LIMIT THEOREMS FOR STOPPED RANDOM WALKS

BY R. H. FARRELL

Cornell University

1. Summary and introduction. In this paper we prove several asymptotic results about the expected length of time until first passage of a vector valued random walk. We suppose $X_i = (X_i^1, \cdots, X_i^k)$, $i \geqq 1$, is a sequence of independently and identically distributed random vectors. We assume throughout that

$$\max_{1 \leqq j \leqq k} E|X_1^j| < \infty; \qquad \mu_j = EX_1^j, 1 \leqq j \leqq k; \qquad \text{and that} \quad 0 < \min_{1 \leqq j \leqq k} \mu_j . \tag{1}$$

We consider the following kind of first passage problem. Let $p(\cdot, \cdots, \cdot)$ be a real valued function of k variables. Given $c > 0$ define a stopping variable $N(c) \geqq 1$ to be the least integer n such that

$$p\left(\sum_{i=1}^{n} X_i^1, \cdots, \sum_{i=1}^{n} X_i^k\right) > c, \tag{2}$$

with $N(c) = \infty$ if the inequality (2) fails for all $n \geqq 1$. The main result of this paper is:

THEOREM 1. *Suppose the function $p(\cdot, \cdots, \cdot)$ is a homogeneous function of degree one. Given the above assumptions, with probability one $\lim_{c \to \infty} c^{-1}N(c) = 1/p(\mu_1, \cdots, \mu_k)$. Suppose there is a real number $\alpha > 0$ such that*

$$\text{if } \min_{1 \leqq j \leqq k} a_j \geqq 0 \quad \text{then} \quad \alpha p(a_1, \cdots, a_k) \geqq \min_{1 \leqq j \leqq k} a_j . \tag{3}$$

Then

$$\lim_{c \to \infty} E|c^{-1}N(c) - 1/p(\mu_1, \cdots, \mu_k)| = 0. \tag{4}$$

If one views Theorem 1 in the light of Doob [2] Theorem 1 becomes very plausible. It is natural to define a continuous parameter process

$$X(c) = c - p\left(\sum_{i=1}^{N(c)-1} X_i^1, \cdots, \sum_{i=1}^{N(c)-1} X_i^k\right). \tag{5}$$

In case the random variables $X_1^1, \cdots, X_1^k$ are nonnegative, using additional assumptions about $p(\cdot, \cdots, \cdot)$ it can be shown that asymptotically as $c \to \infty$ the process $X(\cdot)$ becomes a stationary Markov process. We plan to examine this question further in another place.

The language of this paper is that of random variables and probability. We use freely the idea of stopped random walks. This makes it necessary to define, in the sequel, many different stopping variables. To simplify notation we use

Received 25 March 1963; revised 9 April 1964.

the same notation for different stopping variables and leave it to the context to distinguish the various random variables.

Theorem 1 is a corollary of Theorem 3, stated below. The sequence of the argument is to prove a one-dimensional result, Theorem 2, then to prove by mathematical induction a k-dimensional result, Theorem 3, for the choice $p(x_1, \cdots, x_k) = \min_{1 \leqq j \leqq k} x_j$. The statements of Theorem 2 and Theorem 3 follow.

THEOREM 2. *Suppose* $\{X_i, i \geqq 1\}$ *is a sequence of independently and identically distributed random variables such that*

$$E|X_1| < \infty \quad \text{and} \quad EX_1 = \mu > 0.$$

If $c > 0$ *is a real number define an integer valued random variable* $N(c)$ *to be the least integer* $n \geqq 1$ *such that* $X_1 + \cdots + X_n > c$, *with* $N(c) = \infty$ *if for all* $n \geqq 1$, $X_1 + \cdots + X_n \leqq c$. *Let* $h(\cdot)$ *be a real valued continuous strictly increasing convex function defined on* $[0, \infty)$ *such that* $h(0) = 0$ *and* $\lim_{x\to\infty} h(x)/x = \infty$. *Assume that* $Eh(|X_1|) \leqq \Delta < \infty$. *Then*

$$(6) \qquad P(N(c) < \infty) = 1; \qquad EN(c) < \infty; \qquad c \leqq \mu EN(c) \leqq c + h^{-1}(\Delta EN(c)).$$

Convex functions of the type mentioned in the statement of Theorem 2 always exist. We discuss this question briefly at the beginning of Section 2.

THEOREM 3. *Suppose* $\{X_i, i \geqq 1\}$ *is a sequence of independently and identically distributed* k*-dimensional random vectors satisfying* (1). *Let* $p(x_1, \cdots, x_k) = \min_{1 \leqq j \leqq k} x_j$ *and let* $N(c)$ *be defined as in* (2). *Then*

$$(7) \qquad P(N(c) < \infty) = 1; \qquad EN(c) < \infty; \qquad \lim_{c\to\infty} c^{-1}EN(c) = 1/(\min_{1 \leqq j \leqq k} \mu_j).$$

We prove Theorem 2 in Section 2 for the sake of completeness and in order to have a result including a uniformity condition. Theorem 3 does not have any uniformity condition. Theorem 4 is a partial generalization of Theorem 2 in regard to a uniformity condition.

THEOREM 4. *Assume the hypotheses and definitions of Theorem* 3. *In addition assume that if* $2 \leqq j \leqq k$, $i \geqq 1$, *then* $X_i^j \geqq 0$, *and that* $0 < \mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_k$. *Let* $h(\cdot)$ *be a real valued nonnegative continuous function defined on* $[0, \infty)$ *which satisfies*

$$(8) \qquad h(0) = 0; \qquad \lim_{x\to\infty} h(x)/x = \infty.$$

There exists a function $a_k(\cdot, \cdot, \cdot)$ *of three variables* (*and also depending on* $h(\cdot)$) *such that if* $\mu > 0$, $\Delta > 0$ *then* $\lim_{c\to\infty} a_k(c, \mu, \Delta) = 0$ *and if*

$$(9) \qquad \max_{1 \leqq j \leqq k} Eh(|x_1^j|) \leqq \Delta, \qquad \min_{1 \leqq j \leqq k} EX_1^j \geqq \mu_1,$$

then

$$(10) \qquad 1/\mu_1 \leqq c^{-1}EN(c) \leqq 1/\mu_1 + a_k(c, \mu_1, \Delta).$$

The statement of Theorem 4 is asymmetric in that we require $X_1^2, \cdots, X_1^k$ to be nonnegative random variables. We believe that this restriction can be removed. But we have not been able to do so using the methods of this paper.

The possibility of proving Theorems 2 and 4 (thereby obtaining results with uniformity conditions) was suggested to the author by arguments in a paper by Kiefer and Sacks [4]. Both Theorems have application to the study of the asymptotic behavior of the expected sample size in sequential tests.

In regard to Theorem 2, since $\lim_{x\to\infty} h(x)/x = \infty$, if $d > 0$ then $\lim_{x\to\infty} h^{-1}(dx)/x = d \lim_{y\to\infty} h^{-1}(y)/y = d \lim_{z\to\infty} z/h(z) = 0$. Therefore from (6) it follows that $\lim_{c\to\infty} c^{-1}EN(c) = \mu^{-1}$. This part of Theorem 2 is known.

The results of this paper on the length of time to first passage are related to the problems of renewal theory. We refer the reader to Smith [6] for a summary of results. The only multidimensional result, along the lines of this paper, in print, seems to be that of Chung [1].

In the proofs we use freely the idea of a stopped random walk and in connection with this an identity due to Wald [7]. Because this identity is used repeatedly we state it here.

Suppose $\{Z_n, n \geqq 1\}$ is a sequence of independently and identically distributed random variables and $E|Z_1| < \infty$. Suppose N is a random variable taking only positive integer values and $EN < \infty$. Suppose for each integer $n \geqq 1$ the event $\{N \geqq n\}$ is independent of the random variables $\{Z_m, m \geqq n\}$. Then

$$E|Z_1 + \cdots + Z_N| < \infty \quad \text{and} \quad E(Z_1 + \cdots + Z_N) = (EN)(EZ_1).$$

Throughout the remainder of this paper we refer to this as Wald's identity and omit further mention of the reference.

In the proofs given below many of the arguments require the definition of new stopping variables. For the sake of brevity the definitions of these stopping variables omit the statement that the value is ∞ if the stated condition fails to hold for any positive integer n. In this context the meaning of statements like "$P(N < \infty) = 1$" is that the exceptional set in question has measure zero.

The proof of Theorem 2 is given in Section 2. The proofs of Theorem 3 and Theorem 4 require several lemmas proven in Section 3. The proof of Theorem 3 is given in Section 4, of Theorem 4 in Section 5, and of Theorem 1 in Section 6. The author is indebted to W. L. Smith for the present form of the statement of Lemma 3 and also for a very much shorter proof.

2. Proof of Theorem 2. Given X_1, let $F(\cdot)$ be defined by

$$F(a) = P(|X_1| \leqq a), \qquad -\infty < a < \infty. \tag{11}$$

Define a real number sequence $\{a_n, n \geqq 1\}$ by

$$a_0 = 0; \quad \text{if} \quad i \geqq 0 \quad \text{then} \quad a_i < a_{i+1}; \quad \int_{a_n+}^{\infty} |x| F(dx) < 1/(n+1)^3. \tag{12}$$

Otherwise the sequence $\{a_n, n \geqq 1\}$ may be arbitrary. Let $h(\cdot)$ be the continuous function satisfying $h(0) = 0$ and

$$\text{(13)} \qquad \text{if} \quad i \geqq 0 \quad \text{and} \quad a_i < x < a_{i+1} \quad \text{then} \quad h'(x) = i + 1,$$

where $h'(\cdot)$ is the derivative of $h(\cdot)$. Then $h(\cdot)$ is a convex function defined on $[0, \infty)$. It is easy to verify that $\lim_{x\to\infty} h(x)/x = \infty$ and that if $i \geqq 0$ and $a_i \leqq x \leqq a_{i+1}$ then $h(x) \leqq (i + 1)x$. Therefore

$$\text{(14)} \qquad \begin{aligned} Eh(|X_1|) &= \int_{-\infty}^{\infty} h(|x|)F(dx) \\ &\leqq \sum_{i=0}^{\infty} \int_{a_{i+}}^{a_{i+1}} (i + 1)xF(dx) \leqq \sum_{n=1}^{\infty} 1/n^2 < \infty. \end{aligned}$$

This shows that a convex function with the properties needed for Theorem 2 always exists.

Suppose N is the least integer $n \geqq 1$ such that $X_1 + \cdots + X_n > c$. From the strong law of large numbers it follows that $P(N < \infty) = 1$. Define random variables $\{N_m, m \geqq 1\}$ by, if $m \geqq 1$

$$\text{(15)} \qquad N_m = N \quad \text{if} \quad N \leqq m; \qquad N_m = m \quad \text{if} \quad N > m.$$

Then $EN_m < \infty$. We may apply Wald's identity to $h(|X_1|) + \cdots + h(|X_{N_m}|)$ and obtain, using Jensen's inequality,

$$\text{(16)} \qquad (Eh(|X_1|))(EN_m) \geqq Eh(|X_{N_m}|) \geqq h(E|X_{N_m}|).$$

But using Wald's identity,

$$\text{(17)} \qquad (EX_1)(EN_m) \leqq c + E|X_{N_m}| \leqq c + h^{-1}[(EN_m)(Eh(|X_1|))].$$

Since as noted earlier $\lim_{x\to\infty} h^{-1}(dx)/x = 0$ if $d > 0$, it follows that $\sup_{m\geqq 1} EN_m < \infty$. By the monotone convergence theorem, $EN = \lim_{m\to\infty} EN_m$. Using (17) and the remarks following, it must be the case that $EN < \infty$.

By definition of $N(c)$, $X_1 + \cdots + X_{N(c)} > c$. Using Wald's identity, it follows that $(EX_1)(EN(c)) \geqq c$. If $h(\cdot)$ is any convex function on $[0, \infty)$ such that $h(0) = 0$ and $h(\cdot)$ is strictly increasing, then by an argument like the above, it now follows that if $Eh(|X_1|) \leqq \Delta$ then

$$c \leqq \mu EN(c) \leqq c + h^{-1}(\Delta EN(c)).$$

That completes the proof of Theorem 2.

3. Proofs of some lemmas.

LEMMA 1. *Let $Z_i = (Z_i^1, \cdots, Z_i^k)$, $i \geqq 1$, be a sequence of independently and identically distributed random vectors. Suppose*

$$\text{(18)} \qquad EZ_1^j = \omega_j, \qquad 1 \leqq j \leqq k; \qquad 0 < \omega_1 \leqq \omega_2 \leqq \cdots \leqq \omega_k.$$

Let $N_k(0)$ be defined to be the least integer $n \geqq 1$ such that

$$\text{(19)} \qquad \min_{1\leqq j\leqq k} \sum_{i=1}^{n} Z_i^j > 0.$$

There exists a sequence of nonnegative real valued functions $\{\alpha_k(\cdot), k \geqq 1\}$ *such that*

$$\text{if} \quad k \geqq 1, \qquad EN_k(0) \leqq \alpha_k(F) < \infty, \tag{20}$$

where F *is the joint distribution function of* Z_1.

Proof. By Theorem 2, if $k = 1$ then

$$\omega_1 EN_1(0) \leqq h^{-1}[(Eh(|Z_1|))(EN_1(0))], \tag{21}$$

where $h(\cdot)$ is a suitable convex function. We define the functions $\alpha_k(\cdot)$ by induction, $k \geqq 1$.

In k dimensions define a sequence of stopping variables $\{M_i, i \geqq 1\}$ by,

$$\begin{aligned} &M_0 = 0 \text{ with probability one; if } j \geqq 0 \text{ then given } M_0 + \cdots + M_j = \\ &m, M_{j+1} \text{ is the least integer } n \geqq 1 \text{ such that } Z^1_{m+1} + \cdots + Z^1_{m+n} > 0. \end{aligned} \tag{22}$$

Then $\{M_i, i \geqq 1\}$ is a sequence of independently and identically distributed random variables, and

$$EM_1 \leqq \alpha_1(F_1), \tag{23}$$

F_1 the distribution function of Z^1_1.

Define $k - 1$ dimensional random vectors by, if $i \geqq 1$,

$$Z^*_i = \sum_{q=M_0+\cdots+M_{i-1}+1}^{M_1+\cdots+M_i} (Z^2_q, \cdots, Z^k_q) = (Z^{*2}_i, \cdots, Z^{*k}_i).$$

Then using Wald's identity, $EZ^*_1 = (EM_1)(\omega_2, \cdots, \omega_k)$. Let W be the least integer $n \geqq 1$ such that $\min_{2 \leqq j \leqq k} \sum_{i=1}^n Z^{*j}_i > 0$. Then $\min_{2 \leqq j \leqq k} \sum_{i=1}^{M_1+\cdots+M_W} Z^j_i > 0$, and this implies that $N(0) \leqq M_1 + \cdots + M_W$. Therefore using Wald's identity

$$EN(0) \leqq (EM_1)(EW). \tag{24}$$

By inductive hypothesis, if F^*_{k-1} is the joint distribution function of Z^*_1, then using (23) and (24)

$$EN(0) \leqq \alpha_1(F_1)\alpha_{k-1}(F^*_{k-1}) = \alpha_k(F).$$

That completes the argument.

Lemma 2. *Suppose* $h(\cdot)$ *is a convex function defined on* $[0, \infty)$ *and* $h(0) = 0$. *Then* $h(x)/x$ *is a nondecreasing function of* $x > 0$.

Proof. Suppose $0 \leqq y \leqq x$. Let $y = \alpha x$ so that $0 \leqq \alpha \leqq 1$. Then since $k(\cdot)$ is a convex function, $h(\alpha x) = h((1 - \alpha)0 + \alpha x) \leqq (1 - \alpha)h(0) + \alpha h(x) = \alpha h(x)$. Therefore $h(y)/y = h(\alpha x)/(\alpha x) \leqq h(x)/x$. The proof is complete.

Lemma 3. *Suppose* $\{Z_i, i \geqq 1\}$ *is a sequence of independently and identically distributed random variables with* $E|Z_1| < \infty$, $EZ_1 = 0$. *Suppose* $h(\cdot)$ *is a real-valued nonnegative function defined on* $[0, \infty)$ *that is continuous,* $h(0) = 0$, *and* $\lim_{x\to\infty} h(x)/x = \infty$. *There exists a sequence* $\{b_i(\cdot), i \geqq 1\}$ *of real valued nonnegative nondecreasing functions (depending on* $h(\cdot)$*) satisfying*

(25) *if* $\Delta > 0$ *then* $\lim_{i\to\infty} b_i(\Delta) = 0$; *if* $n \geqq 1$ *and*

$$Eh(|Z_1|) \leqq \Delta \text{ then } E|n^{-1}\sum_{i=1}^{n} Z_i| \leqq b_n(\Delta). \tag{25}$$

PROOF. Define truncated random variables

$$Z'_i = Z_i \text{ if } |Z_i| \leqq n,\ 1 \leqq i, \text{ and}$$
$$Z'_i = 0 \text{ otherwise, } 1 \leqq i.$$

Let $nS_n = Z_1 + \cdots + Z_n$ and $nS'_n = Z'_1 + \cdots + Z'_n$, $n \geqq 1$.

Since $\lim_{x\to\infty} h(x)/x = \infty$ we may find a continuous nonnegative convex function $h^*(\cdot)$ defined on $[0, \infty)$ satisfying if $x \geqq 0$, $h^*(x) \leqq h(x)$; $\lim_{x\to\infty} h^*(x)/x = \infty$. By Lemma 2, $h^*(x)/x$ is a nondecreasing function of x. We define $\rho(\cdot)$ by $\rho(x) = x/h^*(x)$, $0 < x < \infty$. Then $\rho(\cdot)$ is a nonincreasing function and $\lim_{x\to\infty} \rho(x) = 0$.

We will need the following inequalities. Let $G(\cdot)$ be the distribution function of $|Z_1|$. Then

$$\int_n^\infty |x| G(dx) \leqq \int_n^\infty (|x|/h^*(x))h^*(x)G(dx) \leqq \rho(n)\int_n^\infty h(x)G(dx) \leqq \rho(n)\Delta. \tag{26}$$

Also,

$$n^{-1}EZ_1'^2 = n^{-1}\int_0^n x^2 G(dx) \leqq n^{-\frac{1}{3}} + n^{-1}\int_{n^{1/3}}^n h^*(x)\rho(x)xG(dx) \leqq n^{-\frac{1}{3}} + \rho(n^{\frac{1}{3}})\Delta. \tag{27}$$

The existence of the function $b_1(\cdot)$ is obvious. We proceed inductively to define functions $b_2^*(\cdot), \cdots, b_n^*(\cdot), \cdots$ which satisfy (25). Then for each $n \geqq 2$ we may define $b_n(\cdot)$ as the greatest lower bound to all functions $b_n^*(\cdot)$ satisfying (25) in order to obtain a nondecreasing function of Δ.

Suppose $b_1(\cdot), b_2^*(\cdot), \cdots, b_{n-1}^*(\cdot)$ have been chosen to satisfy (25). Then

$$\begin{aligned}
E|S_n| &= P(|Z_i| \leqq n,\ 1 \leqq i \leqq n)E(|S'_n| \mid |Z_i| \leqq n,\ 1 \leqq i \leqq n) \\
&\quad + P(|Z_i| > n \text{ for some } i,\ 1 \leqq i \leqq n) \\
&\quad \cdot E(|S_n| \mid |Z_i| > n \text{ for some } i,\ 1 \leqq i \leqq n) \\
&\leqq E|S'_n| + nP(|Z_n| > n)E(|S_{n-1}| + n^{-1}|Z_n| \mid |Z_n| > n) \\
&\leqq E|S'_n| + nP(|Z_n| > n)b_{n-1}^*(\Delta) \\
&\quad + P(|Z_n| > n)E(|Z_n| \mid |Z_n| > n) \\
&\leqq E|S'_n| + \rho(n)\Delta + \rho(n)\Delta b_{n-1}^*(\Delta),
\end{aligned} \tag{28}$$

by (26). We use here the fact that S_{n-1} and Z_n are independent random variables so that

$$E(|S_{n-1}| \mid |Z_n| > n) = E|S_{n-1}| \leqq b^*_{n-1}(\Delta).$$

Also by (26) and (27),

$$E|S'_n| \leqq (ES'^2_n)^{\frac{1}{2}} = (n^{-1}EZ'^2_1 + (n-1)n^{-1}(EZ'_1)^2)^{\frac{1}{2}}$$
$$\leqq (n^{-\frac{1}{3}} + \rho(n^{\frac{1}{3}})\Delta + \Delta^2\rho^2(n))^{\frac{1}{2}}.$$

Therefore from (28) it follows that there exist real number sequences $\{\alpha_n(\Delta), n \geqq 1\}$ and $\{\beta_n(\Delta), n \geqq 1\}$ such that $\alpha_n(\Delta) \to 0$ and $\beta_n(\Delta) \to 0$ as $n \to \infty$, and

$$E|S_n| \leqq \alpha_n(\Delta) + \beta_n(\Delta)b^*_{n-1}(\Delta).$$

Define

$$b^*_n(\Delta) = \alpha_n(\Delta) + \beta_n(\Delta)b^*_{n-1}(\Delta).$$

Then $\lim_{n\to\infty} b^*_n(\Delta) = 0$ follows at once. To complete the proof of Lemma 3 we define $b_n(\Delta)$ as described earlier.

LEMMA 4. *Given the hypotheses of Lemma 3, if $\delta > 0$ then*

$$P\left(\sup_{n \leqq q} q^{-1} \sum_{i=1}^{q} Z_i > \delta\right) \leqq \delta^{-1}b_n(\Delta).$$

PROOF. Let $\mathfrak{F}_i$ be the least σ-algebra of sets in which $Z_1 + \cdots + Z_i$, Z_{i+1}, Z_{i+2}, $\cdots$ are all measurable functions. Then if $i \geqq 1$, $\mathfrak{F}_i \supset \mathfrak{F}_{i+1}$. It is easily verified that

$$E(Z_1 \mid \mathfrak{F}_n) = n^{-1}(Z_1 + \cdots + Z_n).$$

Further, $\{E(Z_1 \mid \mathfrak{F}_i), i \geqq 1\}$ is a backward martingale. By Theorem 3.2, Chapter 7, Doob [3], it follows that

$$P\left(\sup_{n\leqq q} q^{-1} \sum_{i=1}^{q} Z_i > \delta\right) \leqq \delta^{-1}E\left|n^{-1}\sum_{i=1}^{n} Z_i\right| \leqq \delta^{-1}b_n(\Delta).$$

LEMMA 5. *Let $Z_i = (Z_i^1, \cdots, Z_i^k)$, $i \geqq 1$, be a sequence of independently and identically distributed random variables. Let $0 < \omega_j = EZ_1^j$, $1 \leqq j \leqq k$ and assume that $\omega_1 < \omega_2 \leqq \omega_3 \leqq \cdots \leqq \omega_k$. Suppose that $h(\cdot)$ is a real valued nonnegative convex function defined on $[0, \infty)$ satisfying $h(0) = 0$ and $\lim_{x\to\infty} h(x)/x = \infty$. We suppose that there is a constant $\alpha > 0$ such that*

$$h(x + y) \leqq \alpha(h(x) + h(y)), \qquad x \geqq 0, \qquad y \geqq 0. \tag{29}$$

Let $W^*(c)$ be the least integer $n \geqq 1$ such that

$$\min_{1\leqq j\leqq k} \sum_{i=1}^{n} Z_i^j > c, \tag{30}$$

and let $W_1^*(c)$ be the least integer $n \geqq 1$ such that

$$Z_1^1 + \cdots + Z_n^1 > c.$$

We suppose that if $c > 0$, $q(c)$ *is a positive integer such that* $\lim_{c\to\infty} q(c) = \infty$ *and* $\lim_{c\to\infty} q(c)/c^{\frac{1}{2}} = 0$. *If* $\Delta \geqq \max_{1\leqq j\leqq k} Eh(|Z_i^j|)$, *then*

$$\begin{aligned}(31)\quad P(W_1^*(c) < W^*(c)) \leqq{}& h^{-1}(\Delta)q(c)(q(c)+1)/(2c)\\ &+ (k-1)b_{q(c)}(4\alpha^2\Delta)/(\min_{2\leqq j\leqq k}(\omega_j - \omega_1)).\end{aligned}$$

(See the statement of Lemma 3 for a definition of $b_n(\cdot)$.)

PROOF. The event $W_1^*(c) < W^*(c)$ implies the event that $Z_1^1 + \cdots + Z_q^1 > c$ and $\sum_{i=1}^{c} (Z_i^1 - Z_i^j) > 0$ for some $q \geqq 1$, $2 \leqq j \leqq k$. Also

$$\begin{aligned}P(\text{some } q \leqq q(c), Z_1^1 + \cdots + Z_q^1 > c) &\leqq \sum_{i=1}^{q(c)} P(Z_1^1 + \cdots + Z_i^1 > c)\\ &\leqq \sum_{i=1}^{q(c)} ih^{-1}(\Delta)/c \leqq h^{-1}(\Delta)q(c)(q(c)+1)/(2c).\end{aligned}$$

Also by Lemma 4, and by virtue of (29),

$$\begin{aligned}&P(\text{some } q > q(c), \sum_{i=1}^{q} (Z_i^1 - Z_i^j) > 0)\\ &\quad = P(\text{some } q > q(c), q^{-1}\sum_{i=1}^{q} [(Z_i^1 - Z_i^j) - (\omega_1 - \omega_j)] > (\omega_j - \omega_1))\\ &\quad \leqq b_{q(c)}(Eh(|Z_1^1 - Z_1^j - \omega_1 + \omega_j|))/(\omega_j - \omega_1) \leqq b_{q(c)}(4\alpha^2\Delta)/(\omega_j - \omega_1).\end{aligned}$$

We use here the assumption of convexity in order to assert that $h(\cdot)$ is nondecreasing and that $h(|\omega_j|) \leqq \Delta$. Formula (31) now follows at once from the preceding steps.

4. Proof of Theorem 3. The proof of Theorem 3 is complicated by the fact that $\mu_1 = \mu_2 = \cdots = \mu_k$ is a difficult case. We begin the argument by "breaking the ties." We assume throughout this section that

$$(32)\qquad 0 < \mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_k .$$

Choose $\epsilon > 0$ such that $\mu_1 - \epsilon > 0$. We may then find $\epsilon_1, \cdots, \epsilon_k$ satisfying

$$(33)\qquad \epsilon = \epsilon_1 > \epsilon_2 > \cdots > \epsilon_k \geqq 0; \qquad 0 < \mu_1 - \epsilon_1 < \mu_2 - \epsilon_2 < \cdots < \mu_k - \epsilon_k .$$

We define new random variables

$$(34)\qquad \begin{aligned} Z_i^j &= X_i^j - \epsilon_j, \qquad 1 \leqq j \leqq k, \qquad 1 \leqq i;\\ Z_i &= (Z_i^1, \cdots, Z_i^k), \qquad 1 \leqq i.\end{aligned}$$

We will prove Theorem 3 for the random variables defined in (34).

Let $N(c)$ be as in the statement of Theorem 3. Let $N^*(c)$ be the least integer $n \geqq 1$ such that $\min_{1\leqq j\leqq k} \sum_{i=1}^{n} Z_i^j > c$. Then

$$(35)\qquad N(c) \leqq N^*(c).$$

That the random variables Z_i^j can take negative values also causes difficulty in the argument. To circumvent this trouble we define nonnegative random variables Y_i^j, $1 \leqq j \leqq k$, $i \geqq 1$. Define stopping variables $\{N_i, i \geqq 1\}$ as follows.

$$N_0 = 0 \text{ with probability one}; \quad \text{if } j \geqq 0 \text{ and } N_0 + \cdots + N_j = m \tag{36}$$
$$\text{then } N_{j+1} \text{ is the least integer } n \geqq 1 \text{ such that } \min_{1 \leqq j \leqq k} \sum_{i=m+1}^{m+n} Z_i^j > 0.$$

$\{N_i, i \geqq 1\}$ is a sequence of independently and identically distributed random variables. By Lemma 1, $EN_1 < \infty$. Define

$$Y_i^j = \sum_{q=N_0+\cdots+N_{i-1}+1}^{N_1+\cdots+N_i} Z_q^j, \quad 1 \leqq j \leqq k, \quad i \geqq 1. \tag{37}$$
$$Y_i = (Y_i^1, \cdots, Y_i^k), \quad i \geqq 1.$$

The random vectors $\{Y_i, i \geqq 1\}$ are independently and identically distributed and

$$EY_1 = (EN_1)(\mu_1 - \epsilon_1, \cdots, \mu_k - \epsilon_k),$$

which is finite.

We define $W(c)$ to be the least integer $n \geqq 1$ such that

$$\min_{1 \leqq j \leqq k} \sum_{i=1}^{n} Y_i^j > c.$$

Then

$$\min_{1 \leqq j \leqq k} \sum_{i=1}^{W(c)} \sum_{q=N_0+\cdots+N_{i-1}+1}^{N_1+\cdots+N_i} Z_q^j > c$$

which implies, using (35), that

$$N(c) \leqq N^*(c) \leqq N_1 + \cdots + N_{W(c)}. \tag{38}$$

By Wald's identity, $EN(c) \leqq (EN_1)(EW(c))$.

We now find an upper bound to $EW(c)$. Let $W_j(c)$ be the least integer $n \geqq 1$ such that $Y_1^j + \cdots + Y_n^j > c$, $1 \leqq j \leqq k$. Then

$$W_1(c) \leqq W(c), \tag{39}$$

and

$$\begin{aligned} EW(c) &= E(W(c) \mid W_1(c) = W(c))P(W_1(c) = W(c)) \\ &\quad + E(W(c) \mid W_1(c) < W(c))P(W_1(c) < W(c)) = E(W_1(c)) \\ &\quad + E(W(c) - W_1(c) \mid W_1(c) < W(c))P(W_1(c) < W(c)). \end{aligned} \tag{40}$$

Given an integer $m \geqq 1$ let $T(c, m, k - 1)$ be the least integer $n \geqq 1$ such that

$$\min_{2 \leqq j \leqq k} \sum_{i=m+1}^{m+n} Y_i^j > c.$$

Let $ET(c, m, k-1) = g(c, k-1)$, which is independent of m. Since the random variables of Y_i^j are nonnegative, if $m = W_1(c)$ and $W_1(c) < W(c)$ then $W(c) - W_1(c) \leqq T(c, m, k-1)$. Further the event $\{m = W_1(c), W_1(c) < W(c)\}$ is independent of the random variable $T(c, m, k-1)$. It follows that

$$E(W(c) - W_1(c) \mid W_1(c) < W(c), W_1(c) = m) \leqq ET(c, m, k-1) = g(c, k-1). \tag{41}$$

Therefore from (40), $EW(c) \leqq EW_1(c) + g(c, k-1)P(W_1(c) < W(c))$ and

$$EN(c) \leqq (EN_1)[(EW_1(c)) + g(c, k-1)P(W_1(c) < W(c))]. \tag{42}$$

It is clear that

$$g(c, k-1) \leqq EW_2(c) + \cdots + EW_k(c). \tag{43}$$

Therefore

$$EN(c) \leqq (EN_1)(EW_1(c)) + (\sum_{j=2}^{k} EW_j(c))(EN_1)P(W_1(c) < W(c)). \tag{44}$$

Using Lemma 5, $\lim_{c\to\infty} P(W_1(c) < W(c)) = 0$. By Theorem 2,

$$\limsup_{c\to\infty} c^{-1} \sum_{j=2}^{k} EW_j(c) < \infty.$$

Again by Theorem 2, $\limsup_{c\to\infty} c^{-1}EW_1(c) \leqq 1/(EN_1)(\mu_1 - \epsilon_1)$. Therefore

$$1/\mu_1 \leqq \liminf_{c\to\infty} c^{-1}EN(c) \leqq \limsup_{c\to\infty} c^{-1}EN(c) \leqq 1/(\mu_1 - \epsilon_1). \tag{45}$$

Since $\epsilon_1 > 0$ is arbitrary, the proof of Theorem 3 is complete.

5. Proof of Theorem 4. Choose $\epsilon > 0$ so small that $EX_1^1 - \epsilon > 0$. If $i \geqq 1$ define

$$Z_i^1 = X_i^1 - \epsilon; \qquad Z_i = (Z_i^1, X_i^2, \cdots, X_i^k).$$

Let $W_1(c)$ be the least integer n such that $Z_1^1 + \cdots + Z_n^1 > c$, and $W(c)$ be the least integer $n \geqq 1$ such that $\min(\sum_{i=1}^n Z_i^1, \min_{2\leqq j\leqq k} \sum_{i=1}^n X_i^j) > c$. Then $W_1(c) \leqq W(c)$ and as in the calculation giving (40),

$$\begin{aligned} EN(c) &\leqq EW_1(c) \\ &\quad + E(W(c) - W_1(c) \mid W_1(c) < W(c))P(W_1(c) < W(c)) \\ &\leqq W_1(c) + g(c, k-1)P(W_1(c) < W(c)). \end{aligned} \tag{46}$$

Let $W_j(c)$ be the least integer $n \geqq 1$ such that $X_1^j + \cdots + X_n^j > c$, $2 \leqq j \leqq k$. Then

$$g(c, k-1) \leqq EW_2(c) + \cdots + EW_k(c). \tag{47}$$

Using Theorem 2 there is a function $a_1(\cdot, \cdot, \cdot)$ of three variables such that

$$\lim_{c\to\infty} a_1(c, \mu, \Delta) = 0,$$

and if $2 \leqq j \leqq k$,

$$\mu_j = EY_1^j \geqq \mu, \qquad Eh(|Y_1^j|) \leqq \Delta,$$

then

$$1/\mu_j \leqq c^{-1}EW_j(c) \leqq 1/\mu_j + a_1(c, \mu, \Delta).$$

Using (46) and (47)

$$\begin{aligned} 1/\mu_1 &\leqq c^{-1}EN(c) \leqq c^{-1}EW_1(c) \\ &\quad + P(W_1(c) < W(c)) \left(\sum_{j=2}^{k} (1/\mu_j + a_1(c, \mu_2, \Delta))\right) \\ &\leqq 1/(\mu_1 - \epsilon) + a_1(c, \mu_1 - \epsilon, \Delta) \\ &\qquad + P(W_1(c) < W(c)) \left(\sum_{j=2}^{k} (1/\mu_j + a_1(c, \mu_2, \Delta))\right) \end{aligned} \tag{48}$$

Therefore

$$\begin{aligned} &|c^{-1}EN(c) - 1/\mu_1| \\ &\qquad \leqq \epsilon/(\mu_1(\mu_1 - \epsilon)) + a_1(c, \mu_1 - \epsilon, \Delta) \\ &\qquad\qquad + P(W_1(c) < W(c)) \left(\sum_{j=2}^{k} (1/\mu_j + a_1(c, \mu_2, \Delta))\right). \end{aligned} \tag{49}$$

Referring to (31) in the statement of Lemma 5, we let $\epsilon = \epsilon(c)$ be a function of c such that $\epsilon(c) \to 0$ as $c \to \infty$ so slowly that

$$\lim_{c\to\infty} b_{q(c)}(4\alpha^2\Delta)/\epsilon(c) = 0.$$

Then using Lemma 5 we see that there must be a function $a_k(\cdot, \cdot, \cdot)$ satisfying $\lim_{c\to\infty} a_k(c, \mu, \Delta) = 0$ and

$$|c^{-1}EN(c) - 1/\mu_1| \leqq a_k(c, \mu_1, \Delta). \tag{50}$$

That completes the proof.

6. Proof of Theorem 1. If $p(\cdot, \cdots, \cdot)$ is a homogeneous function of degree one in k variables then using the law of large numbers,

$$\text{with probability one,} \quad p(\mu_1, \cdots, \mu_k) = \lim_{n\to\infty} p\left(n^{-1}\sum_{i=1}^{n} X_i^1, \cdots, n^{-1}\sum_{i=1}^{n} X_i^k\right). \tag{51}$$

Consequently, exactly as for Doob [2],

$$\text{with probability one,} \qquad \lim_{c\to\infty} N_p(c)/c = 1/p(\mu_1, \cdots, \mu_k). \tag{52}$$

We write, in this argument, $N_p(c)$ for the stopping variable of Theorem 1. In the special case that $p(x_1, \cdots, x_k) = \min(x_1, \cdots, x_k)$ then $\{N(c)/c, c > 0\}$

is a family of nonnegative random variables (integrable functions) satisfying (7) and (52). As is well known this implies

$$\lim_{c\to\infty} E|c^{-1}N(c) - (\min(\mu_1, \cdots, \mu_k))^{-1}| = 0. \tag{53}$$

Given $c > 0$, and $n \geqq 1$, $N(c) = n$ implies

$$c < \min_{1\leqq j\leqq k} \sum_{i=1}^{n} X_i^j \leqq \alpha p\left(\sum_{i=1}^{n} X_i^1, \cdots, \sum_{i=1}^{n} X_i^k\right). \tag{54}$$

Therefore $N_p(c/\alpha) \leqq n$ and

$$N_p(c/\alpha) \leqq N(c). \tag{55}$$

It follows that $EN_p(c/\alpha) < \infty$ and that $\{c^{-1}N_p(c), c > 0\}$ is a uniformly integrable family of functions. Using (52), (4) now follows. The proof is complete.

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