. For definiteness let $\phi_{11} = \theta$ and $\theta_{11} = \phi$. Since by (2.3), $p_{0\cdots 0\, 1}(0 \cdots 0\, 1; 0 \cdots 0\, 1) = N_\nu - 2$, it follows that the sets θ_{1i} and ϕ_{1i} have exactly $N_\nu - 2 = n(0 \cdots 0\, 1) - 1$ treatments in common. Hence any two treatments of the set $\{\theta, \phi, \phi_{12}, \cdots, \phi_{1,n(0\cdots 0\, 1)}\}$ are $(0, \cdots, 0, 1)$th associates. This implies that we can divide the $N = N^*N_\nu$ treatments into N^* groups of N_ν treatments each. Denote these groups by

$$(3.1)\qquad \begin{array}{c} (\phi_{11}, \phi_{12}, \cdots, \phi_{1,N_\nu}) \\ (\phi_{21}, \phi_{22}, \cdots, \phi_{2,N_\nu}) \\ \cdot \quad \cdot \quad \cdot \\ (\phi_{N^*,1}, \phi_{N^*,2}, \cdots, \phi_{N^*,N_\nu}). \end{array}$$

Consider now a $(\gamma_1^*, \gamma_2^*, \cdots \gamma_{\nu-1}^*, 0)$th associate of ϕ_{11} for some $(\gamma_1^*, \gamma_2^*, \cdots, \gamma_{\nu-1}^*) \varepsilon \Gamma_{\nu-1}$, where $\Gamma_{\nu-1}$ is the set of all $(\gamma_1, \gamma_2, \cdots, \gamma_{\nu-1})$ except $(0, \cdots, 0)$. Denote this treatment by η, which is contained in some group of (3.1) except the first. By (2.3) we have $p_{\gamma_1\cdots\gamma_{\nu-1}0}(0 \cdots 0\, 1; \gamma_1 \cdots \gamma_{\nu-1}0) = 0$ for all $(\gamma_1, \cdots, \gamma_{\nu-1}) \varepsilon \Gamma_{\nu-1}$. Hence η is the only $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 0)$th associate of ϕ_{11} in this group. In general, any other group except the first contains at most one $(\gamma_1, \gamma_2, \cdots, \gamma_{\nu-1}, 0)$th associate of ϕ_{11} for any $(\gamma_1, \gamma_2, \cdots, \gamma_{\nu-1}) \varepsilon \Gamma_{\nu-1}$. Also, by (2.3), $p_{\gamma_1\cdots\gamma_{\nu-1}0}(0 \cdots 0\, 1; \gamma_1' \cdots \gamma_{\nu-1}'0) = 0$ for all $(\gamma_1, \gamma_2, \cdots, \gamma_{\nu-1})$, $(\gamma_1', \gamma_2', \cdots, \gamma_{\nu-1}') \varepsilon \Gamma_{\nu-1}$; i.e., if a group in (3.1) contains a $(\gamma_1, \cdots, \gamma_{\nu-1}, 0)$th associate of ϕ_{11} it cannot contain a $(\gamma_1', \cdots, \gamma_{\nu-1}', 0)$th associate of ϕ_{11} for $(\gamma_1, \cdots, \gamma_{\nu-1}) \neq (\gamma_1', \cdots, \gamma_{\nu-1}')$.

Now, by (2.1), $\sum n(\gamma_1 \cdots \gamma_{\nu-1}0) = N^* - 1$, where the summation is over all $(\gamma_1, \cdots, \gamma_{\nu-1}) \varepsilon \Gamma_{\nu-1}$. It follows then that the 2nd, 3rd, $\cdots$, N^*th group in (3.1) each contains exactly one $(\gamma_1, \cdots, \gamma_{\nu-1}, 0)$th associate of ϕ_{11} for some $(\gamma_1, \cdots, \gamma_{\nu-1}) \varepsilon \Gamma_{\nu-1}$. The same holds true for $\phi_{12}, \cdots, \phi_{1,N_\nu}$. Since by (2.3) $p_{0\cdots 0\, 1}(\gamma_1 \cdots \gamma_{\nu-1}0; \gamma_1' \cdots \gamma_{\nu-1}'0) = 0$ for all $(\gamma_1, \cdots, \gamma_{\nu-1})$ and $(\gamma_\nu', \cdots, \gamma_{\nu-1}') \varepsilon \Gamma_{\nu-1}$, we can divide the $N = N^*N_\nu$ treatments into N_ν disjoint groups of N^* treatments each. Denote these groups by $G_1, G_2, \cdots, G_{N_\nu}$. Each $G_k (k = 1, 2, \cdots, N_\nu)$ contains then exactly one treatment from each group in (3.1).

From the previous arguments it is clear that for all $(\gamma_1, \cdots, \gamma_{\nu-1}) \varepsilon \Gamma_{\nu-1}$ only the $(\gamma_1, \cdots, \gamma_{\nu-1}, 0)$th associates of any treatment in G_k are also contained in G_k. Further, by (2.1), $n(\gamma_1 \cdots \gamma_{\nu-1}0) = n(\gamma_1 \cdots \gamma_{\nu-1})$ and, by (2.3),

$$p_{\gamma_1 \cdots \gamma_{\nu-1} 0}(\gamma_1' \cdots \gamma_{\nu-1}' 0; \gamma_1'' \cdots \gamma_{\nu-1}'' 0) = p_{\gamma_1 \cdots \gamma_{\nu-1}}(\gamma_1' \cdots \gamma_{\nu-1}'; \gamma_1'' \cdots \gamma_{\nu-1}'')$$

for all $(\gamma_1, \cdots, \gamma_{\nu-1})$, $(\gamma_1', \cdots, \gamma_{\nu-1}')$, $(\gamma_1'', \cdots, \gamma_{\nu-1}'') \varepsilon \Gamma_{\nu-1}$. Hence we have within each $G_k(k = 1, 2, \cdots, N_\nu)$ an EGD/$(2^{\nu-1} - 1)$-PBIB with N^* treatments and, by the induction hypothesis, a unique association scheme among them. Further, the $(\gamma_1, \cdots, \gamma_{\nu-1}, 0)$th associates of a treatment in G_k, considered within the frame of an EGD/$(2^\nu - 1)$-PBIB, can be taken to be the $(\gamma_1, \cdots, \gamma_{\nu-1})$th associates of the same treatment, considered within the frame of an EGD/$(2^{\nu-1} - 1)$-PBIB.

Now denote the treatments in G_k by the $(\nu - 1)$-plets $(i_1^k, i_2^k, \cdots, i_{\nu-1}^k)$ with $i_1^k = 1^k, 2^k, \cdots, N_1^k$; $i_2^k = 1^k, 2^k, \cdots, N_2^k$; $\cdots$; $i_{\nu-1}^k = 1^k, 2^k, \cdots, N_{\nu-1}^k$ for $k = 1, 2, \cdots, N_\nu$. This indexing is done according to the association scheme for an EGD/$(2^{\nu-1} - 1)$-PBIB. This notation, however, is not unique because of possible permutations of the treatments, although the association scheme is unique. We have to show then that the labeling can be done in such a way that it is also in agreement with the association scheme iii(a) for the EGD/$(2^\nu - 1)$-PBIB.

This is obviously true for the $(\gamma_1, \gamma_2, \cdots, \gamma_{\nu-1}, 0)$th association for all $(\gamma_1, \gamma_2, \cdots, \gamma_{\nu-1}) \varepsilon \Gamma_{\nu-1}$, because of the way the treatment notation has been introduced. The agreement becomes complete if we adjoin the group superscript to the $(\nu - 1)$-plet and denote each treatment by a ν-plet $(i_1, i_2, \cdots, i_{\nu-1}, i_\nu)$, where i_ν denotes the group G_{i_ν} the treatment belongs to.

To demonstrate the agreement with the $(0, \cdots, 0, 1)$th association, we have to show that, following the rules for the EGD/$(2^{\nu-1} - 1)$-PBIB, two $(0, \cdots, 0, 1)$th associates can be indexed so that the first $(\nu - 1)$ components in their $(i_1, i_2, \cdots, i_{\nu-1}, i_\nu)$-representation are the same. Without loss of generality we can do this for the first group of (3.1). Consider now ϕ_{11} and ϕ_{12} of the first group in (3.1). Let η be a $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 0)$th associate of ϕ_{11}, and let θ be the $(0, \cdots, 0, 1)$th associate of η in G_2. We shall show then that θ is also $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 0)$th associate of ϕ_{12}. Since, by (2.3), $p_{\gamma_1 \cdots \gamma_{\nu-1} 0}(\gamma_1 \cdots \gamma_{\nu-1} 1;\ 0 \cdots 0\ 1) = N_\nu - 1$ for all $(\gamma_1, \cdots, \gamma_{\nu-1}) \varepsilon \Gamma_{\nu-1}$, it follows that all $(0, \cdots, 0, 1)$th associates of η are also $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$th associates of ϕ_{11}. This holds for all $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 0)$th associates of ϕ_{11}, $\eta = \eta_1, \eta_2, \cdots, \eta_{n(\eta_1^* \cdots \eta_{\nu-1}^* 0)}$ say. Now, from (2.1), $n(\gamma_1 \cdots \gamma_{\nu-1})(N_\nu - 1) = n(\gamma_1 \cdots \gamma_{\nu-1} 1)$ for all $(\gamma_1, \cdots, \gamma_{\nu-1}) \varepsilon \Gamma_{\nu-1}$. Hence all $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$th associates of ϕ_{11} are contained exactly in the groups of (3.1) to which $\eta_1, \eta_2, \cdots, \eta_{n(\gamma_1^* \cdots \gamma_{\nu-1}^* 0)}$ belong. Since $p_{0 \cdots 0\, 1}(\gamma_1 \cdots \gamma_{\nu-1} 1;\ \gamma_1 \cdots \gamma_{\nu-1} 0) = n(\gamma_1 \cdots \gamma_{\nu-1} 0)$ for all $(\gamma_1, \cdots, \gamma_{\nu-1}) \epsilon \Gamma_{\nu-1}$, it follows that all $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 0)$th associates of ϕ_{12} are also $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$th associates of ϕ_{11}. Hence $\theta = \theta_1, \theta_2, \cdots, \theta_{n(\gamma_1^* \cdots \gamma_{\nu-1}^* 0)}$ are $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 0)$th associates of ϕ_{12}, where $\theta_1, \theta_2, \cdots, \theta_{n(\gamma_1^* \cdots \gamma_{\nu-1}^* 0)}$ are $(0, \cdots, 0, 1)$th associates of $\eta_1, \eta_2, \cdots, \eta_{n(\gamma_1^* \cdots \gamma_{\nu-1}^* 0)}$, respectively, in G_2. Since this result holds for every treatment and every $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*) \varepsilon \Gamma_{\nu-1}$, and because of the labeling in the first group of (3.1), we can therefore index the treatments in general so that the components of treatments in the same group of (3.1) are the same except for the i_ν-component.

This then establishes the $(0, \cdots, 0, 1)$th association. Note also that this indexing does not change, of course, the $(\gamma_1, \cdots, \gamma_{\nu-1}, 0)$th association.

Finally, we consider the $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$th association $((\gamma_1^*, \cdots, \gamma_{\nu-1}^*) \varepsilon \Gamma_{\nu-1})$. To fix our thinking, let ϕ be a treatment in G_1. From previous results it follows that the $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$th associates of ϕ cannot be in G_1, since G_1 is exhausted in a unique way by the $(\gamma_1, \cdots, \gamma_{\nu-1}, 0)$th associates of ϕ. Therefore consider any other group, G_2 say, and let Δ be the $(0, \cdots, 0, 1)$th associate of ϕ in G_2. It follows from (2.3) and (2.2) that

$$p_{0\cdots 0\,1}(\gamma_1 \cdots \gamma_{\nu-1}0; \gamma_1 \cdots \gamma_{\nu-1}1) = n(\gamma_1 \cdots \gamma_{\nu-1}0)$$

for all $(\gamma_1, \cdots, \gamma_{\nu-1}) \varepsilon \Gamma_{\nu-1}$; i.e., all $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 0)$th associates of Δ are also $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$th associates of ϕ, and they are elements of G_2 only. Since this holds true for every $(0, \cdots, 0, 1)$th associates of ϕ, $\Delta = \Delta_1, \Delta_2, \cdots, \Delta_{N_\nu-1}$ say, we obtain in this way $n(\gamma_1^* \cdots \gamma_{\nu-1}^*0)\cdot(N_\nu - 1)$, $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$th associates of ϕ. But by (2.1), $n(\gamma_1 \cdots \gamma_{\nu-1}0)\cdot(N_\nu - 1) = n(\gamma_1 \cdots \gamma_{\nu-1}1)$. Hence we have obtained all $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$th associates of ϕ. Since these associates are not elements of G_1, the last component of their $(i_1, i_2, \cdots, i_\nu)$-representation is different from the last component of ϕ in its $(i_1, i_2, \cdots, i_\nu)$-representation. But these $N_\nu - 1$ groups of $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$th associates of ϕ are also $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 0)$th associates of $\Delta_1, \Delta_2, \cdots, \Delta_{N_\nu-1}$, respectively. Because of the uniqueness of the association scheme for $\nu - 1$ factors, the $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 0)$th associates of $\Delta_k (k = 1, 2, \cdots, N_\nu - 1)$ differ from Δ_k exactly in the components corresponding to the unity components of $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*)$. From the way the $(0, \cdots, 0, 1)$th associates of ϕ have been constructed, it follows then that the $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$th associates of ϕ differ from the $(i_1, i_2, \cdots, i_\nu)$-representation of ϕ exactly in the components corresponding to the unity components of $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*, 1)$. Since this holds for every $(\gamma_1^*, \cdots, \gamma_{\nu-1}^*) \varepsilon \Gamma_{\nu-1}$ this implies the association scheme iii(a).

This proves the theorem.

The relationship between EGD/$(2^{\nu-1} - 1)$-PBIB's and EGD/$(2^\nu - 1)$-PBIB's and their association schemes as established in this proof will be used frequently throughout Section 4.

4. Properties of NN′ and non-existence theorems. Let $n_{i_1 i_2 \cdots i_\nu, j} = 1$ if the treatment $(i_1, i_2, \cdots, i_\nu)$ occurs in the jth block, and $n_{i_1 i_2 \cdots i_\nu, j} = 0$ otherwise. The $N \times b$ matrix $\mathbf{N} = (n_{i_1 i_2 \cdots i_\nu, j})$ is called the incidence matrix of the design. The ν-plet $(i_1, i_2, \cdots, i_\nu)$ should be interpreted in this connection as one subscript. By generalizing the results obtained by Vartak [11], we find that, after a suitable arrangement of the treatments, the matrix $\mathbf{NN}'$ has the form

$$\mathbf{NN}' = \begin{bmatrix} \mathbf{A}_{11\cdots 1,0} & \mathbf{A}_{11\cdots 1,1} & \cdots & \mathbf{A}_{11\cdots 1,1} \\ \mathbf{A}_{11\cdots 1,1} & \mathbf{A}_{11\cdots 1,0} & \cdots & \mathbf{A}_{11\cdots 1,1} \\ \vdots & & & \vdots \\ \mathbf{A}_{11\cdots 1,1} & \mathbf{A}_{11\cdots 1,1} & \cdots & \mathbf{A}_{11\cdots 1,0} \end{bmatrix} \tag{4.1}$$

where $\mathbf{A}_{11\cdots 1,0}$ and $\mathbf{A}_{11\cdots 1,1}$ are symmetric matrices of order $N_1 N_2 \cdots N_{\nu-1}$, the structure of which will be given presently, and there are $N_\nu - 1$ matrices $\mathbf{A}_{11\cdots1,1}$ in each row and column of $\mathbf{NN}'$. The number of subscripts of these and the following $\mathbf{A}$-matrices is ν. Equation (4.1) can be written more conveniently in the form of a Kronecker product as follows

$$\mathbf{NN}' = \mathbf{I}_{N_\nu} \times (\mathbf{A}_{11\cdots1,0} - \mathbf{A}_{11\cdots1,1}) + \mathbf{J}_{N_\nu} \times \mathbf{A}_{11\cdots1,1} \tag{4.2}$$

where $\mathbf{I}_{N_\nu}$ is the identity matrix of order N_ν and $\mathbf{J}_{N_\nu}$ is a square matrix of order N_ν with all elements equal to one. For the matrices $\mathbf{A}_{11\cdots1,0}$ and $\mathbf{A}_{11\cdots1,1}$ we have the following recurrence relations

$$\mathbf{A}_{11\cdots1,0} = \mathbf{I}_{N_{\nu-1}} \times (\mathbf{A}_{11\cdots1,00} - \mathbf{A}_{11\cdots1,10}) + \mathbf{J}_{N_{\nu-1}} \times \mathbf{A}_{11\cdots1,10}$$

$$\mathbf{A}_{11\cdots1,1} = \mathbf{I}_{N_{\nu-1}} \times (\mathbf{A}_{11\cdots1,01} - \mathbf{A}_{11\cdots1,11}) + \mathbf{J}_{N_{\nu-1}} \times \mathbf{A}_{11\cdots1,11}$$

where $\mathbf{A}_{1\cdots1,00}$, $\mathbf{A}_{1\cdots1,10}$, $\mathbf{A}_{1\cdots1,01}$ and $\mathbf{A}_{1\cdots1,11}$ are symmetric matrices of order $N_1N_2 \cdots N_{\nu-2}$. These matrices can then be obtained iteratively by using the general recurrence relation

$$\begin{aligned}\mathbf{A}_{1\cdots1,\gamma_k\gamma_{k+1}\cdots\gamma_\nu} = \mathbf{I}_{N_{k-1}} &\times (\mathbf{A}_{1\cdots1,0\gamma_k\cdots\gamma_\nu} - \mathbf{A}_{1\cdots1,1\gamma_k\cdots\gamma_\nu}) \\ &+ \mathbf{J}_{N_{k-1}} \times \mathbf{A}_{1\cdots1,1\gamma_k\cdots\gamma_\nu}\end{aligned} \tag{4.3}$$

where each of the $\gamma_k, \gamma_{k+1}, \cdots, \gamma_\nu (k = 2, \cdots, \nu)$ takes on the value zero or one, and $\mathbf{A}_{\gamma_1\gamma_2\cdots\gamma_\nu} = \lambda(\gamma_1\gamma_2 \cdots \gamma_\nu)$. Equation (4.3) means that the $\mathbf{A}$-matrices have "locally" the same structure as $\mathbf{NN}'$; i.e., each $\mathbf{A}$-matrix consists of two different "elements," one type along the main diagonal and the other type in the off-diagonal positions. In fact, from a formal point of view it would be more logical to denote $\mathbf{NN}'$ by $\mathbf{A}_{1\cdots1}$.

To evaluate the eigenvalues and the determinant of $\mathbf{NN}'$, we generalize the arguments put forward by Vartak [11]. Let $\mathbf{D}_{N_k}$ be a square matrix of order N_k and of the form

$$\mathbf{D}_{N_k} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & & -2 & & \vdots \\ \vdots & & & & 0 \\ 1 & \cdots & & 1 & -(N_k - 1) \end{bmatrix} \qquad (k = 1, 2, \cdots, \nu). \tag{4.4}$$

Furthermore define a square matrix $\mathbf{H}_{N_k}$ of order $N_1N_2 \cdots N_k$ by the following recurrence relation

$$\mathbf{H}_{N_k} = \begin{bmatrix} \mathbf{H}_{N_{k-1}} & \mathbf{H}_{N_{k-1}} & \mathbf{H}_{N_{k-1}} & \cdots & \mathbf{H}_{N_{k-1}} \\ \mathbf{H}_{N_{k-1}} & -\mathbf{H}_{N_{k-1}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{H}_{N_{k-1}} & & -2\mathbf{H}_{N_{k-1}} & & \vdots \\ \vdots & & & & \mathbf{0} \\ \mathbf{H}_{N_{k-1}} & \cdots & & \mathbf{H}_{N_{k-1}} & -(N_k - 1)\mathbf{H}_{N_{k-1}} \end{bmatrix} \tag{4.5}$$

where $\mathbf{0}$ is a null matrix of order $N_1N_2 \cdots N_{k-1}$. Equation (4.5) can also be written as a Kronecker product between $\mathbf{D}_{N_k}$ and $\mathbf{H}_{N_{k-1}}$ as

$$\mathbf{H}_{N_k} = \mathbf{D}_{N_k} \times \mathbf{H}_{N_{k-1}} \tag{4.6}$$

$(k = \nu, \nu - 1, \cdots, 1)$ with $\mathbf{H}_{N_0} = 1$. Because of the relationship (4.6), $\mathbf{H}_{N_k}$ behaves almost in the same way as $\mathbf{D}_{N_k}$ does. In particular, we find

$$\mathbf{H}_{N_k}\mathbf{H}'_{N_k} = \operatorname{diag}\{N_k\mathbf{H}_{N_{k-1}}, 1\cdot 2\mathbf{H}_{N_{k-1}}, 2\cdot 3\mathbf{H}_{N_{k-1}}, \cdots, (N_k - 1)N_k\mathbf{H}_{N_{k-1}}\} \tag{4.7}$$

where $\operatorname{diag}\{\mathbf{M}_1, \mathbf{M}_2, \cdots, \mathbf{M}_m\}$ is a matrix which is block-diagonal; i.e., its diagonal elements are the square matrices $\mathbf{M}_1, \mathbf{M}_2, \cdots \mathbf{M}_m$ and its off-diagonal elements are null matrices of suitable order. For the further development we shall also obtain the determinant $|\mathbf{H}_{N_k}|$ of $\mathbf{H}_{N_k}$. Using the fact that if $\mathbf{U}$ and $\mathbf{V}$ are square matrices of order u and v, respectively, then the determinant of the Kronecker product $\mathbf{W} = \mathbf{U} \times \mathbf{V}$ is given by

$$|\mathbf{W}| = |\mathbf{U}|^v \cdot |\mathbf{V}|^u \tag{4.8}$$

and by applying (4.6) and (4.8) repeatedly, we obtain

$$|\mathbf{H}_{N_k}| = \prod_{i=1}^{k} |\mathbf{D}_{N_i}|^{N/N_i}.$$

But now

$$|\mathbf{D}_{N_i}| = (-1)^{N_i - 1} N_i!$$

so that we obtain the following result.

LEMMA 4.1. *The determinant* $|\mathbf{H}_{N_k}|$ *of the matrix* $\mathbf{H}_{N_k}$ *defined by the recurrence relation* (4.6) *is given by*

$$|\mathbf{H}_{N_k}| = \prod_{i=1}^{k} [(-1)^{N_i - 1} N_i!]^{N/N_i}. \tag{4.9}$$

Before stating and proving a theorem on the eigenvalues of $\mathbf{NN}'$, we introduce a new set of parameters θ which, later on, will be shown to be the eigenvalues of NN'.

Consider an arbitrary ν-plet $\gamma \varepsilon \Gamma_0$. Denote the set of its zero components by $\Omega_\gamma^{(0)}$ and the set of its unity components by $\Omega_\gamma^{(1)}$. Let $\sum_{1,\gamma'}$ denote the summation over all $\gamma' \varepsilon \Gamma_0$ with an even number of unity components such that $\Omega_{\gamma'}^{(0)} \supseteq \Omega_\gamma^{(0)}$; i.e., γ' has at least zero components in the same positions where γ has its zero components. Here the number zero is to be considered as an even number. Similarly, let $\sum_{2,\gamma'}$ denote the summation over all $\gamma' \varepsilon \Gamma$ with an odd number of unity components such that $\Omega_{\gamma'}^{(0)} \supseteq \Omega_\gamma^{(0)}$. Now let γ'' be a ν-plet with $\Omega_{\gamma''}^{(1)} \cap \Omega_\gamma^{(1)} = 0$; i.e., γ'' and γ have no unity components in the same positions. For any such γ'' we denote by $\sum_{1,\gamma'}^{(\gamma'')}$ and $\sum_{2,\gamma'}^{(\gamma'')}$ similar type expressions as $\sum_{1,\gamma'}$ and $\sum_{2,\gamma'}$, respectively, except that the unity components of γ'' replace the corresponding zero components of γ' after the summation according to the rules given for $\sum_{1,\gamma'}$ and $\sum_{2,\gamma'}$ has been performed.

For any $\gamma \varepsilon \Gamma_0$ then define

$$(4.10)\quad \theta(\gamma) = \sum_{1,\gamma'} \lambda(\gamma') - \sum_{2,\gamma'} \lambda(\gamma') + \sum_{\gamma'' \varepsilon \Gamma^*} n(\gamma'') \left[\sum_{1,\gamma'}^{(\gamma'')} \lambda(\gamma') - \sum_{2,\gamma'}^{(\gamma'')} \lambda(\gamma') \right]$$

where $\lambda(\gamma')$ are the parameters of an EGD/$(2^\nu - 1)$-PBIB, $n(\gamma'')$ are given by (2.1), and Γ^* is the collection of all γ'' with $\Omega_{\gamma''}^{(1)} \cap \Omega_\gamma^{(1)} = 0$.

We shall now prove the following theorem.

THEOREM 4.1. *The eigenvalues of the matrix* NN' *of an* EGD/$(2^\nu - 1)$-PBIB *are* $\theta(\gamma)(\gamma \varepsilon \Gamma_0)$, *given by* (4.10), *with their respective multiplicities* $n(\gamma)$ *given by* (2.1). *The determinant* $|NN'|$ *of* NN' *is*

$$(4.11)\quad |NN'| = \prod_{\gamma \varepsilon \Gamma_0} \theta(\gamma)^{n(\gamma)}.$$

PROOF. We shall prove this theorem by induction. For this reason we note first that the theorem is true for $\nu = 2$ as was shown by Vartak [11]. His results can be put into the form of Equations (4.10) and (4.11) if we replace his θ_0, θ_1, θ_2 and θ_3 by our $\theta_{(00)}$, $\theta_{(01)}$, $\theta_{(10)}$ and $\theta_{(11)}$, respectively. Now suppose the theorem has been proved for $\nu - 1$; i.e., for $N^* = N_1N_2 \cdots N_{\nu-1}$. We then have to show that, as a consequence, it holds also for ν; i.e., for $N = N_1N_2 \cdots N_{\nu-1}N_\nu$.

To determine the roots of the determinantal equation in θ

$$|\mathbf{NN}' - \theta \mathbf{I}_N| = 0$$

we consider the expression

$$(4.12)\quad \mathbf{H}_{N_\nu}[\mathbf{NN}' - \theta \mathbf{I}_N]\mathbf{H}'_{N_\nu}.$$

Applying (4.2) we can rewrite (4.12) as

$$(4.13)\quad \begin{aligned} \mathbf{H}_{N_\nu}[\mathbf{NN}' - \theta \mathbf{I}_N]\mathbf{H}'_{N_\nu} = \mathbf{H}_{N_\nu}\{\mathbf{I}_{N_\nu} \times [(\mathbf{A}_{1\cdots1,0} - \theta \mathbf{I}_{N/N_\nu}) \\ - \mathbf{A}_{1\cdots1,1}] + \mathbf{J}_{N_\nu} \times \mathbf{A}_{1\cdots1,1}\}\mathbf{H}'_{N_\nu} \end{aligned}$$

It is easy to verify that for any $k = 1, 2, \cdots, \nu$ we have

$$(4.14)\quad \begin{aligned} &\mathbf{H}_{N_k}\{\mathbf{I}_{N_k} \times [(\mathbf{A}_{1\cdots1,0\gamma_{k+1}\cdots\gamma_\nu} - \theta \mathbf{I}) - \mathbf{A}_{1\cdots1,1\gamma_{k+1}\cdots\gamma_\nu}] \\ &\quad + \mathbf{J}_{N_k} \times \mathbf{A}_{1\cdots1,1\gamma_{k+1}\cdots\gamma_\nu}\}\mathbf{H}'_{N_k} = \text{diag}\,\{N_k\mathbf{H}_{N_{k-1}}[(\mathbf{A}_{1\cdots1,0\gamma_{k+1}\cdots\gamma_\nu} - \theta \mathbf{I}) \\ &\quad + (N_k - 1)\mathbf{A}_{1\cdots1,1\gamma_{k+1}\cdots\gamma_\nu}] \cdot \mathbf{H}'_{N_{k-1}},\ 1\cdot2\mathbf{H}_{N_{k-1}}[(\mathbf{A}_{1\cdots1,0\gamma_{k+1}\cdots\gamma_\nu} - \theta \mathbf{I}) \\ &\quad - \mathbf{A}_{1\cdots1,1\gamma_{k+1}\cdots\gamma_\nu}]\mathbf{H}'_{N_{k-1}},\ 2\cdot3\mathbf{H}_{N_{k-1}}[(\mathbf{A}_{1\cdots1,0\gamma_{k+1}\cdots\gamma_\nu} - \theta \mathbf{I}) \\ &- \mathbf{A}_{1,\cdots1,1\gamma_{k+1}\cdots\gamma_\nu}]\mathbf{H}'_{N_{k-1}},\ \cdots,\ (N_k - 1)N_k\mathbf{H}_{N_{k-1}}[(\mathbf{A}_{1\cdots1,0\gamma_{k+1}\cdots\gamma_\nu} - \theta \mathbf{I}) \\ &\quad - \mathbf{A}_{1\cdots1,1\gamma_{k+1}\cdots\gamma_\nu}]\mathbf{H}'_{N_{k-1}}\} \end{aligned}$$

where $\mathbf{I}$ is the identity matrix of order $N_1N_2 \cdots N_{k-1}$. In the light of (4.14) with $k = \nu$, we can rewrite (4.13) and hence (4.12) as

$$\mathbf{H}_{N_\nu}[\mathbf{NN}' - \theta\mathbf{I}_N]\mathbf{H}'_{N_\nu} = N_\nu \mathbf{H}_{N_{\nu-1}}[(\mathbf{A}_{1\cdots 1,0} + (N_\nu - 1)\mathbf{A}_{1\cdots 1,1}) \tag{4.15}$$
$$- \theta\mathbf{I}_{N/N_\nu}]\mathbf{H}'_{N_{\nu-1}} \dotplus \operatorname{diag}[1\cdot 2, 2\cdot 3, \cdots, (N_\nu - 1)N_\nu]$$
$$\times \mathbf{H}_{N_{\nu-1}}[(\mathbf{A}_{1\cdots 1,0} - \mathbf{A}_{1\cdots 1,1}) - \theta I_{N/N_\nu}]\mathbf{H}'_{N_{\nu-1}}$$

where $\dotplus$ denotes the direct sum of matrices.

We have noted earlier that the **A**-matrices are of a similar structure as $\mathbf{NN}'$. In fact, $\mathbf{A}_{1\cdots 1,0}$ is essentially the $\mathbf{NN}'$-matrix for $\nu - 1$ factors $N_1, N_2, \cdots, N_{\nu-1}$ if one only ignores the last component of the λ's after it has been reduced to its final form by applying (4.3) repeatedly. The same is true for $\mathbf{A}_{1\cdots 1,1}$. Hence we can apply Theorem 4.1 to $(\mathbf{A}_{1\cdots 1,0} + (N_\nu - 1)\mathbf{A}_{1\cdots 1,1})$ and $(\mathbf{A}_{1\cdots 1,0} - \mathbf{A}_{1\cdots 1,1})$ with the only provision that we have to substitute $\theta(\gamma_1 \cdots \gamma_{\nu-1})$ by $\theta(\gamma_1 \cdots \gamma_{\nu-1}0)$ for the first term, and $\theta(\gamma_1 \cdots \gamma_{\nu-1}1)$ for the other term for all $(\gamma_1, \cdots, \gamma_{\nu-1}) \varepsilon \Gamma_{0,\nu-1}$, where $\Gamma_{0,\nu-1}$ is the set of all $(\nu - 1)$-plets $(\gamma_1, \gamma_2, \cdots, \gamma_{\nu-1})$. This can be seen immediately from (4.10) and the fact that, by (4.3), $\mathbf{A}_{1\cdots 1,0}$ reduces to a matrix with elements λ whose νth component is zero, and $\mathbf{A}_{1\cdots 1,1}$ reduces to a matrix whose elements differ from those of $\mathbf{A}_{1\cdots 1,0}$ only in that their νth component is one.

Taking the determinant on both sides of (4.15) we obtain

$$|\mathbf{H}_{N_\nu}[\mathbf{NN}' - \theta\mathbf{I}_N]\mathbf{H}'_{N_\nu}| = N_\nu^{N/N_\nu}|\mathbf{H}_{N_{\nu-1}}|^2$$
$$\cdot \prod_{\Gamma_{0,\nu-1}} (\theta(\gamma_1 \cdots \gamma_{\nu-1}0) - \theta)^{n(\gamma_1 \cdots \gamma_{\nu-1})}[(N_{\nu-1} - 1)!N_\nu!]^{N/N_\nu}$$
$$\cdot|\mathbf{H}_{N_{\nu-1}}|^{2(N_\nu-1)}[\prod_{\Gamma_{0,\nu-1}} (\theta(\gamma_1 \cdots \gamma_{\nu-1}1) - \theta)^{n(\gamma_1 \cdots \gamma_{\nu-1})}]^{N_\nu-1} \tag{4.16}$$
$$= |\mathbf{H}_{N_\nu}|^2 \prod_{\Gamma_{0,\nu-1}} (\theta(\gamma_1 \cdots \gamma_{\nu-1}0) - \theta)^{n(\gamma_1 \cdots \gamma_{\nu-1}0)}$$
$$\cdot \prod_{\Gamma_{0,\nu-1}} (\theta(\gamma_1 \cdots \gamma_{\nu-1}1) - \theta)^{n(\gamma_1 \cdots \gamma_\nu 1)}$$

by using (4.9) and noting that $n(\gamma_1 \cdots \gamma_{\nu-1})$ for $\nu - 1$ factors is the same as $n(\gamma_1 \cdots \gamma_{\nu-1}0)$ for ν factors, and $n(\gamma_1 \cdots \gamma_{\nu-1})(N_\nu - 1) = n(\gamma_1 \cdots \gamma_{\nu-1}1)$. Hence (4.16) yields

$$|\mathbf{NN}' - \theta\mathbf{I}_N| = \prod_{\gamma\varepsilon\Gamma_0} (\theta(\gamma) - \theta)^{n(\gamma)}.$$

This completes the proof.

Since $\mathbf{NN}'$ is positive indefinite Theorem 4.1 leads immediately to the following necessary condition for the existence of an EGD/$(2^\nu - 1)$-PBIB.

THEOREM 4.2. *A necessary condition for the existence of an* EGD/$(2^\nu - 1)$-PBIB *is that*

$$\theta(\gamma) \geqq 0 \tag{4.17}$$

for all $\gamma \varepsilon \Gamma_0$, *where* $\theta(\gamma)$ *is given by* (4.10).

To illustrate Theorem 4.2 consider the following example.

EXAMPLE 4.1. Suppose $N = 18$ and $N_1 = 2$, $N_2 = N_3 = 3$. Let $b = 9$, $k = 4$,

$r = \lambda(000) = 2$, and $\lambda(100) = 2$, $\lambda(010) = 1$, $\lambda(110) = 0$, $\lambda(001) = 1$, $\lambda(101) = \lambda(011) = \lambda(111) = 0$. Then $\theta(000) = 8$, $\theta(100) = 4$, $\theta(010) = 5$, $\theta(110) = 1$, $\theta(001) = 5$, $\theta(101) = 1$, $\theta(011) = 2$, $\theta(111) = -2$. Because of $\theta(111) = -2$, Equation (4.17) is not satistied. Hence the PBIB does not exist.

A necessary condition for the existence of a symmetrical EDG/$(2^{\nu} - 1)$-PBIB with $|\mathbf{N}| \neq 0$ can be derived from the fact that if $|\mathbf{N}| \neq 0$ then $|\mathbf{NN}'| = |\mathbf{N}|^2$ is a perfect square.

THEOREM 4.3. *A necessary condition for the existence of a symmetrical* EGD/$(2^{\nu} - 1)$-PBIB *with a non-singular incidence matrix* $\mathbf{N}$ *is that* $\prod_{\gamma\varepsilon\Gamma_0} \theta(\gamma)^{n(\gamma)}$ *is a perfect square, where* $\theta(\gamma)$ *is given by* (4.10), *and* $n(\gamma)$ *is given by* (2.1).

To illustrate Theorem 4.3 consider the following example.

EXAMPLE 4.2. Suppose we have a symmetrical PBIB with $N = 24$, $N_1 = 2$, $N_2 = 3$, $N_3 = 4$. Then $b = 24$. Let $r = k = 4$, and $\lambda(100) = 2$, $\lambda(010) = 1 = \lambda(110) = \lambda(001) = \lambda(101)$, $\lambda(011) = 0 = \lambda(111)$. The determinant $|\mathbf{NN}'|$ is then given by

$$|NN'| = 16\cdot 2\cdot 10^2\cdot 2^2\cdot 8^3\cdot 2^3\cdot 2^6\cdot 2^6$$

which is not a perfect square in contradiction to Theorem 4.3. Hence the PBIB does not exist.

To prove next a theorem on the Hasse-Minkowski invariant $c_p(\mathbf{NN}')$ of NN'[2] we note first the following results for the c_p invariants of the direct sum and the Kronecker product of matrices (cf. [1] and [10] respectively).

If $\mathbf{P}$ and $\mathbf{Q}$ are symmetric matrices with rational elements whose c_p invariants exist and if $\mathbf{U} = \mathbf{P} \dotplus \mathbf{Q}$ is the direct sum of $\mathbf{P}$ and $\mathbf{Q}$, and $\mathbf{V} = \mathbf{P} \times \mathbf{Q}$ the Kronecker product of $\mathbf{P}$ and $\mathbf{Q}$, then

$$c_p(\mathbf{U}) = (-1, -1)_p c_p(\mathbf{P}) c_p(\mathbf{Q}) (|\mathbf{P}|, |\mathbf{Q}|)_p \tag{4.18}$$

and

$$\begin{aligned} c_p(\mathbf{V}) = (-1, -1)_p^{m+n-1} [c_p(\mathbf{P})]^n [c_p(\mathbf{Q})]^m \\ \cdot (|\mathbf{P}|, -1)_p^{n(n-1)/2} (|\mathbf{Q}|, -1)_p^{m(m-1)/2} (|\mathbf{P}|, |\mathbf{Q}|)_p^{mn-1} \end{aligned} \tag{4.19}$$

where m and n are the orders of $\mathbf{P}$ and $\mathbf{Q}$ respectively, and where $(a, b)_p$ is the Hilbert norm residue symbol of the non-zero rational numbers a and b [4]. If $\mathbf{Q}$ is a non-zero rational number, λ say, then (4.19) reduces to

$$c_p(\lambda\mathbf{P}) = c_p(\mathbf{P}) (\lambda, -1)_p^{m(m-1)/2} (\lambda, |\mathbf{P}|)_p^{m-1}. \tag{4.20}$$

For future reference we now state the following result.

LEMMA 4.2. *If* $\mathbf{L}$, $\mathbf{D}$, *and* $\mathbf{R}$ *are symmetric matrices with rational elements and of order* l, d *and* r, *respectively, whose* c_p *invariants exist, and if* ρ *is a non-zero rational number, then the* c_p *invariant of the matrix*

$$\mathbf{W} = \rho\mathbf{L} \dotplus \mathbf{D} \times \mathbf{R} \tag{4.21}$$

is given by

$$c_p(\mathbf{W}) = (-1, -1)^d c_p(\mathbf{L})[c_p(\mathbf{R})]^d(\rho, -1)_p^{l(l+1)/2}(|\mathbf{D}|, -1)_p^{r(r-1)/2}$$
$$(4.22) \qquad \cdot(|\mathbf{R}|, -1)_p^{d(d-1)/2}(|\mathbf{D}|, |\mathbf{R}|)_p^{dr-1}(|\mathbf{L}|, |\mathbf{D}|)_p^{r}(|\mathbf{L}|, |\mathbf{R}|)_p^{d}$$
$$\cdot(\rho, |\mathbf{L}|)_p^{l-1}(\rho, |\mathbf{D}|)_p^{lr}(\rho, |\mathbf{R}|)_p^{ld}.$$

The proof follows immediately by using (4.18), (4.19), (4.20) and the well-known properties of the Hilbert norm residue symbol (cf. [3], [6]).

Before stating the main theorem on the c_p invariant of $\mathbf{NN}'$, we introduce the following notation. For any ν-plet $\gamma \varepsilon \Gamma$ let ϵ be the ν-plet with $\epsilon_i = 1 - \gamma_i (i = 1, 2, \cdots, \nu)$. Furthermore, for any $\gamma \varepsilon \Gamma$ let

$$(4.23) \qquad m(\gamma) = \prod_{i \varepsilon I(\gamma)} N_i$$

where $I(\gamma)$ was defined in Section 2, and let $m(0) = 1$, where $0 = (0 \cdots 0)$. Finally, let $k(\gamma)$ be the number of unity components in γ for any $\gamma \varepsilon \Gamma$.

We are now in a position to prove the following theorem.

THEOREM 4.4. *The Hasse-Minkowski invariant* $c_p(\mathbf{NN}')$ *of the matrix* $\mathbf{NN}'$ *of an* EGD/$(2^\nu - 1)$-PBIB *whose eigenvalues* $\theta(\gamma)$, *as given by* (4.14), *are positive for all* $\gamma \varepsilon \Gamma_0$ *is of the form*

$$c_p(\mathbf{NN}') = (-1, -1)_p(\theta(0), -N)_p \prod_{\gamma \varepsilon \Gamma} (m(\epsilon)\theta(0), \theta(\gamma))^{n(\gamma)}$$
$$(4.24) \qquad \cdot \prod_{\gamma \varepsilon \Gamma} (\theta(\gamma), -1)_p^{\frac{1}{2}n(\gamma)[\Sigma^* N_i - 2(k(\gamma)-1)]}$$
$$\cdot \prod_{\gamma \varepsilon \Gamma} \prod_{i \varepsilon I(\gamma)} (\theta(\gamma), N_i)_p^{n(\gamma)/(N_i - 1)} \cdot \prod_{\gamma \varepsilon \Gamma} [\textstyle\prod^*(\theta(\gamma'), \theta(\gamma''))_p]^{n(\gamma)}$$

where $n(\gamma)$ *and* $m(\gamma)$ *are defined by* (2.1) *and* (4.23) *respectively,* $k(\gamma)$ *and* $I(\gamma)$ *are as stated previously,* $\prod^*$ *denotes the product over all pairs of* ν*-plets* γ', γ'' *with* $\gamma' \neq \gamma''$ *such that* $\Omega_{\gamma'}^{(1)} \cup \Omega_{\gamma''}^{(1)} = \Omega_\gamma^{(1)}$, *and* $\sum^*$ *denotes the summation over all* $i \varepsilon I(\gamma)$.

PROOF. We shall prove this theorem by induction. The theorem is true for $\nu = 2$. This has been shown by Vartak [11], and his result can be put into the form of (4.24) if we substitute his θ_0, θ_1, θ_2 and θ_3 by our $\theta(00)$, $\theta(01)$, $\theta(10)$ and $\theta(11)$, respectively. Now suppose the theorem is true for $\nu - 1$; i.e., for $N^* = N_1 N_2 \cdots N_{\nu-1}$. We then have to show that it holds also for ν; i.e., for $N = N_1 N_2 \cdots N_{\nu-1} N_\nu$.

Let $\mathbf{NN}'$ be the matrix under consideration for the EGD/$(2^\nu - 1)$-PBIB. We note first that $\mathbf{H}_{N_\nu}\mathbf{NN}'\mathbf{H}_{N_\nu}'$ is rationally congruent to $\mathbf{NN}'$ since $\mathbf{H}_{N_\nu}$ is non-singular as a consequence of (4.9). Then by the Hasse-Minkowski theorem [2], we have $c_p(\mathbf{NN}') = c_p(\mathbf{H}_{N_\nu}\mathbf{NN}'\mathbf{H}_{N_\nu}')$. From (4.13) in connection with (4.14) it follows that $\mathbf{H}_{N_\nu}\mathbf{NN}'\mathbf{H}_{N_\nu}'$ can be written as

$$(4.25) \qquad \begin{aligned} \mathbf{H}_{N_\nu}\mathbf{NN}'\mathbf{H}_{N_\nu}' &= N_\nu \mathbf{H}_{N_{\nu-1}}[\mathbf{A}_{1\cdots1,0} + (N_\nu - 1)\mathbf{A}_{1\cdots1,1}]\mathbf{H}_{N_{\nu-1}}' \\ &\dotplus \operatorname{diag}[1\cdot2, 2\cdot3, \cdots, (N_\nu - 1)N_\nu] \times \mathbf{H}_{N_{\nu-1}}[\mathbf{A}_{1\cdots1,0} - \mathbf{A}_{1\cdots1,1}]\mathbf{H}_{N_{\nu-1}}'. \end{aligned}$$

We notice that (4.25) is of the form (4.21) with

$$\mathbf{L} = \mathbf{H}_{N_{\nu-1}}[\mathbf{A}_{1\cdots1,0} + (N_\nu - 1)\mathbf{A}_{1\cdots1,1}]\mathbf{H}'_{N_{\nu-1}},$$
$$\mathbf{D} = \text{diag}\,[1\cdot2, 2\cdot3, \cdots, (N_\nu - 1)N_\nu],$$
$$\mathbf{R} = \mathbf{H}_{N_{\nu-1}}[\mathbf{A}_{1\cdots1,0} - \mathbf{A}_{1\cdots1,1}]\mathbf{H}'_{N_{\nu-1}}$$

with $l = r = N_1 \cdots N_{\nu-1}$, $d = N_\nu - 1$, and $\rho = N_\nu$.

We have mentioned earlier that the **A**-matrices behave in the same way as the matrix $\mathbf{NN}'$, and so therefore does any linear combination of **A**-matrices. Since **L** and **R** are of the order $N_1N_2 \cdots N_{\nu-1}$, it follows then that (4.24) is true for **L** and **R**. However, by applying (4.24) to **L** and **R** we have to modify the formula by substituting $\theta(\gamma_1 \cdots \gamma_{\nu-1}0)$ for $\theta(\gamma_1 \cdots \gamma_{\nu-1})$ in **L**, and $\theta(\gamma_1 \cdots \gamma_{\nu-1}1)$ for $\theta(\gamma_1 \cdots \gamma_{\nu-1})$ in **R** for every $(\gamma_1, \gamma_2, \cdots, \gamma_{\nu-1}) \,\varepsilon\, \Gamma_{0,\nu-1}$. This can be seen from the form of **L** and **R** and the derivation for the general expression of the eigenvalues. We now can apply (4.22) to (4.25). In doing so we note again that $n(\gamma_1 \cdots \gamma_{\nu-1})$ for EGD/$(2^\nu - 1)$-PBIB is the same as $n(\gamma_1 \cdots \gamma_{\nu-1}0)$ for an EGD/$(2^\nu - 1)$-PBIB with the same factors $N_1, N_2, \cdots, N_{\nu-1}$. The same is true for the parameters m and k. Also, $n(\gamma_1 \cdots \gamma_{\nu-1})(N_\nu - 1) = n(\gamma_1 \cdots \gamma_{\nu-1}1)$. We then obtain (omitting the p-subscript for convenience)

$$\begin{aligned}
c_p(NN') &= (-1, -1)(\theta(0 \cdots 0), -N_1 \cdots N_{\nu-1}) \\
&\cdot \prod_{\Gamma_{\nu-1}} (m(\epsilon_1 \cdots \epsilon_{\nu-1}0)\theta(0 \cdots 0), \theta(\gamma_1 \cdots \gamma_{\nu-1}0))^{n(\gamma_1\cdots\gamma_{\nu-1}0)} \\
&\cdot \prod_{\Gamma_{\nu-1}} (\theta(\gamma_1 \cdots \gamma_{\nu-1}0), -1)^{\frac{1}{2}n(\gamma_1\cdots\gamma_{\nu-1}0)[\Sigma^{**}N_i - 2(k(\gamma_1\cdots\gamma_{\nu-1}0)-1)]} \\
&\cdot \prod_{\Gamma_{\nu-1}} \prod_{i\epsilon I(\gamma_1\cdots\gamma_{\nu-1}0)} (\theta(\gamma_1 \cdots \gamma_{\nu-1}0), N_i)^{n(\gamma_1\cdots\gamma_{\nu-1}0)/(N_i-1)} \\
&\cdot \prod_{\Gamma_{\nu-1}} [\textstyle\prod^*(\theta(\gamma_1' \cdots \gamma_{\nu-1}'0), \theta(\gamma_1'' \cdots \gamma_{\nu-1}''0))]^{n(\gamma_1\cdots\gamma_{\nu-1}0)} \\
&\cdot \prod_{\Gamma_{\nu-1}} (m(\epsilon_1 \cdots \epsilon_{\nu-1}0)\theta(0 \cdots 0), \theta(\gamma_1 \cdots \gamma_{\nu-1}1))^{n(\gamma_1\cdots\gamma_{\nu-1}1)} \\
&\cdot \prod_{\Gamma_{\nu-1}} (\theta(\gamma_1 \cdots \gamma_{\nu-1}1), -1)^{\frac{1}{2}n(\gamma_1\cdots\gamma_{\nu-1}1)[\Sigma^{**}N_i - 2(k(\gamma_1\cdots\gamma_{\nu-1}0)-1)]} \qquad (4.26)\\
&\cdot \prod_{\Gamma_{\nu-1}} \prod_{i\epsilon I(\gamma_1\cdots\gamma_{\nu-1}0)} (\theta(\gamma_1 \cdots \gamma_{\nu-1}1), N_i)^{n(\gamma_1\cdots\gamma_{\nu-1}1)/(N_i-1)} \\
&\cdot \prod_{\Gamma_{\nu-1}} [\textstyle\prod^*(\theta(\gamma_1' \cdots \gamma_{\nu-1}'1), \theta(\gamma_1'' \cdots \gamma_{\nu-1}''1))]^{n(\gamma_1\cdots\gamma_{\nu-1}1)} \\
&\cdot (\prod_{\Gamma_{0,\nu-1}} \theta(\gamma_1 \cdots \gamma_{\nu-1}1)^{n(\gamma_1\cdots\gamma_{\nu-1}0)}, -1)^{(N_\nu-1)(N_\nu-2)/2} \\
&\cdot (N_\nu, \prod_{\Gamma_{0,\nu-1}} \theta(\gamma_1 \cdots \gamma_{\nu-1}1)^{n(\gamma_1\cdots\gamma_{\nu-1}0)}) \\
&\cdot (\prod_{\Gamma_{0,\nu-1}} \theta(\gamma_1 \cdots \gamma_{\nu-1}0)^{n(\gamma_1\cdots\gamma_{\nu-1}0)}, N_\nu) \\
&\cdot (\prod_{\Gamma_{0,\nu-1}} \theta(\gamma_1 \cdots \gamma_{\nu-1}0)^{n(\gamma_1\cdots\gamma_{\nu-1}0)}, \prod_{\Gamma_{0,\nu-1}} \theta(\gamma_1 \cdots \gamma_{\nu-1}1)^{n(\gamma_1\cdots\gamma_{\nu-1}0)})^{N_\nu-1}
\end{aligned}$$

where $\sum^{**}$ denotes the summation over all $i \varepsilon I(\gamma_1 \cdots \gamma_{\nu-1}0)$, and $\prod^*$ has the same meaning as in Equation (4.24). Here we have used various properties of the Hilbert norm residue symbol and the fact that

$$(N_\nu, -1)^{l(l+1)/2}(N_\nu, -1)^{r(r-1)/2}(N_\nu, N_\nu)^{lr} = +1.$$

To obtain $|\mathbf{L}|$ and $|\mathbf{R}|$ we have used (4.11) with the appropriate substitutions. One can then verify that, after some manipulations, (4.26) reduces to (4.24). This completes the proof.

Note that the actual range of the last product in (4.24) is $\Gamma' = \{\Gamma$ minus all γ with only one unity component$\}$, because of the conditions imposed on γ' and γ'', and because of $\theta(0) = r^2$ and $(r^2, \theta)_p = +1$.

We now observe that

$$\mathbf{NN}' = \mathbf{NI}_N\mathbf{N}',$$

i.e., $\mathbf{NN}'$ is rationally congruent to $\mathbf{I}_N$ if $\mathbf{N}$ is non-singular. Since $c_p(\mathbf{I}_N) = +1$ for all odd primes p, this leads to the following necessary condition for the existence of symmetrical EGD/$(2^\nu - 1)$-PBIB's with non-singular incidence matrix $\mathbf{N}$.

THEOREM 4.5. *A necessary condition for the existence of a symmetrical* EDG/$(2^\nu - 1)$-PBIB *with* $|\mathbf{N}| \neq 0$ *is that* $c_p(\mathbf{NN}') = +1$ *for all odd primes* p, *where* $c_p(\mathbf{NN}')$ *is the Hasse-Minkowski invariant of* $\mathbf{NN}'$ *given by* (4.24).

To illustrate Theorem 4.5 consider the following example.

EXAMPLE 4.3. Suppose we want to construct a symmetrical EGD/7-PBIB with $N = 45, N_1 = N_2 = 3, N_3 = 5$. Let $k = r = 9$, and $\lambda(100) = 5$, $\lambda(010) = 5$, $\lambda(110) = 3$, $\lambda(001) = 4$, $\lambda(101) = 2$, $\lambda(011) = 1$, $\lambda(111) = 0$. From (4.10) we obtain $\theta(000) = 81$, $\theta(100) = 24$, $\theta(010) = 36$, $\theta(110) = 6$, $\theta(001) = 31$, $\theta(101) = 4$, $\theta(011) = 1$, $\theta(111) = 1$. Applying (4.24) we find $c_p(\mathbf{NN}') = (6, -3)_p$. For $p = 3$ this becomes

$$c_3(\mathbf{NN}') = (2, -3)_3(3, -3)_3 = (2, -1)_3(2, 3)_3 = (2/3) = -1,$$

where (2/3) is the Legendre symbol. Therefore the condition of Theorem (4.5) is not fulfilled. Hence the PBIB does not exist.

5. Example of an existent EGD/7-PBIB. Suppose $N = 24$ with $N_1 = 2$, $N_2 = 3$, $N_3 = 4$. The treatments are denoted by

(111)	(112)	(113)	(114)
(211)	(212)	(213)	(214)
(121)	(122)	(123)	(124)
(221)	(222)	(223)	(224)
(131)	(132)	(133)	(134)
(231)	(232)	(233)	(234)

Take $k = 3$, $b = 32$, $r = 4 = \lambda(000)$, $\lambda(100) = 0$, $\lambda(010) = 1$, $\lambda(110) = 0$, $\lambda(001) = 1 = \lambda(101)$, $\lambda(011) = 0 = \lambda(111)$. The plan of the design is then as follows:

[(111),	(112),	(213)]	[(131),	(132),	(233)]
[(112),	(214),	(113)]	[(132),	(234),	(133)]
[(113),	(111),	(214)]	[(133),	(131),	(234)]
[(114),	(212),	(111)]	[(134),	(232),	(131)]
[(211),	(114),	(112)]	[(231),	(134),	(132)]
[(212),	(113),	(114)]	[(232),	(133),	(134)]
[(213),	(211),	(212)]	[(233),	(231),	(232)]
[(214),	(213),	(211)]	[(234),	(233),	(231)]
[(121),	(122),	(223)]	[(111),	(121),	(131)]
[(122),	(224),	(123)]	[(112),	(122),	(132)]
[(123),	(121),	(224)]	[(113),	(123),	(133)]
[(124),	(222),	(121)]	[(114),	(124),	(134)]
[(221),	(124),	(122)]	[(211),	(221),	(231)]
[(222),	(123),	(124)]	[(212),	(222),	(232)]
[(223),	(221),	(222)]	[(213),	(223),	(233)]
[(224),	(223),	(221)]	[(214),	(224),	(234)]

6. Acknowledgments. I would like to thank the referee for pointing out an inadequacy in the proof of Theorem 3.1 in the original manuscript. Also, I am grateful to Professor O. Kempthorne for his interest in this work.

REFERENCES

[1] Bose, R. C. and Connor, W. S. (1952). Combinatorial properties of group divisible incomplete block designs. *Ann. Math. Statist.* **23** 367–383.

[2] Hasse, H. (1923). Über die Äquivalenz quadratischer Formen im Körper der rationalen Zahlen. *J. Reine Angew. Math.* **152** 205–224.

[3] Hasse, H. (1949). *Zahlentheorie.* Akademie-Verlag, Berlin.

[4] Hilbert, D. (1932). Die Theorie der algebraischen Zahlkörper. *Gesammelte Abhandlungen,* **1** 63–362. Springer, Berlin.

[5] Hinkelmann, K. and Kempthorne, O. (1963). Two classes of group divisible partial diallel crosses. *Biometrika* **50** 281–291.

[6] Jones, B. W. (1950). *The Arithmetic Theory of Quadratic Forms.* Wiley, New York.

[7] Raghavarao, D. (1960). A generalization of group divisible designs. *Ann. Math. Statist.* **31** 756–771.

[8] Roy, P. M. (1953–1954). Hierarchical group divisible incomplete block designs with m associate classes. *Science and Culture* **19** 210–211.

[9] Vartak, M. N. (1955). On an application of Kronecker product of matrices to statistical designs. *Ann. Math. Statist.* **26** 420–438.

[10] Vartak, M. N. (1958). On the Hasse-Minkowski invariant of the Kronecker product of matrices. *Canad. J. Math.* **10** 66–72.

[11] Vartak, M. N. (1959). The non-existence of certain PBIB designs. *Ann. Math. Statist.* **30** 1051–1060.