TABLES OF DISTRIBUTION-FREE TOLERANCE LIMITS

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Consider an ordered sample $X_1 \leqq X_2 \leqq \cdots \leqq X_n$ and a second finite random sample $Y_1, Y_2, \cdots, Y_N$ from an infinite population with a continuous density function, $f(x)$.

The one-tolerance limit problem is: for any integer r, such that $1 \leqq r \leqq n$, and for any integer N_0 such that $0 \leqq N_0 \leqq N$, find the probability that at least N_0 of the Y_i's are greater than X_r. The two-tolerance limit problem can be similarly stated: for any pair of integers r_1 and r_2, such that $1 \leqq r_1 \leqq r_2 \leqq n$, and for any integer N_0, such that $0 \leqq N_0 \leqq N$, find the probability that at least N_0 of the Y_i's are greater than X_{r_1} and less than X_{r_2}.

The probability that N_0 of the Y_i's lie above X_r is given by

$$(1) \qquad P(N_0) = \binom{N_0 + n - r}{N_0}\binom{N - N_0 + r - 1}{N - N_0} \Big/ \binom{N + n}{N}$$

From the theory of statistically equivalent blocks as defined by Tukey [2], it can be shown that the probability of at least N_0 of the Y_i's lying between X_{r_1} and X_{r_2} is equal to the probability of at least N_0 of the Y_i's being greater than X_r, where $r = r_1 + n + 1 - r_2$, i.e., all two-tolerance limit cases are equal and are reducible to (1), the general one-tolerance limit case.

From Eq. (1), iterative computation was performed using $P(N) = n!\,(N + n - r)!/(n - r)!\,(N + n)!$ and $P(N_0 - 1) = P(N_0)(N - N_0 + r)N_0/(N_0 + n - r)(N - N_0 + 1)$ such that $\sum_{N_k=N_0}^{N} P(N_k) \geqq K$.

The tables show that the probability is at least K that N_0, or more, of a second sample of N will lie between the r_1 and r_2 values of the first sample of size n. Or, equivalently, it gives the least N_0 which lie above the rth lowest.

Values for the proportion of the population covered by the specified interval are also included primarily to illustrate the rapidity with which these limiting values are approached from a finite second sample. These limiting values can be obtained from reference [1]. These proportions do not always appear to converge monotonically to the limiting values due to the discreteness of N_0 and N.

REFERENCES

[1] Murphy, R. B. (1948). Non-parametric tolerance limits. *Ann. Math. Statist.* **19** 581–589.
[2] Tukey, J. W. (1947). Nonparametric estimation II: Statistically equivalent blocks and tolerance regions—the continuous case. *Ann. Math. Statist.* **18** 529–539.

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TABLE I

Distribution-free tolerance limits

K	n	$r = 1$, N							$r = 2$, N						
		5	10	25	50	75	100	∞	5	10	25	50	75	100	∞
0.50	5	5	9	22	44	65	87	0.870	3	7	17	34	52	69	0.686
	10	5	9	23	47	70	93	0.933	4	8	21	42	63	84	0.837
	25	5	10	24	49	73	97	0.972	5	10	23	47	70	94	0.933
	50	5	10	25	49	74	99	0.986	5	10	24	48	73	97	0.966
	75	5	10	25	50	75	99	0.991	5	10	25	49	73	98	0.978
	100	5	10	25	50	75	99	0.993	5	10	25	49	74	98	0.983
0.75	5	4	7	19	38	57	76	0.757	2	5	13	27	41	54	0.545
	10	4	9	22	43	65	87	0.870	4	7	18	37	56	75	0.752
	25	5	9	24	47	71	94	0.946	4	9	22	44	67	89	0.895
	50	5	10	24	49	73	97	0.972	5	9	23	47	71	94	0.947
	75	5	10	25	49	74	98	0.982	5	10	24	48	72	96	0.964
	100	5	10	25	49	74	99	0.986	5	10	24	48	73	97	0.973
0.90	5	3	6	15	31	47	62	0.630	1	3	10	20	30	41	0.416
	10	4	7	19	39	59	79	0.794	3	6	16	32	49	65	0.663
	25	4	9	22	45	68	91	0.912	4	8	21	42	63	84	0.853
	50	5	9	23	47	71	95	0.954	4	9	22	45	69	92	0.924
	75	5	9	24	48	72	96	0.970	4	9	23	47	70	94	0.949
	100	5	10	24	48	73	97	0.977	5	9	23	47	71	95	0.961
0.95	5	2	5	13	27	40	54	0.549	1	3	8	16	25	33	0.342
	10	3	7	18	36	55	73	0.741	2	5	14	29	44	59	0.605
	25	4	8	21	43	66	88	0.887	3	7	20	40	61	81	0.823
	50	4	9	23	46	70	93	0.941	4	8	22	44	67	90	0.908
	75	4	9	23	47	71	95	0.961	4	9	23	46	69	93	0.938
	100	5	9	24	48	72	96	0.970	4	9	23	47	70	94	0.953
0.99	5	1	3	9	19	29	39	0.398	0	1	4	10	15	21	0.222
	10	2	5	14	30	46	62	0.630	1	4	11	23	35	48	0.495
	25	3	7	19	40	61	82	0.831	3	6	17	36	55	74	0.762
	50	4	8	21	44	67	90	0.912	3	7	20	42	64	85	0.874
	75	4	8	22	46	69	92	0.940	4	8	21	44	67	90	0.914
	100	4	9	23	46	70	94	0.954	4	8	22	45	68	92	0.935

K	n	$r = 3$, N							$r = 4$, N						
		5	10	25	50	75	100	∞	5	10	25	50	75	100	∞
0.50	5	2	5	12	25	37	50	0.495	1	3	8	16	23	31	0.313
	10	4	7	19	37	56	74	0.741	3	6	16	32	48	65	0.644
	25	5	9	22	45	67	90	0.894	4	9	21	43	64	86	0.855
	50	5	10	24	47	71	95	0.946	5	9	23	46	70	93	0.927
	75	5	10	24	48	72	97	0.964	5	10	24	48	71	95	0.951
	100	5	10	25	49	73	97	0.973	5	10	24	48	72	96	0.963

TABLE I—Continued

K	n	r = 3, N: 5	10	25	50	75	100	∞	r = 4, N: 5	10	25	50	75	100	∞
0.75	5	1	3	9	18	27	36	0.359	1	2	4	9	14	19	0.193
	10	3	6	16	32	48	64	0.644	2	5	13	27	40	54	0.542
	25	4	8	21	42	63	84	0.849	4	8	20	40	60	80	0.804
	50	4	9	23	46	69	92	0.923	4	9	22	45	67	90	0.899
	75	5	9	23	47	71	94	0.948	5	9	23	46	70	93	0.932
	100	5	9	24	48	72	96	0.961	5	9	23	47	71	95	0.949
0.90	5	1	2	5	12	18	24	0.246	0	1	2	5	8	11	0.112
	10	2	5	13	27	40	54	0.550	1	4	10	21	33	44	0.448
	25	3	7	19	39	59	79	0.800	3	7	18	37	55	74	0.751
	50	4	8	22	44	66	89	0.897	4	8	21	43	64	86	0.871
	75	4	9	23	46	69	92	0.930	4	8	22	45	67	90	0.913
	100	4	9	23	47	70	94	0.947	4	9	23	46	69	92	0.934
0.95	5	0	1	4	9	13	18	0.189	0	0	1	3	5	7	0.076
	10	2	4	11	23	36	48	0.493	1	3	9	18	28	38	0.393
	25	3	7	18	37	56	75	0.768	3	6	17	34	52	70	0.718
	50	4	8	21	43	65	87	0.897	3	7	20	41	62	84	0.852
	75	4	8	22	45	68	90	0.918	4	8	21	44	66	88	0.899
	100	4	9	22	46	69	92	0.938	4	8	22	45	68	91	0.924
0.99	5	0	0	2	4	7	9	0.105	0	0	0	1	2	2	0.032
	10	1	2	8	18	27	37	0.388	0	2	6	13	20	28	0.297
	25	2	5	16	33	51	68	0.704	2	5	14	30	46	63	0.651
	50	3	7	19	40	61	82	0.842	3	6	18	38	59	79	0.812
	75	3	7	20	43	65	87	0.892	3	7	20	41	63	85	0.872
	100	3	8	21	44	67	90	0.918	3	8	21	43	65	88	0.903

K	n	r = 5, N: 5	10	25	50	75	100	∞	r = 6, N: 5	10	25	50	75	100	∞
0.50	5	0	1	3	6	10	13	0.129	—	—	—	—	—	—	——
	10	3	5	14	27	41	55	0.548	2	4	11	23	34	45	0.451
	25	4	8	20	41	61	82	0.815	4	8	19	39	58	78	0.776
	50	5	9	23	45	68	91	0.907	5	9	22	44	67	89	0.887
	75	5	10	24	47	70	94	0.937	5	9	23	46	69	93	0.924
	100	5	10	24	48	72	95	0.953	5	10	24	47	71	94	0.943
0.75	5	0	0	1	2	4	5	0.055	—	—	—	—	—	—	——
	10	2	4	11	22	33	44	0.444	1	3	8	17	26	35	0.350
	25	3	7	19	37	56	75	0.760	3	7	17	35	53	71	0.717
	50	4	8	22	43	65	87	0.877	4	8	21	42	64	85	0.855
	75	4	9	23	45	68	91	0.917	4	9	22	45	67	90	0.902
	100	5	9	23	46	70	93	0.937	4	9	23	46	69	92	0.926

TABLE I—Continued

K	n	r = 5, N = 5	10	25	50	75	100	∞	r = 6, N = 5	10	25	50	75	100	∞
0.90	5	0	0	0	1	1	2	0.020	—	—	—	—	—	—	—
	10	1	3	8	17	26	34	0.354	1	2	6	12	19	26	0.267
	25	3	6	17	34	52	69	0.705	2	6	15	32	48	65	0.660
	50	4	8	20	44	62	83	0.846	3	7	19	40	60	81	0.822
	75	4	8	21	44	66	88	0.896	4	8	21	43	65	87	0.879
	100	4	9	22	45	68	91	0.921	4	8	22	44	67	90	0.909
0.95	5	0	0	0	0	0	1	0.010	—	—	—	—	—	—	—
	10	1	2	6	14	21	29	0.303	0	1	4	10	15	21	0.222
	25	2	5	15	32	49	65	0.670	2	5	14	30	45	61	0.624
	50	3	7	19	40	60	81	0.826	3	7	18	38	58	78	0.801
	75	4	8	21	43	65	86	0.882	3	8	20	42	63	85	0.864
	100	4	8	21	44	67	89	0.910	4	8	21	43	66	88	0.897
0.99	5	0	0	0	0	0	0	0.003	—	—	—	—	—	—	—
	10	0	1	4	9	15	20	0.218	0	0	2	6	10	13	0.150
	25	1	4	13	28	43	58	0.602	1	4	12	25	39	53	0.555
	50	2	6	17	37	56	76	0.785	2	6	17	35	54	73	0.758
	75	3	7	19	40	61	83	0.852	3	7	19	39	60	81	0.834
	100	3	7	20	42	64	86	0.888	3	7	20	41	63	85	0.874

K	n	r = 7, N = 5	10	25	50	75	100	∞	r = 8, N = 5	10	25	50	75	100	∞
0.50	10	2	3	9	18	27	35	0.355	1	2	6	13	19	26	0.258
	25	4	7	18	37	55	74	0.736	4	7	17	35	52	70	0.697
	50	4	9	22	43	65	87	0.867	4	9	21	42	64	85	0.847
	75	5	9	23	46	68	91	0.911	5	9	23	45	67	90	0.898
	100	5	10	23	47	70	93	0.933	5	9	23	46	69	92	0.923
0.75	10	1	2	6	13	19	26	0.260	0	1	4	8	13	17	0.175
	25	3	6	16	33	50	67	0.675	3	6	15	31	47	63	0.633
	50	4	8	20	41	62	83	0.833	4	8	20	40	60	80	0.811
	75	4	9	22	44	66	88	0.887	4	8	21	43	65	87	0.873
	100	4	9	22	45	68	91	0.915	4	9	22	45	67	90	0.904
0.90	10	0	1	4	8	13	18	0.187	0	0	2	5	8	11	0.115
	25	2	5	14	30	45	60	0.616	2	5	13	27	42	56	0.574
	50	3	7	19	39	58	78	0.798	3	7	18	37	57	76	0.775
	75	4	8	20	42	63	85	0.863	4	8	20	41	62	83	0.848
	100	4	8	21	44	66	88	0.897	4	8	21	43	65	87	0.885

TABLE I—Continued

K	n	r = 7, N: 5	10	25	50	75	100	∞	r = 8, N: 5	10	25	50	75	100	∞
	10	0	1	3	6	10	14	0.150	0	0	1	3	6	8	0.087
	25	2	4	13	27	42	56	0.580	2	4	12	25	38	52	0.537
0.95	50	3	6	18	37	56	76	0.776	3	6	17	36	54	73	0.753
	75	3	7	20	41	62	83	0.848	3	7	19	40	60	81	0.831
	100	4	8	21	43	65	87	0.885	3	8	20	42	63	85	0.872
	10	0	0	1	3	6	8	0.093	0	0	0	1	2	4	0.047
	25	1	3	10	23	36	48	0.511	1	3	9	21	33	44	0.469
0.99	50	2	5	16	34	52	70	0.732	2	5	15	33	50	68	0.708
	75	3	6	18	38	58	79	0.816	2	6	17	37	57	77	0.799
	100	3	7	19	40	62	83	0.860	3	7	19	40	61	82	0.847

K	n	r = 9, N: 5	10	25	50	75	100	∞	r = 10, N: 5	10	25	50	75	100	∞
	10	1	1	4	8	12	16	0.164	0	0	2	3	5	7	0.069
	25	3	7	16	33	49	66	0.657	3	6	15	31	46	62	0.618
0.50	50	4	8	21	41	62	83	0.827	4	8	20	40	61	81	0.807
	75	5	9	22	44	66	89	0.884	5	9	22	44	65	87	0.871
	100	5	9	23	46	69	91	0.913	5	9	23	45	68	90	0.903
	10	0	1	2	4	7	9	0.096	0	0	0	1	2	2	0.028
	25	3	5	14	29	44	59	0.592	2	5	13	27	41	54	0.552
0.75	50	4	7	19	39	59	78	0.790	3	7	19	38	57	76	0.768
	75	4	8	21	42	64	85	0.858	4	8	21	42	63	84	0.844
	100	4	9	22	44	66	89	0.893	4	8	22	43	65	88	0.882
	10	0	0	1	2	3	5	0.054	0	0	0	0	0	1	0.010
	25	2	4	12	25	38	52	0.532	2	4	11	23	35	48	0.492
0.90	50	3	6	18	36	55	74	0.753	3	6	17	35	53	71	0.730
	75	3	7	20	40	61	82	0.832	3	7	19	39	60	80	0.817
	100	4	8	21	42	64	86	0.873	4	8	20	42	63	85	0.861
	10	0	0	0	1	2	3	0.036	0	0	0	0	0	0	0.005
	25	1	4	11	23	35	48	0.496	1	3	10	21	32	44	0.456
0.95	50	3	6	16	34	53	71	0.729	2	6	16	33	51	68	0.706
	75	3	7	19	39	59	79	0.815	3	7	18	38	58	78	0.799
	100	3	7	20	41	62	84	0.860	3	7	20	40	61	83	0.848
	10	0	0	0	0	0	1	0.015	0	0	0	0	0	0	0.004
	25	1	2	8	19	29	40	0.428	0	2	7	17	27	36	0.389
0.99	50	2	5	14	31	48	65	0.683	2	4	14	30	46	63	0.660
	75	2	6	17	36	55	75	0.782	2	6	16	35	54	73	0.765
	100	3	6	18	39	59	80	0.834	2	6	18	38	58	79	0.821