# APPLICATIONS OF THE CALCULUS FOR FACTORIAL ARRANGEMENTS II: TWO WAY ELIMINATION OF HETEROGENEITY[1]

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**1. Introduction.** This paper is the second in a series of papers which applies the special notations and operations developed by Kurkjian and Zelen (1962) to problems in experimental design. This body of notation and operations has been applied previously to block and direct product designs by Kurkjian and Zelen (1963).

In this paper we consider general methods for both the analysis and construction of designs which can be used for two-way elimination of heterogeneity. General methods of analysis are derived in Section 2 and applied to balanced incomplete and group divisible designs in Section 3. Section 4 discusses new methods of constructing designs and their analysis.

**2. The analysis of designs for two-way elimination of heterogeneity.** Consider a block design with $v$ treatments in $b$ blocks such that each block contains $k$ experimental units and every treatment is replicated $r$ times. For visualization purposes, it is convenient to regard the design as an array with $b$ columns and $k$ rows where the entries in the array consist of the treatment numbers. Define the matrices $N = (n_{ij})$ and $\tilde{N} = (\tilde{n}_{ih})$ to be of dimension $v \times b$ and $v \times k$ respectively where $n_{ij}$ = number of times treatment $i$ occurs in block $j$ and $\tilde{n}_{ih}$ = number of times treatment $i$ occurs in row $h$. The matrix $N$ is the incidence matrix for the design which relates the treatments to the (columns) blocks; the matrix $\tilde{N}$ similarly relates the treatments to the rows. In this paper we shall call $N$ the column incidence matrix and $\tilde{N}$ will be termed the row incidence matrix. Let $Y_{jh}(j = 1, 2, \cdots, b; h = 1, 2, \cdots, k)$ denote the measurement made in the $j$th block and $h$th row. When treatment $i$ is made in block $j$ and row $h$, the random variable $Y_{jh}$ will be assumed to have the expected value

$$E\{Y_{jh}\} = \mu + t_i + b_j + r_h \tag{2.1}$$

where $\mu$ is a constant, and $t_i$, $b_j$, $r_h$ are the (fixed) effects associated respectively with the treatments, blocks, and rows. These parameters satisfy the restraints $\sum_{i=1}^{v} t_i = \sum_{j=1}^{b} b_j = \sum_{h=1}^{k} r_h = 0$. Furthermore, we shall assume that the $\{Y_{jh}\}$ are uncorrelated with common variance $\sigma^2$.

When analyzing such an experiment design, interest is usually focused on estimating the treatment effects $t_i$. The solution for the estimates of the treatment effects can be obtained by solving a set of $v$ simultaneous linear equations

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which depend on $N$, $\tilde{N}$, and the adjusted treatment totals which are a function of the observations. The adjusted treatment totals are obtained by first defining

$$T_i = \text{Total for treatment } i;$$

$$B_j = \sum_{h=1}^{k} Y_{jh} = \text{Total for } j\text{th block};$$

$$R_h = \sum_{j=1}^{b} Y_{jh} = \text{Total for } h\text{th row};$$

$$g = \sum_{i=1}^{v} T_i = \sum_{j=1}^{b} B_j = \sum_{h=1}^{k} R_h = \sum_{j=1}^{b} \sum_{h=1}^{k} Y_{jh} = \text{Total for all observations.}$$

Then the $i$th adjusted treatment total is defined to be

$$Q_i = T_i - \sum_{j=1}^{b} n_{ij}B_j/k - \sum_{h=1}^{k} \tilde{n}_{ih}R_h/b + g/v$$

or expressed in matrix notation

$$Q = T - NB/k - \tilde{N}R/b + 1g/v \tag{2.2}$$

where $T(v \times 1)$, $B(b \times 1)$, and $R(k \times 1)$ are the respective column vectors of the treatment, block, and row totals, and 1 denotes a $v \times 1$ vector of all unity elements. It can be shown cf. Tocher (1952) that the (reduced) normal equations for estimating the treatment effect vector $t' = (t_1, t_2, \cdots t_v)$ can be written as

$$[rI - NN'/k - \tilde{N}\tilde{N}'/b + J(r/v)]\hat{t} = Q \tag{2.3}$$

where $I$ is the identity matrix of order $v$ and $J = 1\,1'$. Furthermore the estimate of the variance with $\nu_e = (bk - b - v - k + 2)$ degrees of freedom is $s^2 = [Y'Y - \hat{t}'Q - R'R/b - B'B/k + g^2/vr]/\nu_e$.

The paper by Kurkjian and Zelen (1963) introduced a structural property of the design which was related to the (block) incidence matrix $N$ of the design. This structural property was termed Property (A) and is defined as follows. Let $v = \prod_{i=1}^{n} m_i$ and denote the $m_i \times m_i$ identity matrix by $I_i$, and the $m_i \times m_i$ matrix with all elements unity by $J_i$. We then define $D_i^{\delta_i}$ by

$$\begin{aligned} D_i^{\delta_i} &= I_i && \text{if } \delta_i = 0 \\ &= J_i && \text{if } \delta_i = 1. \end{aligned} \tag{2.4}$$

Then a block design will be said to have Property (A) if

$$\text{(A)} \quad NN' = \sum_{s=0}^{n} \Big\{ \sum_{\delta_1+\delta_2+\cdots+\delta_n=s} h(\delta_1, \delta_2, \cdots, \delta_n)[D_1^{\delta_1} \times D_2^{\delta_2} \times \cdots \times D_n^{\delta_n}]\Big\}$$

where the expression in square brackets is the direct (or Kronecker) product of the matrices $D_i^{\delta_i}$ and the $h(\delta_1, \delta_2, \cdots, \delta_n)$ are constants. Property A may be written a bit more concisely by regarding $\delta = (\delta_1, \delta_2, \cdots, \delta_n)$ as a $n$-digit binary number and writing

$$D^{\delta} = D_1^{\delta_1} \times D_2^{\delta_2} \times \cdots \times D_n^{\delta_n}.$$

Then the Property (A) can be written as

$$\text{(A)} \qquad NN' = \sum_{\delta} h(\delta) D^{\delta}$$

where the summation goes through all $n$-digit binary numbers $\delta$. Property (A) as defined earlier by Kurkjian and Zelen refers only to a structural property of the block incidence matrix. We can define a similar property for the row incidence matrix $\tilde{N}$. A design will be said to have property (B) if

$$\text{(B)} \qquad \tilde{N}\tilde{N}' = \sum_{\delta} \tilde{h}(\delta) D^{\delta}$$

holds, where $\tilde{h}(\delta) = \tilde{h}(\delta_1, \delta_2, \cdots, \delta_n)$ denote known constants. When Properties (A) and (B) both hold, we have

$$(2.5) \qquad rI - NN'/k - \tilde{N}\tilde{N}'/b + J(r/v) = \sum_{\delta} g(\delta) D^{\delta}$$

where

$$(2.6) \qquad \begin{aligned} g(\delta) &= r - h(\delta)/k - \tilde{h}(\delta)/b && \text{for } \delta = (0, 0, \cdots, 0) \\ &= -[h(\delta)/k + \tilde{h}(\delta)/b] && \text{for } \delta \neq (0, 0, \cdots, 0), (1, 1, \cdots, 1) \\ &= -[h(\delta)/k + \tilde{h}(\delta)/b - r/v] && \text{for } \delta = (1, 1, \cdots, 1). \end{aligned}$$

Consequently, substituting (2.5) in (2.3), the (reduced) normal equations take the form

$$(2.7) \qquad (\sum_{\delta} g(\delta) D^{\delta})\hat{t} = Q.$$

It will be assumed that the design is connected both by rows and columns separately, so there will always be $(v - 1)$ linearly independent estimable functions of the treatment effects. These will be constrasts. Otherwise the number of estimable function of the treatment effects will be reduced. To solve for the $\hat{t}$, one usually introduces the non-estimable restraint $\sum_{i=1}^{v} \hat{t}_i = 0$. With this non-estimable restraint the term having $\delta = (1, 1, \cdots, 1)$ drops out and hence the summation over all $n$-digit binary numbers $\delta$ in (2.7) need not include $\delta = (1, 1, \cdots, 1)$. The solution of the system of linear equations (2.7) has been shown by Kurkjian and Zelen (1963) to be

$$(2.8) \qquad \hat{t} = \sum_{s=1}^{n} \Big\{ \sum_{x_1+x_2+\cdots x_n = s} [I_1^{\bar{x}_1} M_1^{x_1} \times I_2^{\bar{x}_2} M_2^{x_2} \times \cdots \times I_n^{\bar{x}_n} M_n^{x_n}] / rvE(x_1, x_2, \cdots, x_n) \Big\} Q$$

where $x_i$ take the values 0 or 1,

$$(2.9) \qquad \begin{aligned} I_i^{\bar{x}_i} M_i^{x_i} &= J_i && \text{for } x_i = 0 \\ &= M_i = m_i I_i - J_i && \text{for } x_i = 1. \end{aligned}$$

and $E(x_1, x_2, \cdots, x_n)$ are efficiency factors defined by

$$rE(x_1, x_2, \cdots, x_n) = \sum_{s=0}^{n-1} \left( \sum_{\delta_1+\delta_2+\cdots\delta_n=s} g(\delta_1, \delta_2, \cdots, \delta_n) \prod_{i=1}^{n} m_i^{(1-x_i)\delta_i}(1 - x_i\delta_i) \right). \tag{2.10}$$

Defining $x = (x_1, x_2, \cdots, x_n)$ to be an $n$-digit binary number, the solution of the reduced normal equations can be written as

$$\hat{t} = \{ {\sum_x}' I^x M^x / rvE(x) \} Q \tag{2.11}$$

where

$$I^x M^x = I_1^{x_1} M_1^{x_1} \times I_2^{x_2} M_2^{x_2} \times \cdots \times I_n^{x_n} M_n^{x_n}$$

and the summation ${\sum_x}'$ ranges over all $n$-digit binary numbers except $x = (0, 0, \cdots, 0)$. Thus (2.8) or (2.11) is the solution of the treatment estimates for all designs which allow for two-way elimination of heterogeneity when Properties (A) and (B) hold.

The paper by Kurkjian and Zelen (1963) also gives the explicit expression for the variance of the difference between two treatment estimates. This result is summarized as follows. Let the $i$th treatment be indexed by an $n$-tuple $i = (i_1, i_2, \cdots, i_n)$ where $i_s = 1, 2, \cdots, m_s$ and $s = 1, 2, \cdots, n$. Then the expression for the variance of $(\hat{t}_i - \hat{t}_i')$ is

$$\text{var}\,(\hat{t}_i - \hat{t}_i') = \frac{2\sigma^2}{rv} \sum_{s=1}^{n} \left\{ \sum_{x_1+x_2+\cdots+x_n=s} \frac{\prod_{r=1}^{n} (m_r - 1)^{x_r} + (-1)^{s+1} \prod_{r=1}^{n} (1 - m_r)^{x_r\delta_r}}{E(x_1, x_2, \cdots, x_n)} \right\} \tag{2.12}$$

where

$$\delta_r = 0 \quad \text{if} \quad i_r \neq i_r'$$
$$= 1 \quad \text{if} \quad i_r = i_r'$$

and the $x_r$ take the values zero or one. The assignment of the $n$-tuples to the treatments is made in such a way that the treatment numbers $i = 1$ through $m_n$ correspond to the $n$-tuples $i = (1, 1, \cdots, 1, i_n)$ for $i_n = 1, 2, \cdots, m_n$; treatment numbers $i = m_n + 1$ through $2m_n$ are given by $i = (1, 1, \cdots, 2, i_n)$ for $i_n = 1, 2, \cdots, m_n$. This procedure continues replacing $i_{n-1}$ by $3, \cdots, m_{n-1}$ which assigns the treatments up to $m_{n-1}m_n$. The element $i_{n-3} = 1$ is then changed to $i_{n-3} = 2$ and the process is continued, etc. until the $v$th treatment is given by $(m_1, m_2, \cdots, m_n)$.

**3. Latin Squares (LS), Balanced Incomplete Block (BIB), and Group Divisible Designs (GD).** Block designs such as the randomized blocks (RB), the BIB's and partially balanced incomplete blocks (PBIB) are characterized by a treatment occurring at most one time in any block. We use the convention that when the design is referred to by name, it characterizes the column incidence matrix. The problem in using these designs for elimination of heterogeneity for both columns (blocks) and rows is to arrange the treatments in the rows such that Property (B) holds, while at the same time preserving the same column incidence matrix. Often this results in a treatment occurring more than once in some of the rows. In this section we discuss the analysis and row re-arrangement for the LS, BIB, and GD designs.

The GD designs introduced by Bose and his associates (1952a, b), (1954) can be characterized as being PBIB designs with two associate classes such that the $v = m_1 m_2$ treatments can be divided into $m_1$ groups of $m_2$ treatments each. Furthermore the column incidence matrix has Property (A) and can be written

$$\text{(3.1)} \quad \text{GD:} \quad NN' = (r - \lambda_1)I_1 \times I_2 + (\lambda_1 - \lambda_2)I_1 \times J_2 + \lambda_2 J_1 \times J_2$$

where $\lambda_1$, $\lambda_2$ are parameters associated with the design. When $m_2 = 1$ and $\lambda_1 = \lambda_2 = \lambda$ the design can then be regarded as a BIB design with

$$\text{(3.2)} \qquad \text{BIB:} \quad NN' = (r - \lambda)I + \lambda J$$

where $I$ and $J$ are now of dimension $v$. Specializing the parameters still further so that $b = r = \lambda$, the design will be a RB design with

$$\text{(3.3)} \qquad \text{RB:} \quad NN' = rJ.$$

Thus one can specialize the analysis for GD's and obtain the results for the BIB and RB designs. These same remarks clearly hold for row incidence matrices. Therefore one can consider a GD design with row incidence matrix having the property that

$$\text{(3.4)} \qquad \tilde{N}\tilde{N}' = \tilde{h}(0, 0)I_1 \times I_2 + \tilde{h}(0, 1)I_1 \times J_2 + \tilde{h}(1, 1)J_1 \times J_2$$

and specialize the solutions to obtain various different combinations of row and column incidence forms.

In this paper we shall be dealing only with column and row incidence matrices having the form (3.4). The constants associated with the column incidence matrix are denoted by $h(\delta_1, \delta_2)$. When the three constants are non-zero, we call this the GD form. If $h(0, 0) = 0$, we will term the incidence matrix a modified GD (MGD) form. When $h(0, 1) = 0$, the form will be BIB. If $h(0, 0) = h(0, 1) = 0$, $NN' = h(1, 1)J_1 \times J_2$ and will be called the RB form. When each treatment is repeated in every column (row) $p$ times we have $NN' = p^2 bJ_1 \times J_2$, $\tilde{N}\tilde{N}' = p^2 kJ_1 \times J_2$. If $p = 1$, this corresponds to a RB design.

It will be convenient to adopt a means of identifying designs by specifying the type of column and row incidence matrices. The identification will be (column incidence type)/(row incidence type). Thus the symbol GD/RB corresponds to a design with GD column incidence and RB row incidence matrix.

We also remark that since $\sum_{i=1}^{v} \tilde{n}_{ih} = b$ for all $h$, and $\sum_{h=1}^{k} \tilde{n}_{ih} = r$ for all $i$, we have

$$1'\tilde{N}\tilde{N}' = rb1' = 1'\{\tilde{h}(0, 0)I_1 \times I_2 + \tilde{h}(0, 1)I_1 \times J_2 + \tilde{h}(1, 1)J_1 \times J_2\}$$
$$= \{\tilde{h}(0, 0) + m_2\tilde{h}(0, 1) + m_1m_2\tilde{h}(1, 1)\}1'$$

and thus $rb = \tilde{h}(0, 0) + m_2\tilde{h}(0, 1) + m_1m_2\tilde{h}(1, 1)$.

3.1 *The case* $\tilde{N}\tilde{N}' = p^2kJ$. Suppose that each treatment occurs in every row of a design exactly $p$ times. Therefore the row-incidence matrix is of the RB form; i.e.

$$\tilde{N}\tilde{N}' = p^2kJ. \tag{3.5}$$

When the design is such that $b = pv$, then $r = pk$ (by virtue of $vr = bk$) and Hartley, Shrikhande, and Taylor (1953) have shown that the treatments can be arranged so that each treatment occurs $p$ times in every row. Substituting this value for $\tilde{N}\tilde{N}'$ in the l.h.s. of the reduced normal equations (2.7) results in

$$\{rI - NN'/k - \tilde{N}\tilde{N}'/b + (r/v)J\}\hat{t} = \{rI - NN'/k\}\hat{t}$$

as $J\hat{t} = 0$ due to $\sum_{i=1}^{v} \hat{t}_i = 0$. Therefore the reduced normal equations become

$$\{rI - NN'/k\}\hat{t} = Q \tag{3.6}$$

which is independent of $\tilde{N}$. The l.h.s. is the same as the reduced normal equations for block designs with one-way elimination of heterogeneity.

The most widely used designs for two-way elimination of heterogeneity are the LS and Youden Square (YS) designs. These are such that each treatment occurs exactly once in every row. When the LS design is repeated $p$ times so that the parameters are $(v, r = pk, b = pk, k)$, the incidence matrices become

$$NN' = pbJ, \qquad \tilde{N}\tilde{N}' = p^2kJ. \tag{3.7}$$

The case when a BIB has $b = pv$ has been discussed by Shrikhande (1951), and as can be seen from (3.6) presents no problems in obtaining the treatment estimates.

3.2 GD/GD. Consider a GD design with row incidence matrix having the property

$$\tilde{N}\tilde{N}' = \tilde{h}(0, 0)I_1 \times I_2 + \tilde{h}(0, 1)I_1 \times J_2 + \tilde{h}(1, 1)J_1 \times J_2. \tag{3.8}$$

Using (2.6) results in

$$g(0, 0) = r - (r - \lambda_1)/k - \tilde{h}(0, 0)/b,$$
$$g(0, 1) = -[(\lambda_1 - \lambda_2)/k + \tilde{h}(0, 1)/b],$$
$$g(1, 1) = -[\lambda_2/k + \tilde{h}(1, 1)/b - r/v].$$

It is convenient to define the quantities

$$\begin{aligned} E_0(1, 0) &= [r(k - 1) + \lambda_1 - m_2(\lambda_1 - \lambda_2)]/rk \\ E_0(0, 1) &= [r(k - 1) + \lambda_1]/rk. \end{aligned} \tag{3.9}$$

Then using (2.10) the efficiency factors are

$$\begin{aligned} E(1,0) &= r^{-1}[g(0,0) + m_2 g(0,1)] \\ &= [E_0(1,0) - 1] + [rb - \tilde{h}(0,0) - \tilde{h}(0,1)m_2]/rb \\ E(0,1) = E(1,1) &= g(0,0)/r = [E_0(0,1) - 1] + [rb - \tilde{h}(0,0)]/rb. \end{aligned} \tag{3.10}$$

The quantities $E_0(1, 0)$ and $E_0(0, 1)$ denote the efficiency factors of the GD when it is used only for one-way elimination of heterogeneity.

Using (2.11) and (2.12) the treatment estimates are

$$\hat{t} = \frac{1}{rv}\left\{\frac{M_1 \times J_2}{E(1,0)} + \frac{J_1 \times M_2}{E(0,1)} + \frac{M_1 \times M_2}{E(1,1)}\right\} Q \tag{3.11}$$

and can be simplified to

$$\begin{aligned} \hat{t} &= \frac{1}{rv}\left\{\frac{m_1 I_1 \times J_2}{E(1,0)} + \frac{vI_1 \times I_2 - m_1 I_1 \times J_2}{E(0,1)}\right\} Q \\ &= \frac{1}{rv}\left\{\frac{vI_1 \times I_2}{E(0,1)} + \left[\frac{1}{E(1,0)} - \frac{1}{E(0,1)}\right] m_1 I_1 \times J_2\right\} Q. \end{aligned}$$

In order to obtain the variance for the difference between two treatment estimates one associates the $i$th treatment with the 2-tuple $(i_1, i_2)$ where $i_s = 1, 2, \cdots, m_s$ for $s = 1, 2$. Then if $i = (i_1, i_2)$ and $i' = (i_1', i_2')$ are associated with treatments $i$ and $i'$, the variance $(\hat{t}_i - \hat{t}_i')$ is obtained from (2.12) with $n = 2$. This results in

$$\begin{aligned} \operatorname{var}(\hat{t}_i - \hat{t}_i') &= (2\sigma^2/rv)\{m_1/E(1,0) + m_1(m_2 - 1)/E(0,1)\} && \text{for } i_1 \neq i_1' \\ \operatorname{var}(\hat{t}_i - \hat{t}_i') &= 2\sigma^2/rE(0,1) && \text{for } i_1 = i_1'. \end{aligned} \tag{3.12}$$

If the design is a BIB ($\lambda_1 = \lambda_2 = \lambda$, $m_2 = 1$) with the row incidence matrix being of the form (3.8), the efficiency factors (3.10) become

$$\begin{aligned} E(1,0) &= [E_0 - 1] + [rb - \tilde{h}(0,0) - \tilde{h}(0,1)m_2]/rb, \\ E(0,1) &= [E_0 - 1] + [rb - \tilde{h}(0,0)]/rb, \\ E_0 &= [r(k-1) + \lambda]/rk. \end{aligned} \tag{3.13}$$

3.3 *Row arrangements for* $r = pk \pm 1$. Hartley *et al.* (1953) have shown that for a BIB design, if $r = pk + 1$ or $r = pk - 1 (p \geqq 2)$, the treatments can be arranged in rows such that $\tilde{N}\tilde{N}'$ is MGD; i.e.

$$\tilde{N}\tilde{N}' = I_1 \times J_2 + \tilde{h}(1,1) J_1 \times J_2 \tag{3.14}$$

where $m_1 = k$ and $m_2 = v/k$. The value of the constant $\tilde{h}(1, 1)$ is

$$\begin{aligned} \tilde{h}(1,1) &= p^2k + 2p && \text{if } r = pk + 1 \\ &= p^2k - 2p && \text{if } r = pk - 1. \end{aligned}$$

This rearrangement of the treatments within the rows results in dividing the $v$ treatments into $k$ groups of $v/k$ treatments each so that in each row all the treatments of one group occur $p + 1$ $(p - 1)$ times while the remaining treatments occur $p$ times when $r = pk + 1$ $(pk - 1)$. The same result will also hold for GD designs if $m_1 = k$ and $m_2 = v/k$ and $b > v$. Therefore with $\tilde{N}\tilde{N}'$ given by (3.14), $\tilde{h}(0, 0) = 0$, $\tilde{h}(0, 1) = 1$, and the efficiency factors in (3.13) and (3.10) reduce to

$$\text{(3.15)} \quad \begin{aligned} &\text{BIB/MGD:} \quad E(1, 0) = E_0 - m_2/rb, \\ &\qquad\qquad E(0, 1) = E_0 = [r(k - 1) + \lambda]/rk \end{aligned}$$

$$\text{(3.16)} \quad \begin{aligned} &\text{GD/MGD:} \quad E(1, 0) = E_0(1, 0) - m_2/rb, \\ &\qquad\qquad E(0, 1) = E(1, 1) = E_0(0, 1). \end{aligned}$$

Some simplification occurs in writing the treatment estimates for the BIB and GD designs. From (3, 11) we have

$$\text{(3.17)} \qquad \text{BIB:} \quad \hat{t} = \hat{t}_0 - \{I_1 \times J_2/rE_0(1 - rbE_0)\}Q$$

$$\text{(3.18)} \qquad \text{GD:} \quad \hat{t} = \hat{t}_0 - \{I_1 \times J_2/rE_0(1, 0)[m_2 - rbE_0(1, 0)]\}Q$$

where $\hat{t}_0$ is the estimate obtained without eliminating row heterogeneity with the adjusted treatment totals defined by (2.2).

In general special methods are needed for arranging the rows of the design so that the row incidence matrix has Property (B). These methods involve finding a design with the parameters

$$\text{(3.19)} \qquad \tilde{v} = v, \qquad \tilde{r} = r, \qquad \tilde{b} = k, \qquad \tilde{k} = b$$

where a treatment is allowed to appear more than once in a block. This design is then used to arrange the treatments in the rows.

One useful method for constructing such a design is when the parameters satisfy the conditions

$$\text{(3.20)} \qquad \tilde{k} = pv + k^*, \qquad \tilde{r} = p\tilde{b} + r^* = pk + r^*.$$

With these parameters one forms a design in two stages. The first stage consists of constructing a design with parameters $v_1 = v$, $r_1 = pk$, $b_1 = k$, $k_1 = pv$ by replicating each of the $v$ treatments $p$ times in each block. The second stage consists of constructing a design with parameters $v_2 = v$, $r_2 = r^*$, $b_2 = k$, $k_2 = k^*$. This second design need not be connected. This second design is then appended to the first, block to block to make the total number of experimental units in a block $k_1 + k_2 = pv + k^* = \tilde{k}$.

Shrikhande (1951) discussed a special case of the above method for the class of BIB designs with parameters

$$\text{(3.21)} \quad v = s^2, \qquad b = s^2 + s, \qquad r = s + 1, \qquad k = s, \qquad \lambda = 1.$$

In order to arrange the rows, we consider a design with the parameters (3.21);

i.e. $\tilde{v} = s^2$, $\tilde{r} = s + 1$, $\tilde{b} = s$, $\tilde{k} = s^2 + s$. Note that $\tilde{h}$ and $\tilde{r}$ are of the form (3.20) with $p = 1$, $k^* = s$ and $r^* = 1$. The design in the second step of the procedure is to construct a design with parameters $v_2 = s^2$, $r_2 = 1$, $b_2 = s$, $k_2 = s$. Since $r_2 = 1$, this design must be disconnected and consists of the $s \times s$ array of $s^2$ treatments.

Another two stage method of constructing a design for row arrangements when $\tilde{v} > \tilde{b}$ is to construct the first stage design with parameters $v_1 = \tilde{b} = k$, $r_1 = r$, $b_1 = \tilde{b} = k$, $k_1 = r$. The second stage design is constructed with the parameters $v_2 = v - k$, $r_2 = r$, $b_2 = k$, $k_2 = b - r$ where the treatments in this second design are different from that of the first design. This second design is then appended to the first, block to block to make the total number of treatments $v$ and block size $\tilde{k} = b$.

It should be stressed that when the GD designs are used, the row incidence matrix must be of the GD form where the treatments fall into the same association scheme. In many instances such an arrangement may be impossible.

**4. New classes of designs with two-way elimination of heterogeneity.** When the number of blocks is not a multiple of the number of treatments $b \neq pv$, special methods are needed to arrange the treatments within the rows so that the row incidence matrix has Property (B). In this section we discuss classes of incomplete block designs which have Properties (A) and (B); yet $b \neq pv$. We construct these new designs by taking the direct product of the incidence matrices of two separate incidence matrices. Such constructions have been discussed by Kurkjian and Zelen (1963), Shah (1959), Rao (1961), and Vartak (1955). However these earlier papers have only constructed these designs for one-way elimination of heterogeneity. Mandel and Zelen (1954) have reported on such constructions of which this section is a generalization.

Consider two designs with respective column and row incidence matrices $N_i$, $\tilde{N}_i (i = 1, 2)$. A new design is constructed by taking the direct product of these incidence matrices. This results in $N = N_1 \times N_2$ and $\tilde{N} = \tilde{N}_1 \times \tilde{N}_2$. Furthermore if both these designs possess Properties (A) and (B), it is clear that the derived design will also possess such properties. When two designs are used to form a new design we shall designate the resulting design by $(\cdot/\cdot) \times (\cdot/\cdot)$. For example, the design formed by GD/RB and BIB/MGD will be denoted by GD/RB $\times$ BIB/MGD. In this section we shall investigate the analysis of such designs where the results for the GD/GD $\times$ BIB/MGD are found and various special cases are derived from it.

An example will serve to clarify the construction of new designs. Consider the two designs

$$D_1: \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{matrix} \qquad D_2: \begin{matrix} 1 & 2 \\ 2 & 2. \\ 3 & 1 \end{matrix}$$

The designs are respectively a GD/RB and RB/BIB. The new design obtained by taking the direct product of the respective row and incidence matrices can be easily constructed by using the operation of the symbolic direct product (SDP) (see Kurkjian and Zelen (1962)). An illustration will clarify the details. Taking the SDP of the two designs results in

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{matrix} \otimes \begin{matrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{matrix} = \begin{matrix} (11)(12)(21)(22)(31)(32)(41)(42) \\ (12)(13)(22)(23)(32)(33)(42)(43) \\ (13)(11)(23)(21)(33)(31)(43)(41) \\ (21)(22)(31)(32)(41)(42)(11)(12) \\ (22)(23)(32)(33)(42)(43)(12)(13) \\ (23)(21)(33)(31)(43)(41)(13)(11) \end{matrix}.$$

The resulting design has parameters $v = 12$, $r = 4$, $k = 6$, $b = 8$ and is a PBIB (3 associate classes) for its columns and is a GD design for its rows. A treatment is associated with the 2-tuple $i = (i_1, i_2)$ where $i_1 = 1, 2, 3, 4$ and $i_2 = 1, 2, 3$. Note that the SDP operation consisted simply of constructing an array by adjoining the second design to each treatment of the first design.

4.1. GD/GD × BIB/MGD: Let $N_1$ and $\tilde{N}_1$ correspond to the respective column and row incidence matrices of a GD design having parameters $(v_1 = m_1 m_2, r_1, k_1, b_1, \lambda_{11}, \lambda_{12})$ where $\tilde{N}_1 \tilde{N}_1'$ is assumed to be of the form

$$\tilde{N}_1 \tilde{N}_1' = \tilde{h}_1(0, 0) I_1 \times I_2 + \tilde{h}_1(0, 1) I_1 \times J_2 + \tilde{h}_1(1, 1) J_1 \times J_2. \tag{4.1}$$

Denote by $N_2$ and $\tilde{N}_2$ the respective column and row incidence matrices of a BIB with parameters $(v_2 = m_3 m_4, r_2, b_2, k_2, \lambda_2)$ where $\tilde{N}_2 \tilde{N}_2'$ is assumed to be of the form

$$\tilde{N}_2 \tilde{N}_2' = \tilde{h}_2(0, 1) I_3 \times J_4 + \tilde{h}_2(1, 1) J_3 \times J_4. \tag{4.2}$$

When a new design is formed by taking the direct product of the two incidence matrices the resulting design will have parameters $(v = v_1 v_2, r = r_1 r_2, b = b_1 b_2, k = k_1 k_2)$ and also will have Properties (A) and (B) with $n = 4$. (In some instances $n$ can be $n = 3$ with $m_4 = 1$). The respective values of the non-zero $h(\delta)$ and $\tilde{h}(\delta)$ constants are

$$h(0, 0, 0, 0) = (r_1 - \lambda_{11})(r_2 - \lambda_2), \quad h(0, 1, 0, 0) = (\lambda_{11} - \lambda_{12})(r_2 - \lambda_2),$$

$$h(0, 0, 1, 1) = (r_1 - \lambda_{11})\lambda_2, \quad h(0, 1, 1, 1) = (\lambda_{11} - \lambda_{12})\lambda_2,$$

$$h(1, 1, 0, 0) = \lambda_{12}(r_2 - \lambda_2), \quad h(1, 1, 1, 1) = \lambda_{12}\lambda_2,$$

$$\tilde{h}(\delta_1, \delta_2, \delta_3, \delta_4) = \tilde{h}_1(\delta_1, \delta_2)\tilde{h}_2(\delta_3, \delta_4).$$

Using (2.6) to find the $g(\delta)$ values and (2.10) to evaluate the efficiency factors results, after some lengthy algebra, in

$$
\begin{aligned}
E(1,0,0,0) &= E(1,0), \\
E(0,1,0,0) &= E(1,1,0,0) = E(0,1), \\
E(0,0,1,0) &= E_2 + (1 - E_{20})(E_0(1,0) - E_{10}), \\
E(0,0,0,1) &= E(0,0,1,1) = E_{20} + (1 - E_{20})(E_0(1,0) - E_{10}), \\
E(1,0,1,0) &= E(1,0) + E_2(1 - E(1,0)) \\
&\qquad + [E_0(1,0) - E(1,0)][1 - E_2] \\
&\qquad\qquad + [E_{20} - E_2][1 - E(1,0)], \\
E(1,0,1,1) &= E(1,0,0,1) = E_0(1,0) + E_{20}(1 - E_0(1,0)), \\
E(0,1,1,0) &= E(1,1,1,0) = E(0,1) + E_2[1 - E(0,1] \\
&\qquad + [E_0(0,1) - E(0,1)][1 - E_2] \\
&\qquad\qquad + [E_{20} - E_2][1 - E(0,1)], \\
E(1,1,1,1) &= E(1,1,0,1) = E(0,1,1,1) = E(0,1,0,1) = E_0(0,1) \\
&\qquad + E_{20}[1 - E_0(0,1)]
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
E_0(1,0) &= [r_1(k_1 - 1) + \lambda_{11} - m_2(\lambda_{11} - \lambda_{12})]/r_1k_1, \\
E_0(0,1) &= [r_1(k_1 - 1) + \lambda_{11}]/r_1k_1, \\
E_{10} &= v_1\lambda_{12}/r_1k_1, \\
E_{20} &= v_2\lambda_2/r_2k_2, \\
E(1,0) &= [E_0(1,0) - 1] + [r_1b_1 - \tilde{h}_1(0,0) + \tilde{h}_1(0,1)m_2]/r_1b_1, \\
E(0,1) &= [E_0(0,1) - 1] + [r_1b_1 - \tilde{h}_1(0,0)]/r_1b_1, \\
E_2 &= [E_{20} - 1] + [r_2b_2 - \tilde{h}_2(0,1)m_2]/r_2b_2.
\end{aligned}
\tag{4.4}
$$

Note that the efficiency factors with a zero subscript refer to efficiency factors used with one-way elimination.

The treatment estimates can be obtained by applying (2.8). After consolidating terms having the same efficiency factors, the estimate can be written as

$$
\begin{aligned}
\hat{t} = \frac{1}{rv}\Bigg\{ & \frac{m_1I_1 \times J_2 \times J_3 \times J_4}{E(1,0,0,0)} + \frac{m_1I_1 \times M_2 \times J_3 \times J_4}{E(0,1,0,0)} \\
& + \frac{J_1 \times J_2 \times m_3I_3 \times J_3}{E(0,0,1,0)} + \frac{J_1 \times J_2 \times m_3I_3 \times M_4}{E(0,0,0,1)} + \frac{M_1 \times J_2 \times M_3 \times J_4}{E(1,0,1,0)} \\
& + \frac{M_1 \times J_2 \times m_3I_3 \times M_4}{E(1,0,1,1)} + \frac{m_1I_1 \times M_2 \times M_3 \times J_4}{E(0,1,1,0)} \\
& + \frac{m_1I_1 \times M_2 \times m_3I_3 \times M_4}{E(1,1,1,1)}\Bigg\} Q.
\end{aligned}
\tag{4.5}
$$

There will be eight distinct variances for comparing differences between treatments. These can be obtained by applying (2.12) with $n = 4$. The variances can be written explicitly by recalling that the two treatments $i$ and $i'$ are associated with $i = (i_1, i_2, i_3, i_4)$ and $i' = (i_1', i_2', i_3', i_4')$. Define for $s = 1, 2, 3, 4$

$$\delta_s = 1 \qquad \text{if } i_s = i_s'$$
$$= 0 \qquad \text{if } i_s \neq i_s'$$
$$\varphi_s = 1 - (1 - m_s)^{\delta_s}.$$

Then the variance for comparing the two treatment estimates $(\hat{t}_i - \hat{t}_i')$ is

$$\begin{aligned}\text{var}(\hat{t}_i - \hat{t}_i') = \frac{2\sigma^2}{rv}\Bigg\{&\frac{m_1 - \varphi_1}{E(1,0,0,0)} + \frac{m_1(m_2 - 1) - \varphi_1(\varphi_2 - 1)}{E(0,1,0,0)} \\ &+ \frac{m_3 - \varphi_3}{E(0,0,1,0)} + \frac{m_3(m_4 - 1) - \varphi_3(\varphi_4 - 1)}{E(0,0,0,1)} \\ &+ \frac{(m_1 - 1)(m_3 - 1) - (\varphi_1 - 1)(\varphi_3 - 1)}{E(1,0,1,0)} \\ &+ \frac{(m_1 - 1)m_3(m_4 - 1) - (\varphi_1 - 1)\varphi_3(\varphi_4 - 1)}{E(1,0,1,1)} \\ &+ \frac{m_1(m_2 - 1)(m_3 - 1) - \varphi_1(\varphi_2 - 1)(\varphi_3 - 1)}{E(0,1,1,0)} \\ &+ \frac{m_1(m_2 - 1)m_3(m_4 - 1) - \varphi_1(\varphi_2 - 1)\varphi_3(\varphi_4 - 1)}{E(1,1,1,1)}\Bigg\}.\end{aligned} \tag{4.6}$$

There will be eight distinct variances corresponding to the treatment comparisons where

(1) $\delta_1 = \delta_3 = 0$, (2) $\delta_1 = 1$, $\delta_2 = \delta_3 = 0$, (3) $\delta_1 = \delta_4 = 0$, $\delta_3 = 1$,

(4) $\delta_1 = \delta_2 = 1$, $\delta_3 = 0$, (5) $\delta_1 = 0$, $\delta_3 = \delta_4 = 1$,

(6) $\delta_1 = \delta_2 = \delta_4 = 0$, $\delta_3 = 1$, (7) $\delta_1 = \delta_3 = \delta_4 = 1$, $\delta_2 = 0$,

(8) $\delta_1 = \delta_2 = \delta_3 = 1$, $\delta_4 = 0$.

A design of this complexity may not be useful in a practical sense. However it supplies the basic results for the analysis of designs which can be derived from it. Examples are discussed in the following sub-sections.

4.2 RB/GD × BIB/MGD. The design RB/GD has column incidence structure $N_1N_1' = p^2k_1J$ and GD row structure. It can be obtained by taking a GD with $(pv)$ blocks and using the rows as columns, etc. The value of the $\tilde{h}(\delta_1, \delta_2)$ are thus $\tilde{h}_1(0, 0) = (r_1 - \tilde{\lambda}_{11})$, $\tilde{h}_1(0, 1) = (\tilde{\lambda}_{11} - \tilde{\lambda}_{12})$, $\tilde{h}_1(1, 1) = \tilde{\lambda}_{12}$ where a tilde ($\sim$) is used to denote that these parameters are associated with the row incidence matrix. A GD goes into a RB design when $\lambda_1 = \lambda_2 = r$. Therefore the GD efficiency factors become

$$\begin{aligned} &E_{10} = E_0(0, 1) = E_0(1, 0) = 1, \\ (4.7) \qquad &E(1, 0) = [r_1(b_1 - 1) + \tilde{\lambda}_{11} - m_2(\tilde{\lambda}_{11} - \tilde{\lambda}_{12})]/r_1b_1, \\ &E(0, 1) = [r_2b_2 - \tilde{h}_2(0, 1)]/r_2b_2 \end{aligned}$$

which enables one to find the efficiency factors of this design. These can be found by substitution in (4.3) and results in

$$\begin{aligned} &E(1, 0, 0, 0) = E(1, 0), \qquad E(0, 1, 0, 0) = E(1, 1, 0, 0) = E(0, 1), \\ &E(0, 0, 1, 0) = E_2, \qquad E(0, 0, 0, 1) = E(0, 0, 1, 1) = E_{20}, \\ (4.8) \qquad &E(1, 0, 1, 0) = 1 + [1 - E(1, 0)][E_{20} - E_2], \\ &E(1, 0, 1, 1) = E(1, 0, 0, 1) = E(1, 1, 1, 1) = E(1, 1, 0, 1) \\ &\qquad = E(0, 1, 1, 1) = E(0, 1, 0, 1) = E(0, 1, 1, 0) \\ &\qquad = E(1, 1, 1, 0) = 1 + [1 - E(0, 1)][E_{20} - E_2]. \end{aligned}$$

The treatment estimates and variances for comparing treatments can now be obtained with (4.5) and (4.6).

4.3 RB/GD × BIB/RB. The design RB/GD × BIB/RB will have

$$\tilde{h}_2(0, 1) = 0, \qquad \tilde{h}_2(1, 1) = p_2^2k_2J.$$

Therefore from (4.4) $E_2 = E_{20}$ and we have

$$\begin{aligned} &E(1, 0, 0) = E(1, 0), \qquad E(0, 1, 0) = E(1, 1, 0) = E(0, 1), \\ (4.9) \qquad &E(0, 0, 1) = E_{20}, \\ &E(1, 0, 1) = E(0, 1, 1) = E(1, 1, 1) = 1, \end{aligned}$$

where $E(1, 0)$ and $E(0, 1)$ are defined as in (4.14). When $m_4 = 1$, we have $M_4 = 0$ and $J_4 = 1$. Hence the treatment estimate can be obtained from (4.5) by ignoring those efficiency factors for which $x_4 = 1$ and deleting the $J_4$ which appears in the terms where $x_4 = 0$. This leads to

$$\hat{t} = \frac{1}{rv}\left\{\frac{m_1I_1 \times J_2 \times J_3}{E(1,0,0)} + \frac{m_1I_1 \times M_2 \times J_3}{E(0,1,0)} + \frac{J_1 \times J_2 \times m_3I_3}{E(0,0,1)} + \frac{M_1 \times J_2 \times M_3 + m_1I_1 \times M_2 \times M_3}{E(1,1,1)}\right\} Q.$$

Similarly the variances can be obtained from (4.6) by ignoring those terms for which $i_4 \neq i_4'$. This results in

$$(4.10) \qquad \text{var}\,(\hat{t}_i - \hat{t}_i') = \frac{2\sigma^2}{rv}\left\{\frac{m_1}{E(1,0,0)} + \frac{m_1(m_2 - 1)}{E(0,1,0)} + \frac{m_3}{E(0,0,1)} + \frac{(m_1m_2 - 1)(m_3 - 1) - 1}{E(1,1,1)}\right\} \quad \text{for } i_1 \neq i_1', i_3 \neq i_3';$$

$$\text{(4.11)}\quad \operatorname{var}(\hat{t}_i - \hat{t}'_i) = \frac{2\sigma^2}{rv}\left\{\frac{m_1 m_2}{E(0,1,0)} + \frac{m_3}{E(0,1,0)} + \frac{(m_1 m_2 - 1)(m_3 - 1) - 1}{E(1,1,1)}\right\} \quad \text{for } i_1 = i'_1,\ i_2 \neq i'_2,\ i_3 \neq i'_3;$$

$$\text{(4.12)}\quad \operatorname{var}(\hat{t}_i - \hat{t}'_i) = \frac{2\sigma^2}{rv}\left\{\frac{m_3}{E(0,0,1)} + \frac{(m_1 m_2 - 1)m_3}{E(1,1,1)}\right\} \quad \text{for } i_1 = i'_1,\ i_2 = i'_2,\ i_3 \neq i'_3;$$

$$\text{(4.13)}\quad \operatorname{var}(\hat{t}_i - \hat{t}'_i) = \frac{2\sigma^2}{rv}\left\{\frac{m_1}{E(1,0,0)} + \frac{m_1(m_2 - 1)}{E(0,1,0)} + \frac{m_1 m_2(m_3 - 1)}{E(1,1,1)}\right\} \quad \text{for } i_1 \neq i'_1,\ i_3 = i'_3;$$

$$\text{(4.14)}\quad \operatorname{var}(\hat{t}_i - \hat{t}'_i) = \frac{2\sigma^2}{rv}\left\{\frac{m_1 m_2}{E(0,1,0)} + \frac{m_1 m_2(m_3 - 1)}{E(1,1,1)}\right\} \quad \text{for } i_1 = i'_1,\ i_2 \neq i'_2,\ i_3 = i'_3.$$

4.4. RB/BIB × BIB/RB. The design RB/BIB will have $h_1(0, 0) = h_1(0, 1) = 0$, $h_1(1, 1) = p_1^2 b_1$, $\tilde{h}_1(0, 0) = (r_1 - \tilde{\lambda}_1)$, $\tilde{h}_1(1, 1) = \tilde{\lambda}_1$. All results may be derived from the RB/GD × BIB/RB by taking $\tilde{\lambda}_{11} = \tilde{\lambda}_{12} = r_1$ and $m_2 = 1$. The efficiency factors may be obtained from (4.9) by ignoring those factors for which $x_2 = 1$ and deleting the terms having $x_2 = 0$. This gives

$$\text{(4.15)}\quad \begin{aligned} E(1,0) &= [r_1(b_1 - 1) + \tilde{\lambda}_1]/r_1 b_1 \\ E(0,1) &= E_{20}; \qquad E(1,1) = 1. \end{aligned}$$

Therefore

$$\text{(4.16)}\quad \begin{aligned} \hat{t} &= \frac{1}{rv}\left\{\frac{M_1 \times J_3}{E(1,0)} + \frac{J_1 \times M_3}{E(0,1)} + \frac{M_1 \times M_3}{E(1,1)}\right\} Q \\ &= \frac{1}{rv}\left\{\frac{m_1 I_1 \times J_3}{E(1,0)} + \frac{J_1 \times m_3 I_3}{E(0,1)} + \frac{M_1 \times M_3}{E(1,1)}\right\} Q \end{aligned}$$

and the variances for comparing treatments are:

$$\text{(4.17)}\quad \operatorname{var}(\hat{t}_i - \hat{t}'_i) = \frac{2\sigma^2}{rv}\left\{\frac{m_1}{E(1,0)} + \frac{(m_1 - 1)m_3}{E(1,1)}\right\} \quad \text{for } i_1 \neq i'_1,\ i_3 = i'_3;$$

$$\text{(4.18)}\quad \operatorname{var}(\hat{t}_i - \hat{t}'_i) = \frac{2\sigma^2}{rv}\left\{\frac{m_3}{E(0,1)} + \frac{(m_1 - 1)m_3}{E(1,1)}\right\} \quad \text{for } i_1 = i'_1,\ i_3 \neq i'_3;$$

$$\text{(4.19)}\quad \operatorname{var}(\hat{t}_i - \hat{t}'_i) = \frac{2\sigma^2}{rv}\left\{\frac{m_1}{E(1,0)} + \frac{m_3}{E(0,1)} + \frac{[(m_1 - 1)(m_3 - 1) - 1]}{E(1,1)}\right\} \quad \text{for } i_1 \neq i'_1,\ i_3 \neq i'_3.$$

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