COMPARISON OF REPLACEMENT POLICIES, AND RENEWAL THEORY IMPLICATIONS

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1. Introduction and preliminaries. Among the most useful replacement policies currently in popular use are the age replacement policy and the block replacement policy. Under an *age replacement* policy a unit is replaced upon failure or at age T, a specified positive constant, whichever comes first. Under a *block replacement* policy a unit is replaced upon failure and at times T, $2T$, $3T$, $\cdots$. We shall assume for both policies that units fail permanently, independently, and that the time required to perform replacement is negligibly small. Block replacement is easier to administer since the planned replacements occur at regular intervals and so are readily scheduled. This type of policy is commonly used with digital computers and other complex electronic systems. On the other hand, age replacement seems more flexible since under this policy planned replacement takes into account the age of the unit. It is therefore of some interest to compare these two policies with respect to the number of failures, number of planned replacements, and number of removals. ("Removal" refers to both failure replacement and planned replacement.)

Block replacement policies have been investigated by E. L. Welker, 1959, R. F. Drenick, 1960, and B. J. Flehinger, 1962. Age replacement policies have been studied by G. Weiss, 1956, and Barlow and Proschan, 1962, among others.

The results of this paper depend heavily on the properties of distributions with monotone failure rate (Barlow, Marshall, and Proschan, 1963). If a unit failure distribution F has a density f, it can be verified by differentiating $\log \bar{F}(x)$ that the failure rate $r(x) = f(x)/\bar{F}(x)$ is increasing (decreasing) if $\log \bar{F}(x)$ is concave when finite (is convex on $[0, \infty)$). We consistently use $\bar{F}$ for $1 - F$. For mathematical convenience and added generality, we use this concavity (convexity) property as the definition of increasing (decreasing) failure rate whether a density exists or not. We shall refer to increasing failure rate by IFR and decreasing failure rate by DFR. It is also easy to show that F is IFR(DFR) if and only if

$$F_x(\Delta) = [F(x + \Delta) - F(x)]/\bar{F}(x)$$

is increasing (decreasing) for all x such that $\Delta > 0$ and $\bar{F}(x) > 0$. This implies F is IFR(DFR) if and only if

$$\bar{F}(x - \Delta)/\bar{F}(x) \tag{1.1}$$

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is increasing (decreasing) in x for all $\Delta > 0$. This property will be needed in Theorem 2.1.

The evaluation of the replacement policies considered also depends heavily on the theory of renewal processes (e.g., Smith, 1958, and Cox, 1962). A renewal process is a sequence $\{X_k\}_{k=1}^{\infty}$ of independent random variables with common distribution F. We also assume $F(0^-) = 0$. If the random variables are not identically distributed we call this a renewal sequence. Let us write $N(t)$ for the largest value of n for which $X_1 + X_2 + \cdots + X_n \leqq t$; in other words $N(t)$ (the renewal random variable) is the number of renewals that will have occurred by time t. The process $\{N(t); t \geqq 0\}$ is known as a renewal counting process. We will be primarily concerned with bounding the renewal function, $M(t) = E[N(t)]$. Previously, Feller, 1948, has given methods for bounding $M(t)$ using bounds on F.

In this paper we show that, assuming an IFR(DFR) unit failure distribution, the number of failures in $[0, t]$ is stochastically larger (smaller) under an age policy than under a block policy (Theorem 2.1). The number of planned replacements and the total number of removals is always stochastically smaller under an age policy than under a block policy (Theorem 2.2). By considering the number of failures and the number of removals per unit of time as the duration of the replacement operation becomes indefinitely large, we obtain in Theorem 2.4 simple useful bounds on the renewal function for any failure distribution, and an improvement on these bounds by assuming IFR(DFR) failure distributions. In particular we show that the moments, binomial moments, and variance of a renewal random variable associated with an IFR(DFR) renewal process are dominated (subordinated) by the corresponding moments and variance of a related Poisson random variable. Inequalities for renewal sequence are also obtained.

2. Contrast between age and block replacement. Denote the number of renewals in $[0, t]$ when replacement occurs only at failure by $N(t)$ and let $M(t) = E[N(t)]$. Denote the number of failures in $[0, t]$ under a block policy by $N_B^*(t)$ and under an age policy by $N_A^*(t)$, both having replacement interval T. The following theorem provides a stochastic comparison of the number of failures experienced under these policies. We assume $F(0^-) = 0$ throughout.

THEOREM 2.1. *If F is* IFR(DFR), *then*

$$P[N(t) \geqq n] \geqq (\leqq)\ P[N_A^*(t) \geqq n] \geqq (\leqq)\ P[N_B^*(t) \geqq n] \tag{2.1}$$

for $t \geqq 0$, and $n = 0, 1, 2, \cdots$. Equality is attained for the exponential distribution $F(x) = 1 - e^{-x/\mu_1}$ where μ_1 denotes the mean of F.

We defer the proof of Theorem 2.1 to Section 3. The following bounds on $M(t)$ are an immediate consequence of Theorem 2.1 and will be useful later on. (See Theorem 2.4 and Theorem 4.2.)

COROLLARY 2.1. *If F is* IFR(DFR), *then*

$$(2.2)\quad \begin{array}{lll} \text{(i)} & M(t) \geqq (\leqq)\ E[N_A^*(t)] \geqq (\leqq)\ E[N_B^*(t)] & \\ \text{(ii)} & M(t) \geqq (\leqq)\ kM(t/k) & k = 1, 2, \cdots \\ \text{(iii)} & M(t) \leqq (\geqq)\ t/\mu_1 & \\ \text{(iv)} & M(h) \leqq (\geqq)\ M(t+h) - M(t) & h \geqq 0 \end{array}$$

for all $t \geqq 0$.

PROOF.

(i) This follows from Theorem 2.1 and the fact that

$$M(t) = E[N(t)] = \sum_{n=1}^{\infty} P[N(t) \geqq n].$$

(ii) Let $T = t/k$ and observe that for this replacement interval

$$M(t) \geqq (\leqq)\ E[N_B^*(t)] = kM(T) = kM(t/k).$$

(iii) By (ii)

$$M(kT)T/kT \geqq (\leqq)\ M(T) \qquad k = 1, 2, 3, \cdots .$$

Letting $k \to \infty$ we obtain $M(T) \leqq (\geqq)\ T/\mu_1$, since $\lim_{t\to\infty} M(t)/t = 1/\mu_1$ by the elementary renewal theorem (e.g., Smith, 1958).

(iv) Define $\delta(t) = t - [X_1 + X_2 + \cdots + X_{N(t)}]$. Then

$$M(t+h) - M(t) = \int_0^t \int_0^h [1 + M(h-u)]\, dF_x(u)\, d_x P[\delta(t) \leqq x]$$

$$\geqq (\leqq) \int_0^t \int_0^h [1 + M(h-u)]\, dF(u)\, d_x P[\delta(t) \leqq x]$$

since $F_x(u)$ is increasing (decreasing) in x. Therefore

$$M(t+h) - M(t) \geqq (\leqq) M(h) \int_0^t d_x P[\delta(t) \leqq x] = M(h). \|$$

It is easy to show that (ii) is not true in general if k is allowed to be rational. The following formula, true for all distributions with second moment $\mu_2 < \infty$, provides an interesting comparison with (iii) of (2.2)

$$M(t) = t/\mu_1 + \mu_2/2\mu_1^2 - 1 + o(1),$$

(see e.g. Smith, 1958). As we shall show, inequality (iii) of (2.2) can be improved by assuming somewhat more. It is also true under weaker assumptions.

Let $N_A(t)$ and $N_B(t)$ denote the *total* number of removals in $[0, t]$ following an age and a block replacement policy respectively. The following theorem, true for all distributions, is intuitively obvious.

THEOREM 2.2.

$$P[N(t) \geqq n] \leqq P[N_A(t) \geqq n] \leqq P[N_B(t) \geqq n]$$

for all $t \geqq 0,\ n = 0, 1, 2, \cdots$.

We defer the proof to Section 3.

COROLLARY 2.2.

$$\begin{aligned} &\text{(i)} && M(t) \leqq E[N_A(t)] \leqq E[N_B(t)] \\ (2.3)\quad &\text{(ii)} && M(t) \leqq kM(t/k) + k \qquad k = 1, 2, \cdots \\ &\text{(iii)} && M(t) \geqq t/\mu_1 - 1 \end{aligned}$$

for all $t \geqq 0$.

PROOF.

(i) This is an immediate consequence of Theorem 2.2.

(ii) Let $T = t/k$ and observe that for this replacement interval

$$M(t) \leqq E[N_B(t)] = kM(T) + k = kM(t/k) + k.$$

(iii) This follows from the elementary renewal theorem $\lim_{t\to\infty} M(t)/t = 1/\mu_1$. ||

The following theorem summarizes some well-known limit results from renewal theory.

THEOREM 2.3.

(i) $\lim_{t\to\infty} N(t)/t = \lim_{t\to\infty} M(t)/t = 1/\mu_1$

(ii) $\lim_{t\to\infty} N_A^*(t)/t = \lim_{t\to\infty} E[N_A^*(t)]/t = F(T)/\int_0^T \bar{F}(x)\,dx$

(iii) $\lim_{t\to\infty} N_B^*(t)/t = \lim_{t\to\infty} E[N_B^*(t)]/t = M(T)/T.$

PROOF.

(i) See e.g. Smith, 1958.

(ii) The times between failures $\{Y_i\}_{i=1}^{\infty}$ for an age replacement policy constitute a renewal process with distribution

$$\bar{H}_T(t) = P[Y_i \geqq t] = [\bar{F}(T)]^n\, \bar{F}(t - nT) \tag{2.4}$$

for $nT \leqq t < (n+1)T$. The expected value of Y_i can be calculated from (2.4) to be $E[Y_i] = \int_0^T \bar{F}(x)\,dx/F(T)$. (Weiss, 1956, calculated higher moments.) Hence $\lim_{t\to\infty} N_A^*(t)/t = F(T)/\int_0^T \bar{F}(x)\,dx$ by (i).

(iii) Let $N_{B_i}^*(T)$ denote the number of failures in $[(i-1)T, iT]$ following a block replacement policy. Then

$$\lim_{t\to\infty} N_B^*(t)/t = \lim_{n\to\infty} \sum_{i=1}^{n} N_{B_i}^*(T)/nT = E[N_{B_i}^*(T)]/T = M(T)/T. \;||$$

For a generalization of (iii), see, e.g., Flehinger, 1962.

Using these limit results we can improve (iii) of (2.2) and (iii) of (2.3).

THEOREM 2.4.

$$\begin{aligned} &\text{(i)} && M(t) \geqq t/\int_0^t \bar{F}(x)\,dx - 1 \geqq t/\mu_1 - 1. \\ (2.5)\quad &\text{(ii)} && \textit{If } F \textit{ is } \text{IFR(DFR)}, \textit{ then} \end{aligned}$$

$$M(t) \leqq (\geqq)\; tF(t) \Big/ \int_0^t \bar{F}(x)\,dx \;\; \leqq (\geqq)\; t/\mu_1 \tag{2.6}$$

for all $t \geqq 0$.

PROOF.

(i) By Corollary 2.2 (i), $E[N_A(t)] \leqq E[N_B(t)]$. By Theorem 2.3 (ii) and (iii)

$$\lim_{t\to\infty} E[N_A(t)]/t = 1 \Big/ \int_0^t \bar{F}(x)\,dx \leqq \lim_{t\to\infty} E[N_B(t)]/t = M(T)/T + 1/T,$$

which implies

$$M(T) \geqq T \Big/ \int_0^T \bar{F}(x)\,dx - 1 \qquad \text{for all} \quad T > 0.$$

Obviously $\int_0^T \bar{F}(x)\,dx \leqq \mu_1$ implies $T/\int_0^T \bar{F}(x)\,dx - 1 \geqq T/\mu_1 - 1$.

(ii) By Corollary 2.1 (i), $E[N_A^*(t)] \geqq (\leqq)\ E[N_B^*(t)]$. By Theorem 2.3 (ii) and (iii)

$$M(T)/T = \lim_{t\to\infty} E[N_B^*(t)]/t \leqq (\geqq) \lim_{t\to\infty} E[N_A^*(t)]/t$$
$$= F(T) \Big/ \int_0^T \bar{F}(x)\,dx \leqq (\geqq)\ 1/\mu_1 .$$

The last inequality follows by noting that $\bar{H}_T(t)$ [see (2.4)] is decreasing (increasing) in T if F is IFR(DFR). ||

Smith, 1961, has given the inequality $M(t) < 1/\bar{F}(t)$ which is true for all F. Since

$$tF(t) \Big/ \int_0^t \bar{F}(x)\,dx \leqq t \Big/ \int_0^t \bar{F}(x)\,dx \leqq 1/\bar{F}(t).$$

(ii) of Theorem 2.4 is an improvement on Smith's result when F is IFR. (This observation is due to the referee.)

Equality is approximately attained in (2.5) for intervals $(t - \epsilon, t)$ where $t = k\mu_1 (k = 1, 2, \cdots)$ by the distribution degenerate at the mean μ_1, and of course $M(t) \geqq 0$ is sharp for $t < \mu_1$. Equality is attained in (2.6) for the Poisson process. Note that inequality (2.6) is an improvement on (2.5) in the DFR case since

$$tF(t) \Big/ \int_0^t \bar{F}(x)\,dx \geqq \int_0^t F(x)\,dx \Big/ \int_0^t \bar{F}(x)\,dx = t \Big/ \int_0^t \bar{F}(x)\,dx - 1.$$

From (2.5) and (2.6) we see that

$$1/\mu_1 - 1/T \leqq M(T)/T \leqq F(T) \Big/ \int_0^T \bar{F}(x)\,dx \leqq 1/\mu_1$$

when F is IFR. Hence when F is IFR the expected numbers of failures per unit of time under the two replacement policies do not differ by more then $1/T$ in the limit as $t \to \infty$.

3. Proofs of the Theorems of Section 2.

Proof of Theorem 2.1. Assume F is IFR. First let us suppose $0 \leqq t \leqq T$, where T is the replacement interval. Let $P[N(t) \geqq n \mid x]$ denote the probability that

$N(t) \geqq n$, given that the age of the unit in operation at time 0 is x. Then we shall show that

$$P[N(t) \geqq n \mid x] \geqq P[N_A^*(t) \geqq n \mid x] \geqq P[N_B^*(t) \geqq n]. \tag{3.1}$$

For $n = 0$, (3.1) is trivially true. For $n > 0$, we can rewrite (3.1) as

$$\begin{aligned} \int_0^t F^{(n-1)}(t-u)\, dF_x(u) &\geqq \int_0^t F^{(n-1)}(t-u)\, dF_x^T(u) \\ &\geqq \int_0^t F^{(n-1)}(t-u)\, dF(u) \end{aligned} \tag{3.1'}$$

where $F^{(n)}(t)$ denotes the n-fold convolution of F with itself and

$$F_x(u) = [F(x+u) - F(x)]/\bar{F}(x).$$

$F_x^T(u)$ is the distribution of the time to the first failure when the age of the unit in operation at time 0 is x and planned replacement is scheduled for $T - x$ if no failure intervenes. We need specify the distribution $F_x^T(u)$ only on $[0, t]$:

$$\begin{aligned} F_x^T(u) &= [F(x+u) - F(x)]/\bar{F}(x) && \text{if} \quad u \leqq T - x \\ &= [F(T) - F(x) + \bar{F}(T)F(u - T + x)]/\bar{F}(x) && \text{if} \quad T - x \leqq u \leqq t. \end{aligned}$$

To prove (3.1′) we need only show

$$F_x(u) \geqq F_x^T(u) \geqq F(u) \qquad \text{for } 0 \leqq u \leqq t \tag{3.2}$$

since $F^{(n-1)}(t-u)$ is decreasing in u. For $u \leqq T - x$

$$F_x(u) = F_x^T(u) = [F(x+u) - F(x)]/\bar{F}(x) \geqq F(u)$$

since F is IFR. For $t - x \leqq u \leqq t$

$$[F(x + u - T + T) - F(T)]/\bar{F}(T) \geqq F(x + u - T)$$

implies $F(x+u) \geqq F(T) + \bar{F}(T)\, F(u - T + x)$ and so

$$F_x(u) = \frac{F(x+u) - F(x)}{\bar{F}(x)} \geqq \frac{\bar{F}(x) - \bar{F}(T) + \bar{F}(T)F(u - T + x)}{\bar{F}(x)} = F_x^T(u)$$

proves the first inequality in (3.2). Also for $T - x < u \leqq t$, $\bar{F}(u - T + x)/\bar{F}(u)$ is increasing in u by (1.1) since we may assume $x \leqq T$. Therefore

$$\bar{F}(u - T + x)/\bar{F}(u) \leqq \bar{F}(x)/\bar{F}(T)$$

since $0 \leqq u \leqq T$. Rearrangement yields

$$\bar{F}(u) \geqq \bar{F}(T)\, \bar{F}(u - T + x)/\bar{F}(x),$$

so that

$$F_x^T(u) = [\bar{F}(x) - \bar{F}(T) + \bar{F}(T)F(u - T + x)]/\bar{F}(x) \geqq F(u)$$

which completes the proof of (3.2). From (3.2) we deduce that for $x > 0$ and $0 \leqq t \leqq T$

$$P[N(t) \geqq n \mid x] \geqq P[N_A^*(t) \geqq n \mid x] \geqq P[N_B^*(t) \geqq n].$$

Now suppose $kT < t \leqq (k+1)T$, where $k \geqq 1$. The proof proceeds by induction on k. Assume (3.1) is true for $0 \leqq t \leqq kT$. For $n = 0$, (3.1) is trivially true. For $n > 0$, write

$$P[N(t) \geqq n] = \sum_{r=0}^{n} \int_0^T \{P[N(T) = r \mid \delta(T) = x]P[N(t-T) \geqq n - r \mid \delta(T) = x]\}\, d_x\, P[\delta(T) \leqq x]$$

$$P[N_A^*(t) \geqq n] = \sum_{r=0}^{n} \int_0^T \{P[N_A^*(T) = r \mid \delta(T) = x]P[N_A^*(t-T) \geqq n - r \mid \delta(T) = x]\}\, d_x\, P[\delta(T) \leqq x]$$

and

$$P[N_B^*(t) \geqq n] = \sum_{r=0}^{n} \int_0^T \{P[N_B^*(T) = r \mid \delta(T) = x]P[N_B^*(t-T) \geqq n - r]\}\, d_x\, P[\delta(T) \leqq x]$$

where $\delta(T)$ is a random variable denoting the age of the unit in use at time T. By inductive hypothesis

$$P[N(t-T) \geqq n-r \mid \delta(T) = x] \geqq P[N_A^*(t-T) \geqq n-r \mid \delta(T) = x] \geqq P[N_B^*(t-T) \geqq n-r].$$

Also

$$P[N(T) = r \mid \delta(T) = x] = P[N_A^*(T) = r \mid \delta(T) = x] = P[N_B^*(T) = r \mid \delta(T) = x]$$

since all three policies coincide on $[0, T]$. Hence (3.1) follows for $kT \leqq t \leqq (k+1)T$ for all $k \geqq 1$ by the axiom of mathematical induction.

For F DFR the proof is similar with the inequalities reversed. ||

Proof of Theorem 2.2 (Due to Albert W. Marshall). Let $\{X_k\}_{k=1}^{\infty}$ denote a realization of the lives of successive components. We shall compute what would have occurred under an age and under a block replacement policy. Let $T_A^n(T_B^n)$ denote the time of the nth removal under an age (block) replacement policy. Then

$$T_A^n = \min(T_A^{n-1} + T, T_A^{n-1} + X_n), \qquad T_B^n = \min(T_B^{n-1} + \alpha, T_B^{n-1} + X_n)$$

where $\alpha (0 \leqq \alpha \leqq T)$ is the remaining life to a scheduled replacement. Since initially $T_A^1 = T_B^1$, we have by induction $T_A^n \geqq T_B^n$. Thus for any realization

$\{X_k\}_{k=1}^{\infty}$ $N_A(t)$ is smaller than $N_B(t)$. By a similar argument $N(t)$ is smaller than $N_A(t)$ for any realization. ||

4. Renewal theory consequences. A renewal process is an IFR(DFR) renewal process if the underlying distribution F is IFR(DFR). This does not imply that $N(t)$, the renewal random variable associated with an IFR(DFR) renewal process, is IFR(DFR). (See Barlow, Marshall, Proschan, 1963.) However, just as the geometric (exponential) distribution is a natural comparison distribution for IFR and DFR discrete (continuous) random variables, the Poisson process serves as a natural comparison process for IFR and DFR renewal processes. In Corollary 2.1 we saw that the mean of an IFR(DFR) renewal random variable is dominated (subordinated) by the mean of an associated Poisson random variable. This is also true of the binomial moments and, indeed, even the variance.

We define the mth binomial moment, $B_m(t)$, as

$$B_m(t) = \sum_{j=0}^{\infty} \binom{j}{m} P[N(t) = j].$$

The following result is well known and we omit the proof.

LEMMA 4.1. *For any renewal counting process* $\{N(t); t \geqq 0\}$,

$$B_m(t) = M^{(m)}(t)$$

where $M^{(m)}(t)$ *denotes the m-fold convolution of* $M(t) = E[N(t)]$.

A stationary renewal process $\{\hat{X}_k\}_{k=1}^{\infty}$ is one for which $\hat{X}_1$ has distribution

$$\hat{F}(t) = \int_0^t \bar{F}(x)\, dx/\mu_1$$

and $\hat{X}_k (k = 2, 3, \cdots)$ are independently distributed according to F. Denote a stationary renewal counting process by $\{\hat{N}(t); t \geqq 0\}$. It is known (Cox, 1962, page 46) that $E[\hat{N}(t)] = t/\mu_1$ for this process. A useful comparison can be made between renewal counting processes and their associated stationary processes.

THEOREM 4.1. *Let* $\{N(t) : t \geqq 0\}$ *denote a renewal counting process governed by* F *with mean* μ_1. *Let* $\{\hat{N}(t); t \geqq 0\}$ *denote the associated stationary process. If*

$$\int_t^{\infty} \bar{F}(x)\, dx/\bar{F}(t) \leqq (\geqq)\, \mu_1 \qquad \text{for all} \quad t \geqq 0 \tag{4.1}$$

then

(i) $$P[\hat{N}(t) \geqq n] \geqq (\leqq) P[N(t) \geqq n]$$

(ii) $$B_m(t) = E\binom{N(t)}{m} \leqq (\geqq) \frac{t^m}{m!(\mu_1)^m}$$

(iii) $$M_n(t) = E(N^n(t)) \leqq (\geqq) \sum_{j=0}^{\infty} j^n (t/\mu_1)^j e^{-t/\mu_1}/j!$$

for $n, m = 0, 1, 2, \cdots$ *and* $0 \leqq t < \infty$.

PROOF. By hypothesis

$$\hat{F}(t) = \int_0^t \bar{F}(x)\, dx/\mu_1 \geqq (\leqq)\ F(t).$$

Therefore

$$P[\hat{N}(t) \geqq n] = \int_0^t F^{(n-1)}(t - x)\, d\hat{F}(x) \geqq \int_0^t F^{(n-1)}(t - x)\, dF(x)$$
$$= P[N(t) \geqq n],$$

proving (i).

By (i), $E[N(t)] = M(t) \leqq E[\hat{N}(t)]$. Since $E[\hat{N}(t)] = t/\mu_1$ (Cox, 1962, p. 46), (ii) follows from Lemma 4.1.

To obtain (iii) note that $x^n = \sum_{m=1}^{n} x(x-1) \cdots (x-m+1) S_n^m$ where S_n^m are Stirling numbers of the second kind (Jordan, 1950, p. 168). Therefore $E[N^n(t)] = \sum_{m=1}^{n} m!\, B_m(t) S_n^m$. Since S_n^m are positive (Jordan, 1950, p. 169), (iii) follows from (ii). ||

The importance of Theorem 4.1 stems from the interpretation of (4.1). Note that (4.1) implies that the mean residual life of a unit aged t is less (greater) than the mean life of a new component. If F is a IFR(DFR) with mean μ_1, then (4.1) is true. Of course, the converse is not true and (4.1) is a significant weakening of the IFR(DFR) assumption.

THEOREM 4.2. *If* $\{N(t);\, t \geqq 0\}$ *is an* IFR(DFR) *renewal counting process, then*

$$\mathrm{Var}[N(t)] \leqq (\geqq)\ E[N(t)] = M(t).$$

The inequality is sharp.

PROOF. Assume $\{N(t);\, t \geqq 0\}$ is an IFR renewal counting process. Since

$$\mathrm{Var}[N(t)] = \int_0^t [2M(t-x) + 1 - M(t)]\, dM(x)$$

we need only show

$$\int_0^t [2M(t-x) - M(t)]\, dM(x) \leqq 0.$$

But $M(x) \leqq M(t) - M(t-x)$ by (iv) of Corollary 2.1 implies that we need only show $\int_0^t [M(t-x) - M(x)]\, dM(x) \leqq 0$. Clearly

$$\int_0^t [M(t-x) - M(x)]\, dM(x) = \int_0^{t/2} [M(t-x) - M(x)]\, dM(x)$$
$$+ \int_{t/2}^t [M(t-x) - M(x)]\, dM(x).$$

Let $y = t - x$, then

$$\int_{t/2}^t [M(t-x) - M(x)]\, dM(x) = \int_0^{t/2} [M(t-y) - M(y)]\, dM(t-y).$$

Hence we need only show

$$\int_0^{t/2} [M(t-x) - M(x)]\, dM(x) \leq \int_0^{t/2} [M(t-x) - M(x)]\, d[M(t) - M(t-x)].$$

This follows immediately, since $M(t - x) - M(x)$ is non-increasing in x, $[M(t - x) - M(x)] \geq 0$ for $0 \leq x \leq t/2$ and $M(x) \leq M(t) - M(t - x)$.

All inequalities are reversed if F is DFR. Equality is attained by the Poisson process. ||

It is well known (Feller, 1948) that

$$\lim_{t\to\infty} \operatorname{Var}[N(t)]/E[N(t)] = \sigma^2/\mu_1^2 \leq 1$$

when F is IFR. Therefore, Theorem 4.2 is in a sense an extension of this result to all t.

5. Generalizations. Next we obtain a generalization of the inequality $M(t) \leq (\geq)\ t/\mu_1$ which holds when successive replacements have *different* failure distributions but a common mean. The method of proof is quite different from that used in Theorem 2.1 or Theorem 4.1. We will need to define the generalized renewal function

$$M_0(t) = F_1(t) + F_1 * F_2(t) + F_1 * F_2 * F_3(t) + \cdots. \tag{5.1}$$

Note that $M_0(t)$ is the expected number of renewals in a stochastic process in which the first unit has distribution F_1, its replacement has distribution F_2, etc.

THEOREM 5.1. *Let $F_1, F_2, F_3, \cdots$ be non-degenerate* IFR(DFR) *distributions with common mean μ_1 and assume*

$$F_i(t) \not\equiv G(t) = 1 - e^{-t/\mu_1} \qquad \textit{for } t \geq 0$$

and $i = 1, 2, \cdots$. Then

$$M_0(t) < (>)\ t/\mu_1 \qquad \textit{for } t > 0. \tag{5.2}$$

PROOF. Assume F_i $(i = 1, 2, \cdots)$ are IFR. First suppose $F_1 = F_2 = \cdots$. Then

$$M_0(t) = \sum_{k=1}^{\infty} F_1^{(k)}(t) < \sum_{k=1}^{\infty} G^{(k)}(t) = t/\mu_1$$

for $0 < t \leq \mu_1$. (See Barlow, Marshall, 1963.) Suppose there exists $\tau > \mu_1$ such that $M_0(\tau) = \tau/\mu_1$. Then, since F has mass in $[0, \tau]$

$$\tau/\mu = M_0(\tau) = \int_0^{\tau} [1 + M_0(\tau - x)]\, dF_1(x) < \int_0^{\tau} \left[1 + \frac{\tau - x}{\mu_1}\right] dF_1(x)$$

or

$$\tau/\mu_1 < F_1(\tau) + \int_0^\tau F_1(x)\, dx/\mu_1$$

which implies

$$\left[\int_0^\tau \bar{F}_1(x)\, dx/F_1(\tau)\right] < \mu_1 .$$

But this contradicts (2.6) of Theorem 2.4. Hence $M_0(\tau) < \tau/\mu_1$, $0 < \tau < \infty$, for this special case.

The argument proceeds by induction. Suppose the theorem is true for all sequences of distributions of the form $H_1, H_2, \cdots, H_k = H_{k+1} = H_{k+2} = \cdots$ where the H_i satisfy the IFR assumption.

$$M_1(t) = F_2(t) + F_2 * F_3(t) + \cdots + F_2 * F_3 * \cdots * F_{k+1}(t) + F_2 * F_3 * \cdots * F_{k+1} * F_{k+1}(t) + \cdots$$

and

$$M_0(t) = \int_0^t [1 + M_1(t - x)]\, dF_1(x).$$

As before, $M_0(t) < t/\mu_1$ for $0 < t \leqq \mu_1$. Suppose there exists $\tau > \mu_1$ such that $M_0(\tau) = \tau/\mu_1$. This implies

$$\tau/\mu_1 = \int_0^\tau [1 + M_1(\tau - x)]\, dF_1(x) < F_1(\tau) + \int_0^\tau F_1(x)\, dx$$

and

$$\int_0^\tau \bar{F}_1(x)/F_1(\tau) < \mu_1 .$$

This is a contradiction. Theorem 5.1 follows by the axiom of mathematical induction.

All inequalities are reversed for DFR distributions. ||

We note that Smith (1960) and Chow and Robbins (1963) have considered sequences of non-identically distributed random variables and have given conditions under which $\lim_{t\to\infty} M(t)/t = 1/\mu_1$.

The method of proof used in Theorem 5.1 can be used to generalize the bound on $M(t)$ in yet another direction.

THEOREM 5.2. *Assume F has density f, failure rate $r(x) = f(x)/[\bar{F}(x)]$, and mean μ_1.*

(i) *If $r(x) \geqq \alpha$ for all x, then $M(t) \leqq t/\mu_1 + 1/\alpha\mu_1 - 1$.*

(ii) *Suppose there exists $\delta > 0$ such that $f(x) > 0$ for $0 < x < \delta$. If $r(x) \leqq \beta$, then $M(t) \geqq t/\mu_1 + 1/\beta\mu_1 - 1$. Equality is attained in* (i) *and* (ii) *for the Poisson process.*

PROOF. If $F(x) = 1 - e^{-x/\mu_1}$ the bounds are attained. Hence suppose $F(x) \not\equiv 1 - e^{-x/\mu_1}$. Then

$$\inf_x r(x) < 1/\mu_1 < \sup_x r(x) \tag{5.3}$$

(see Barlow Marshall, Proschan, 1963). (5.3) implies $\alpha < 1/\mu_1 < \beta$, and $1/\alpha\mu_1 - 1 > 0$, $1/\beta\mu_1 - 1 < 0$. Since $M(0) = 0$,

$$t/\mu_1 + 1/\mu_1\beta - 1 < M(t) < t/\mu_1 + 1/\alpha\mu_1 - 1.$$

for t sufficiently small.

(i) If $\alpha = 0$, we are done. Hence assume $\alpha > 0$ and suppose there exists $0 < \tau < \infty$ such that

$$M(\tau) = \tau/\mu_1 + 1/\alpha\mu_1 - 1$$

and

$$M(t) < t/\mu_1 + 1/\alpha\mu_1 - 1 \qquad \text{for } t < \tau.$$

Then

$$\begin{aligned} \tau/\mu_1 + 1/\alpha\mu_1 - 1 = M(\tau) &= \int_0^\tau [1 + M(\tau - x)]\, dF(x) \\ &< \int_0^\tau [(\tau - x)/\mu_1 + 1/\alpha\mu_1]\, dF(x) \end{aligned} \tag{5.4}$$

since $r(x) \geqq \alpha$ implies $F(t) \geqq 1 - e^{-\alpha t}$ and hence F has mass in $[0, \tau]$. (5.4) implies

$$\tau/\mu_1 + 1/\alpha\mu_1 - 1 < \frac{1}{\alpha\mu_1} F(\tau) + \int_0^\tau \frac{F(x)}{\mu_1}\, dx.$$

But $r(x) \geqq \alpha$ implies $f(x) \geqq \alpha\bar{F}(x)$ and so

$$\bar{F}(\tau) = \int_\tau^\infty f(x)\, dx \geqq \alpha \int_\tau^\infty \bar{F}(x) dx$$

which implies

$$\frac{1}{\alpha\mu_1} F(\tau) + \int_0^\tau \frac{F(x)\, dx}{\mu_1} \leqq \frac{\tau}{\mu_1} + \frac{1}{\alpha\mu_1} - 1,$$

a contradiction. Hence, actually

$$M(t) < t/\mu_1 + 1/\alpha\mu_1 - 1$$

when $F(t) \not\equiv 1 - e^{-t/\mu_1}$.

(ii) If $\beta = \infty$, the inequality follows from Corollary 2.2 (iii). Hence suppose $\beta < \infty$. There exists $0 < \tau < \infty$ such that

$$M(\tau) = \tau/\mu_1 + 1/\beta\mu_1 - 1$$

and

$$M(t) > t/\mu_1 + 1/\beta\mu_1 - 1 \qquad \text{for } t < \tau.$$

Then

$$\frac{\tau}{\mu_1} + \frac{1}{\beta\mu_1} - 1 = M(\tau) = \int_0^\tau [1 + M(\tau - x)]\, dF(x)$$

$$> \int_0^\tau \left[\frac{\tau - x}{\mu_1} + \frac{1}{\beta\mu_1}\right] dF(x)$$

since F has mass in $[0, \tau]$. Therefore

$$\frac{\tau}{\mu_1} + \frac{1}{\beta\mu_1} - 1 > \frac{F(\tau)}{\beta\mu_1} + \int_0^\tau \frac{F(x)\, dx}{\mu_1} \geqq \frac{\tau}{\mu_1} + \frac{1}{\beta\mu_1} - 1$$

since $r(x) \leqq \beta$. This is a contradiction and therefore

$$M(t) > t/\mu_1 + 1/\beta\mu_1 - 1$$

for $0 < t < \infty$ when $F(t) \not\equiv 1 - e^{-t/\mu_1}$. ||

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