UNIFORM APPROXIMATION OF MINIMAX POINT ESTIMATES

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1. Summary. In this paper the problem of minimax point estimation of a function $g(\theta)$ of a parameter θ is considered, when the loss function is of the form $W(u(x), g(\theta)) = |u(x) - g(\theta)|^p$, $(p > 1)$ and $u(x)$ is an estimate with bounded risk. When Conditions A and B stated later hold, it is shown that a unique minimax estimate $u_0(x)$ exists and if $\{u_n(x)\}$ is any uniformly bounded minimax sequence then the risk functions of $u_n(x)$ converges uniformly to the risk function of $u_0(x)$, so that no almost subminimax estimate can exist which, though not a minimax estimate, has for a wide range of values of the parameter θ, a lower value of the risk than that of the minimax estimate. Under some additional conditions, it is shown that an approximation to the minimax estimate $u_0(x)$ in the space $\mathfrak{F}_\infty^{(p)}$ of functions with bounded risk, may be obtained by the minimax estimate $\bar{u}_N(x)$ in the finite dimensional linear space spanned by N basis vectors $v_1, \cdots v_N$ of $\mathfrak{F}_\infty^{(p)}$, so that the maximum risk of $\bar{u}_N(x)$ converges to that of $u_0(x)$. This may help in finding an approximation to a minimax estimate in non-standard problems, where it is difficult to guess a minimax estimate from invariance or other considerations and specially when the problem is a perturbation of a standard problem.

2. Introduction. It has been pointed out by Hodges and Lehmann [3] and Robbins [8], that in certain cases there exist estimates which are not minimax so that their maximum risks may be slightly greater than that of a minimax estimate but their risk functions are considerably less than that of a minimax estimate, for a range of values of the parameter θ. Such estimates have been called ϵ-minimax or subminimax estimates [8], [11].

Since ϵ is an unspecified small quantity above, the situation may be characterised by the existence of a minimax sequence of estimates $\{u_n(x)\}$ (which is a minimax solution in the wide sense in Wald's terminology), whose risk functions do not uniformly converge to that of the minimax estimate $u_0(x)$. Frank and Kiefer [1], have given examples from the theory of testing of hypotheses, of such almost subminimax solutions. An example is given in the appendix where for any ϵ however small subminimax estimates of this type, exist for a squared error loss function with unbounded range of the parameter. A second difficulty in the choice of a suitable estimate arises when there is a multiplicity of admissible minimax estimates.

In the first part of the paper it is shown that Conditions A and B stated later ensures that a unique minimax estimate $u_0(x)$ exists and any minimax sequence of estimates $\{u_n(x)\}$ converges with respect to a suitable norm topology (Theorem 1) to $u_0(x)$ so that the corresponding risk functions also converge uniformly to

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the risk function of the minimax estimate. Thus no almost subminimax estimate can exist for sufficiently small ϵ, which is substantially better than the minimax estimate for any value of the parameter.

In the second part of the paper we consider the problem of approximating a minimax estimate $u_0(x)$ in the space $\mathfrak{F}_\infty^{(p)}$ of functions $u(x)$ for which $\text{Sup}_\theta E\{|u(x) - g(\theta)|^p \mid \theta\}$ is bounded, by minimax estimates $\bar{u}_N(x)$ in finite dimensional linear spaces V_N, formed by the basis $v_1, v_2, \cdots v_N$ of $\mathfrak{F}_\infty^{(p)}$. This provides a sequence $\{\bar{u}_N(x)\}$ whose risk functions are such that $\{\text{Sup}_\theta\, r(\bar{u}_N(x), \theta)\}$ forms a monotonic decreasing sequence with the limit $\text{Sup}_\theta\, r(u_0(x), \theta)$.

It is of course possible to approximate a decision problem by replacing the space Ω of a priori distributions, the space of terminal decisions D^t and the sample space X by finite number of points, when the problem is sufficiently regular. Thus Wolfowitz [11] considers a finite number of distributions $F_1, F_2 \cdots F_n$ and terminal decisions $d^{t_1} \cdots d^{t_m}$ which are ϵ-dense in the spaces Ω and D^t, in the intrinsic sense and represents the risk function $r(F_i, \delta)$ as a bounded convex domain in an Euclidean space and then selects a finite number of points in the periphery of the domain, to constitute an essentially ϵ-complete class for the decision problem. The choice of a finite number of points in the decision space is possible by introducing ϵ-nets in the sample space and for each cell C_i in the sample space, considering the distributions $\{p_{ij}\}_{j=1,\cdots,m; i=1,\cdots,N}$ on the terminal decisions $d^{t_1} \cdots d^{t_m}$. For sufficiently smooth distributions, the decision problem with distributions $F_1, \cdots F_n$ and decision functions with distributions $\{p_{ij}\}$ on the terminal decisions defined for each cell C_i in the sample space, will be an approximation to the original decision problem and a minimax estimate for the reduced problem will be a ϵ-minimax estimate for the original problem. The reduced problem can be solved as a two person zero-sum game with finite number of strategies for both players. However, the number of strategies, will have to be very large for any reasonable approximation. Moreover, as in the reduced problem the terminal decisions are discrete, the advantage of a convex loss function is lost and one has to consider probability distributions on $d^{t_1} \cdots d^{t_m}$ (mixed strategies) in order that the minimax theorem holds, whereas in the linear spaces considered in this paper, only non-randomised estimates (pure strategies) are required. A second advantage of the method of approximating by minimax estimates in finite dimensional subspaces is that when a problem represents slight variation from a standard problem in which a conventional minimax estimate $v(x)$ is known, one could start off with a basis in which $v(x)$ is the first element. Thus in Example 1 in Section 5, we consider the problem of estimating the mean of a normal population $N(\theta, 1)$, $(-b < \theta < b)$ and we chose a basis $\bar{x}$, $\bar{x}^2$, $\bar{x}^3$, etc., so that at any stage in the approximation, the estimate is better than the conventional estimate $\bar{x}$ and the maximum risk gradually decreases to the maximum risk of the minimax estimate. An appropriate analogy would be with the perturbation methods in the solution of Schrödinger equation of a many particle system, where one starts with a solution for a simpler system and then modifies it taking into consideration the mutual interaction of the particles, rather than trying to solve the equation *ab initio* in which the advantage of a known solution for a simpler problem is lost.

3. Uniform minimax sequences. Let R_n be the Euclidean space of n dimensions which is the sample space, B, a Borel field of subsets of R_n, on which a separable σ-finite measure μ is defined (see Zaanen [12], page 74). Let $f(x, \theta)$ be a class of functions of x measurable with respect to B for any given θ and so that $f(x, \theta) > 0$ holds except for a set P of points of μ-measure zero, the set P being independent of θ, and so that $\int_{R_n} f(x, \theta)\, d\mu(x) = 1$, for all θ. Here θ is a parameter lying in a space Ω and the problem is to estimate a bounded real function $g(\theta)$ of θ, when the loss function for an estimate $u(x)$ is

$$W(u(x), g(\theta)) = |u(x) - g(\theta)|^p, \qquad p > 1$$

where $u(x)$ is measurable with respect to the Borel field B.

We may obviously restrict ourselves to estimates $u(x)$ for which $r_u(\theta) = \int |u(x) - g(\theta)|^p f(x, \theta)\, d\mu(x)$ is bounded. Denote the class of functions for which $\text{Sup}_\theta\, r_u(\theta) \leqq K$ by $\mathfrak{F}_K^{(p)}$ and the class of functions for which $\text{Sup}_u\, r_u(\theta)$ is bounded by $\mathfrak{F}_\infty^{(p)}$. For any function $u(x)$ let

$$\phi(u) = \text{Sup}_\theta\, r_u(\theta). \tag{1}$$

We consider the following two conditions which will be imposed on the problem wherever necessary.

Condition A. There exists a $\theta = \theta_0$, so that for any bounded region M in R_n, except for a set of μ-measure zero, independent of θ,

$$(1/A_M)f(x, \theta_0) < f(x, \theta) < A_M f(x, \theta_0) \qquad \text{for all } \theta \tag{2}$$

or alternatively

$$(1/A^2_M)f(x, \theta') < f(x, \theta'') < A^2_M f(x, \theta') \qquad \text{for all } \theta' \text{ and } \theta'' \tag{3}$$

when $x \,\varepsilon\, M$, and A_M is a constant depending only on M. Condition A holds e.g. when θ is the mean vector of a nondegenerate p-variate normal distribution and Ω is a compact region in the Euclidean p-dimensional space. It may be noted that Condition A prevents the type of situation described by Frank and Kiefer [1] in which an almost subminimax solution exists, for the problem of testing of hypothesis with (0, 1) loss function.

Condition B.

(i) $g(\theta)$ is a continuous function of θ,

(ii) Ω is closed and compact,

(iii) $f(x, \theta)$ is a continuous function of θ,

for given x, except possibly for a set of values of x of μ-measure zero.

When range of values of $g(\theta)$ is bounded, by $a \leqq g(\theta) \leqq b$, we shall consider the truncated estimate $u'(x)$ corresponding to any $u(x)$, where

$$u'(x) = u(x) \qquad \text{when } a \leqq u(x) \leqq b$$
$$= u(a) \qquad \text{when } u(x) < a$$
$$= u(b) \qquad \text{when } u(x) > b.$$

We also consider the following condition for a set of functions.

CONDITION U_p. A set of functions $\{u_i(x)\}$ satisfy the Condition U_p, when for any given $\epsilon > 0$, it is possible to find a bounded region $M(\epsilon)$ in R_n so that

$$(4) \qquad \int_{x \varepsilon R_n - M(\epsilon)} |u_i(x) - g(\theta)|^p f(x, \theta)\, d\mu(x) < \epsilon \qquad \text{for all } \theta \text{ and } i.$$

We then prove the following lemma.

LEMMA 1. *If $\{u_n(x)\}$ is a set of uniformly bounded functions and the Condition B holds, then the set $\{u_n(x)\}$ satisfy the Condition U_p.*

PROOF. For any θ, we chose a region M_θ in R_n so that $\int_{R_n - M_\theta} f(x, \theta)\, d\mu(x) < \epsilon'$. Also as $\int_{M_\theta} f(x, \theta)\, d\mu(x)$ is a continuous function of θ,

$$\left| \int_{M_\theta} f(x, \theta)\, d\mu(x) - \int_{M_\theta} f(x, \theta')\, d\mu(x) \right| < \epsilon'$$

for $\theta' \varepsilon N(\theta)$, a neighbourhood of θ. As $\int_{R_n} f(x, \theta')\, d\mu(x) = 1$ we have $\int_{R_n - M_\theta} f(x, \theta')\, d\mu(x) < 2\epsilon'$ when $\theta' \varepsilon N(\theta)$.

As $u_n(x)$ and $g(\theta)$ are bounded,

$$(5) \quad \int_{R_n - M_\theta} |u_n(x) - g(\theta')|^p f(x, \theta')\, d\mu(x) \leqq \text{Sup}_{x,\theta} |u(x) - g(\theta)|^p \cdot 2\epsilon' = \epsilon.$$

Since Ω is compact we can cover it by a finite number of neighbourhoods $N(\theta_1), \cdots N(\theta_k)$ and then

$$(6) \qquad \int |u_n(x) - g(\theta)|^p f(x, \theta)\, d\mu(x) \leqq \epsilon \quad \text{for all } \theta \varepsilon \Omega \text{ and all } n,$$

where the integral is taken over $R_n - M_{\theta_1} \cup \cdots \cup M_{\theta_k}$.

We now consider the following modification of the notion of a complete class of estimates.

DEFINITION. An estimate $u_1(x)$ will be called uniformly strictly better than an estimate $u_2(x)$ if the risk function $r_{u_1}(\theta)$ of u_1 is less than the risk function $r_{u_2}(\theta)$ of u_2 for all θ. A class C of estimates will be called strictly complete if for any estimate $u(x)$ outside C, there is an estimate $v(x)$ in C, which is uniformly strictly better than $u(x)$.

The above definition imposes the condition $r_{u_1}(\theta) < r_{u_2}(\theta)$ for all θ, so that u_1 is uniformly strictly better than u_2, while in Wald's definition ([10] page 26) u_1 is uniformly better than u_2 if $r_{u_1}(\theta) \leqq r_{u_2}(\theta)$ for all θ, and the strict inequality holds for at least one θ. This implies that while a complete class in the usual sense may not contain all minimax estimates, e.g. minimax estimates which are not admissible, a strictly complete class must contain all minimax estimates.

We may then restate the theorem of Hodges and Lehmann [3], for strictly increasing loss functions.

LEMMA 2. *If $g(\theta)$ has values in a bounded interval (a, b) and C, the class of estimates which have values in (a, b) except for sets of measure zero, and the loss function $W(u, g(\theta))$ is a strictly increasing function of $|u - g(\theta)|$ when $u > g(\theta)$ and when $u < g(\theta)$, then the class C of estimates is strictly complete and contains all minimax estimates.*

PROOF. The proof is the same as given by Hodges and Lehmann [3], noting that $W(u, g(\theta))$ is strictly increasing so that if $u(x)$ has values outside the interval (a, b) with positive probability then the truncated estimate $u'(x)$ satisfies $r_u'(\theta) < r_u(\theta)$ for all θ.

We now have

LEMMA 3. *If the Condition B holds, then a minimax estimate of $g(\theta)$ exists and is unique.*

PROOF. The loss function $|z - g(\theta)|^p$ is continuous in z and θ and thus the space of terminal decisions which is a subset of the bounded interval (a, b) is compact according to the intrinsic topology

$$R(z_1, z_2) = \text{Sup}_\theta \, ||z_1 - g(\theta)|^p - |z_2 - g(\theta)|^p|.$$

Thus Wald's assumptions [10] are satisfied and a minimax estimate $u_0(x)$ exists. From Lemma 2, all minimax estimates are included in the class C of estimates lying in the range (a, b) except for sets of measure zero. Let u_1 and u_2 be two such minimax estimates in C, then as $|u - g(\theta)|^p$ is a strictly convex function of $u(x)$

$$E\{|\tfrac{1}{2}(u_1 + u_2) - g(\theta)|^p \mid \theta\} < \tfrac{1}{2}[E\{|u_1 - g(\theta)|^p \mid \theta\} + E\{|u_2 - g(\theta)|^p|\theta\}]$$

unless $u_1 = u_2$ holds except for a set of measure zero with respect to $f(x, \theta)\, d\mu$ i.e. unless $u_1 = u_2(\mu)$.

From Lemma 1, there exists a set $M \,\varepsilon\, R_n$ so that

$$\int_{R_n - M} |u(x) - g(\theta)|^p f(x, \theta)\, d\mu < \epsilon \quad \text{for all } \theta \text{ and } u(x) \,\varepsilon\, C.$$

Also as $f(x, \theta)$ and $|u(x) - g(\theta)|^p$ are continuous functions of θ except for sets of measure zero of x, $\int_M |u(x) - g(\theta)|^p \, d\mu$ is a continuous function of θ and finally $E\{|u(x) - g(\theta)|^p \mid \theta\}$ is a continuous function of θ. Since Ω is closed and compact, there exists a θ_0 for which

$$\begin{aligned} \text{Sup}_\theta \, E\{|\tfrac{1}{2}(u_1 + u_2) - g(\theta)|^p \mid \theta\} &= E\{|\tfrac{1}{2}(u_1 + u_2) - g(\theta_0)|^p \mid \theta_0\} \\ &< \tfrac{1}{2}\{E[|u_1 - g(\theta_0)|^p \mid \theta_0] + E[|u_2 - g(\theta_0)|^p \mid \theta_0]\} \\ &< \tfrac{1}{2}\{\text{Sup}_\theta \, E[|u_1 - g(\theta)|^p \mid \theta] + \text{Sup}_\theta \, E[|u_2 - g(\theta)|^p \mid \theta]\} \end{aligned}$$

unless $u_1 = u_2(\mu)$. Thus the minimax estimate is unique up to sets of measure zero.

For any given value of the parameter $\theta = \theta_0$, we consider functions $u(x)$ for which $\int |u(x)|^p f(x, \theta_0)\, d\mu(x)$ is bounded. The class of such functions form a Banach space $L_{\theta_0}^{(p)}$ with the norm,

$$\|u(x)\|_{\theta_0} = \left\{ \int |u(x)|^p f(x, \theta_0)\, d\mu(x) \right\}^{1/p}. \tag{7}$$

We shall show that the Banach space $L_{\theta_0}^{(p)}$ is separable. As the measure μ is separable and σ-finite, for any μ-measurable set E of finite μ-measure, there is a μ-measurable set F belonging to an enumerable family of sets $S \subset B$ so that

$$\mu\{(E - E \cap F) \cup (F - E \cap F)\} = \mu(E - F) < \epsilon.$$

Also since $\int f(x, \theta_0)\, d\mu(x) = 1$ and $f(x, \theta_0) \geqq 0$

$$\int_N f(x, \theta_0)\, d\mu(x) < \epsilon'$$

where $N = [x: f(x) > M]$ when M is sufficiently large. Thus in the set $C_1 = C \cap (R_n - N)$

$$\int_{C_1} f(x, \theta_0)\, d\mu(x) < M\mu(C_1)$$

when C is a set of the Borel field B. Thus

$$\int_{E-F} f(x, \theta_0)\, d\mu(x) < M\epsilon + \epsilon' < 2\epsilon'$$

when ϵ is sufficiently small. Hence the measure $\int_C f(x, \theta_0)\, d\mu(x)$ is separable and thus the Banach space $L_{\theta_0}^{(p)}$ is separable (see Zaanen [12] p. 75).

Now for the Banach space $L_{\theta_0}^{(p)}$, we have the inequalities

(8) $$\|\tfrac{1}{2}(u_1 + u_2)\|_{\theta_0}^p + \|\tfrac{1}{2}(u_1 - u_2)\|_{\theta_0}^p \leqq \tfrac{1}{2}[\|u_1\|_{\theta_0}^p + \|u_2\|_{\theta_0}^p] \qquad \text{when } p \geqq 2$$

and

(9) $$\|\tfrac{1}{2}(u_1 + u_2)\|_{\theta_0}^q + \|\tfrac{1}{2}(u_1 - u_2)\|_{\theta_0}^q \leqq \{\tfrac{1}{2}[\|u_1\|_{\theta_0}^p + \|u_2\|_{\theta_0}^p]\}^{q-1} \text{ when } 1 < p \leqq 2$$

where $p^{-1} + q^{-1} = 1$. (See Zaanen [12], p. 130).

We thus have

(10) $$\|\tfrac{1}{2}(u_1 + u_2)\|_{\theta_0}^p \leqq \tfrac{1}{2}[\|u_1\|_{\theta_0}^p + \|u_2\|_{\theta_0}^p] \qquad \text{for } 1 < p < \infty.$$

Thus if $\|u_1\|_{\theta_0}^p < K$, $\|u_2\|_{\theta_0}^p < K$, $\|\tfrac{1}{2}(u_1 + u_2)\|_{\theta_0}^p < K$ holds, i.e. if $u_1, u_2 \,\varepsilon\, \mathfrak{F}_K^{(p)}$, then $\tfrac{1}{2}(u_1 + u_2) \,\varepsilon\, \mathfrak{F}_K^{(p)}$. Also from (10)

(11) $$\begin{aligned}\|u\|_{\theta_0}^p &\leqq 2^{p-1}\{\|u - g(\theta_0)\|_{\theta_0}^p + \|g(\theta_0)\|_{\theta_0}^p\} \\ &\leqq 2^{p-1}\{K + |g(\theta_0)|^p\} \quad \text{when } u \,\varepsilon\, \mathfrak{F}_K^{(p)}.\end{aligned}$$

As $g(\theta)$ is bounded, $u \,\varepsilon\, \mathfrak{F}_\infty^{(p)}$ implies that $u \,\varepsilon\, L_{\theta_0}^{(p)}$, and

(12) $$\mathfrak{F}_\infty^{(p)} \subset L_{\theta_0}^{(p)}.$$

In the class $\mathfrak{F}_\infty^{(p)}$ we now introduce a metric ρ where

(13) $$\rho(u_1, u_2) = \text{Sup}_\theta \left[\int |u_1 - u_2|^p f(x, \theta)\, d\mu(x)\right]^{1/p}.$$

Then $\rho(u_1, u_2)$ satisfies the conditions of a metric since

(i) $\rho(u_1, u_3) \leqq \rho(u_1, u_2) + \rho(u_2, u_3)$

(ii) $\rho(u_1, u_2) = \text{Sup}_\theta [\int |u_1 - u_2|^p f(x, \theta) \, d\mu(x)]^{1/p} \leqq 2^{p-1}[\phi(u_1) + \phi(u_2)]$ is bounded for all $u_1, u_2 \, \varepsilon \, \mathfrak{F}_\infty^{(p)}$

(iii) $\rho(u_1, u_2) = 0$ implies $u_1 = u_2$ except for a set Q of μ-measure zero, since $f(x, \theta) > 0$ except for the set P of μ-measure zero. Identifying all functions $u(x)$ which differ in sets of μ-measure zero, we find that $\rho(u_1, u_2) = 0$ if and only if $u_1 = u_2$.

We shall denote the metric given by the norm of $L_{\theta_0}^{(p)}$ as ρ_{θ_0}. Then the topology defined in $\mathfrak{F}_\infty^{(p)}$ by ρ_{θ_0} is weaker than the topology defined by ρ. We now introduce the following definitions.

Definition.

(i) A sequence of estimates $\{u_n(x)\}$ will be called a minimax sequence when

$$\lim \phi(u_n) = \text{Inf}_{u \varepsilon \mathfrak{F}_\infty^{(p)}} \phi(u).$$

(ii) If the minimax estimate is unique and the risk function of u_n converges uniformly to the risk function of the minimax estimate, then we call $\{u_n(x)\}$ a uniform minimax sequence.

(iii) An estimation problem in which a unique minimax estimate exists and there is a class C of estimates, which is a complete class and so that every minimax sequence of elements of C is a uniform minimax sequence, will be called a regular C-uniform problem.

We shall prove the following theorem.

THEOREM 1. *If the Condition* A *holds, then any minimax sequence which satisfies Condition* U_p, *converges to a limit* $u_\infty \, \varepsilon \, \mathfrak{F}_\infty^{(p)}$ *according to the metric* ρ, *where* u_∞ *is a minimax estimate.*

Before proving the theorem we shall prove the following lemmas.

LEMMA 4. $E\{|u - g(\theta)|^p \mid \theta\}$ *is a continuous function of u according to the metric* ρ *for* $p > 1$, *when* $u \, \varepsilon \, \mathfrak{F}_\infty^{(p)}$ *and is uniformly continuous when* $u \, \varepsilon \, \mathfrak{F}_K^{(p)}$. *If a sequence* $\{u_n\}$ *where* $u_n \, \varepsilon \, \mathfrak{F}_K^{(p)}$ *converges to* u_∞ *according to* ρ, *then* $E\{|u_n - g(\theta)|^p \mid \theta\}$ *converges to* $E\{|u_\infty - g(\theta)|^p \mid \theta\}$ *uniformly for* θ.

PROOF. For any two elements u_1, u_2 in $L_\theta^{(p)}$ we have

$$\left| \|u_1\|_\theta - \|u_2\|_\theta \right| \leqq \|u_1 - u_2\|_\theta . \tag{14}$$

Thus if $u_1, u_2 \, \varepsilon \, \mathfrak{F}_\infty^{(p)}$, $u_1 - g(\theta)$, $u_2 - g(\theta) \, \varepsilon \, L_\theta^{(p)}$ and

$$\begin{aligned} &|[E\{|u_1 - g(\theta)|^p \mid \theta\}]^{1/p} - [E\{|u_2 - g(\theta)|^p \mid \theta\}]^{1/p}| \\ &\qquad \leqq [E\{|u_1 - u_2|^p \mid \theta\}]^{1/p} \leqq \rho(u_1, u_2). \end{aligned} \tag{15}$$

Thus $[E\{|u - g(\theta)|^p \mid \theta\}]^{1/p}$ is a uniformly continuous function of u according to ρ. Since the mapping $x \to x^p$ is continuous for $x > 0$, it follows that $E\{|u - g(\theta)|^p \mid \theta\}$ is a continuous function of u. Also as x^p is a uniformly continuous function of x for $x < M < \infty$, it follows that $E\{|u - g(\theta)|^p \mid \theta\}$ is uniformly continuous in u, when $u \, \varepsilon \, \mathfrak{F}_K^{(p)}$.

The last part of the lemma follows from the fact that the right hand side of the inequality (15) does not involve θ. (This proof was suggested by the referee.)

LEMMA 5. $\phi(u)$ *is a uniformly continuous function of* $u \, \varepsilon \, \mathfrak{F}_K^{(p)}$ *according to the metric* ρ.

PROOF. Let u_1, $u_2 \varepsilon \mathfrak{F}_K^{(p)}$ then from

$$\phi(u_i) = \text{Sup}_\theta E\{|u_i - g(\theta)|^p \mid \theta\} \qquad i = 1, 2 \tag{16}$$

we can find θ_1, θ_2 so that for all θ,

$$E\{|u_i - g(\theta)|^p \mid \theta\} < \phi(u_i) < E\{|u_i - g(\theta_i)|^p \mid \theta_i\} + \epsilon \quad \text{for } i = 1, 2. \tag{17}$$

From Lemma 4, if $\rho(u_1, u_2) < \eta$

$$|E\{|u_1 - g(\theta_i)|^p \mid \theta_i\} - E\{|u_2 - g(\theta_i)|^p \mid \theta_i\}| < \epsilon' \quad \text{for } i = 1, 2. \tag{18}$$

Thus from (17) and (18)

$$\begin{aligned} E\{|u_2 - g(\theta_2)|^p \mid \theta_2\} &< E\{|u_1 - g(\theta_2)|^p \mid \theta_2\} \\ &+ \epsilon' < E\{|u_1 - g(\theta_1)|^p \mid \theta_1\} \\ &+ \epsilon + \epsilon' < E\{|u_2 - g(\theta_1)|^p \mid \theta_1\} + \epsilon + 2\epsilon'. \end{aligned} \tag{19}$$

From (17) and (19)

$$|E\{|u_2 - g(\theta_2)|^p \mid \theta_2\} - E\{|u_2 - g(\theta_1)|^p \mid \theta_1\}| < \epsilon + 2\epsilon'.$$

Thus

$$\begin{aligned} |\phi(u_r) - \phi(u_2)| &< |E\{|u_1 - g(\theta_1)|^p \mid \theta_1\} - E\{|u_2 - g(\theta_2)|^p \mid \theta_2\}| \\ &+ 2\epsilon < |E\{|u_2 - g(\theta_2)|^p \mid \theta_2\} - E\{|u_2 - g(\theta_1)|^p \mid \theta_1\}| \\ &+ |E\{|u_1 - g(\theta_1)|^p \mid \theta_1\} - E\{|u_2 - g(\theta_1)|^p \mid \theta_1\}| + 2\epsilon < 3(\epsilon + \epsilon') \end{aligned} \tag{20}$$

hence the result.

PROOF OF THEOREM 1. Let $\{u_n\}$ be a minimax sequence satisfying Condition U_p. Since $\{u_n\}$ is a minimax sequence

$$\lim \phi(u_n) = \text{Inf}_{u \varepsilon \mathfrak{F}_\infty^{(p)}} \phi(u) = Q \qquad \text{we may take} \quad u_n \varepsilon \mathfrak{F}_K^{(p)} \tag{21}$$

for some K.

When $p \geqq 2$ we have, considering the elements $u_n - g(\theta)$ and $u_{n+r} - g(\theta)$, from (8)

$$\begin{aligned} \|\tfrac{1}{2}(u_n - u_{n+r})\|_\theta^p \leqq \tfrac{1}{2}[\|u_n - g(\theta)\|_\theta^p + \|u_{n+r} - g(\theta)\|_\theta^p] \\ - \|\tfrac{1}{2}(u_n + u_{n+r}) - g(\theta)\|_\theta^p. \end{aligned} \tag{22}$$

Since $\frac{1}{2}(u_n + u_{n+r}) \varepsilon \mathfrak{F}_K^{(p)}$ we have

$$\phi[\tfrac{1}{2}(u_n + u_{n+r})] \geqq \text{Inf}_{u \varepsilon \mathfrak{F}_\infty^{(p)}} \phi(u) = Q.$$

Since $\phi(u) = \text{Sup}_\theta \|u - g(\theta)\|_\theta^p$, we can also find θ' so that

$$\phi[\tfrac{1}{2}(u_n + u_{n+r})] < \|\tfrac{1}{2}(u_n + u_{n+r}) - g(\theta')\|_{\theta'}^p + \tfrac{1}{2}\epsilon \tag{23}$$

i.e.,

$$\|\tfrac{1}{2}(u_n + u_{n+r}) - g(\theta')\|_{\theta'}^p \geqq Q - \tfrac{1}{2}\epsilon. \tag{24}$$

Also

$$\|u_n - g(\theta')\|_{\theta'}^p \leqq \phi(u_n), \qquad \|u_{n+r} - g(\theta')\|_{\theta'}^p \leqq \phi(u_{n+r}). \tag{25}$$

If $n > n_0(\epsilon)$ we have

$$\phi(u_{n+i}) < Q + \tfrac{1}{2}\epsilon \qquad \text{for} \quad i = 0, 1, 2, 3, \cdots . \tag{26}$$

Thus we have from (24), (25), (26)

$$\|\tfrac{1}{2}(u_n - u_{n+r})\|_{\theta'}^p \leqq \epsilon, \quad \text{i.e.,} \quad \|u_n - u_{n+r}\|_\theta^p \leqq 2^p\epsilon = \epsilon'. \tag{27}$$

When $1 < p \leqq 2$ we have from (9)

$$\begin{aligned} \|\tfrac{1}{2}(u_n - u_{n+r})\|_\theta^{p/(p-1)} &\leqq (\tfrac{1}{2}[\|u_n - g(\theta)\|_\theta^p + \|u_{n+r} - g(\theta)\|_\theta^p])^{1/(p-1)} \\ &\qquad - \|\tfrac{1}{2}(u_n + u_{n+r}) - g(\theta)\|_\theta^{1/(p-1)}. \end{aligned} \tag{28}$$

Substituting from above we get

$$\|\tfrac{1}{2}(u_n - u_{n+r})\|_{\theta'}^{p/(p-1)} \leqq (Q + \tfrac{1}{2}\epsilon)^{1/(p-1)} - (Q - \tfrac{1}{2}\epsilon)^{1/(p-1)} \leqq \epsilon_1 \tag{29}$$

i.e., when ϵ is small.

$$\|u_n - u_{n+r}\|_{\theta'}^p \leqq \epsilon' \tag{30}$$

when n is large.

Thus in any case for some θ'

$$\|u_n - u_{n+r}\|_{\theta'}^p \leqq \epsilon' \qquad \text{for} \quad n > n_1(\epsilon'). \tag{31}$$

Since $\{u_n\}$ satisfies the Condition U_p

$$\int_{R_n - M(\epsilon)} |u_{n+r} - g(\theta)|^p f(x, \theta)\, d\mu(x) < \epsilon \qquad \text{for all } \theta \text{ and } i \tag{32}$$

$$\begin{aligned} &\int_{R_n - M(\epsilon)} |u_{n+r} - u_n|^p f(x, \theta)\, d\mu(x) \\ &\qquad \leqq 2^{p-1} \int_{R_n - M(\epsilon)} [|u_n - g(\theta)|^p + |u_{n+r} - g(\theta)|^p] f(x, \theta)\, d\mu(x) \leqq 2^p\epsilon. \end{aligned} \tag{33}$$

Now as Condition A holds we can find a constant $A_{M(\epsilon)}$ so that for any θ

$$f(x, \theta') < A_{M(\epsilon)}^2 f(x, \theta) \qquad \text{when} \quad x \,\varepsilon\, M(\epsilon). \tag{34}$$

Thus from (31), (33) and (34)

$$\begin{aligned} E\{|u_{n+r} - u_n|^p \mid \theta\} &\leqq 2^p\epsilon + A_{M(\epsilon)}^2 \int_{M(\epsilon)} |u_{n+r} - u_n|^p f(x, \theta')\, d\mu\,(x) \\ &\leqq 2^p\,\epsilon + A_{M(\epsilon)}^2\,\epsilon' = \epsilon'' \end{aligned} \tag{35}$$

which holds for all θ, i.e.

$$\rho^p(u_n\,, u_{n+r}) \leqq \epsilon'' \qquad \text{when} \quad n > n(\epsilon''). \tag{36}$$

Thus the sequence $\{u_n\}$ converges according to the metric ρ and therefore converges also according to ρ_{θ_0} and from the completeness of the Banach space $L_{\theta_0}^{(p)}$ there is an element $u_\infty \varepsilon L_{\theta_0}^{(p)}$, which is the limit a.e. (μ) of a subsequence $\{u_{n_i}\}$ and is the limit of $\{u_n\}$ according to ρ_{θ_0}. This limit is determined up to sets of μ-measure zero. From Fatou's lemma

$$\int | u_\infty - u_n |^p f(x, \theta)\, d\mu(x) \leqq \liminf_i \int | u_{n_i} - u_n |^p f(x, \theta)\, d\mu(x)$$

$$\leqq \max_{r \geqq 0} \rho^p(u_n, u_{n+r})$$

$$\lim_n \operatorname{Sup}_\theta \int | u_\infty - u_n |^p f(x, \theta)\, d\mu(x) = 0. \tag{37}$$

Thus $\{u_n\}$ converges to u_∞ according to the metric ρ and thus from (10)

$$\operatorname{Sup}_\theta \|u_\infty - g(\theta)\|_\theta^p \leqq 2^{p-1}\{\operatorname{Sup}_\theta \|u_\infty - u_n\|_\theta^p + \operatorname{Sup}_\theta \|u_n - g(\theta)\|_\theta^p\} \tag{38}$$

so that $u_\infty \varepsilon \mathfrak{F}_{K'}^{(p)}$ for some K'. Also from

$$\operatorname{Sup}_\theta \|u_n - g(\theta)\|_\theta^p \leqq 2^{p-1}\{\operatorname{Sup}_\theta \|u_\infty - u_N\|_\theta^p + \operatorname{Sup}_\theta \|u_\infty - g(\theta)\|_\theta^p\} \tag{39}$$

$u_N \varepsilon \mathfrak{F}_{K''}^{(p)}$ holds uniformly for N. Thus from Lemma 5, $\phi(u)$ is a uniformly continuous function of u and

$$\phi(u_\infty) = \lim \phi(u_n) = \operatorname{Inf}_{u \varepsilon \mathfrak{F}_\infty^{(p)}} \phi(u) = Q.$$

Thus u_∞ is a minimax estimate and $\{u_n(x)\}$ converges to u_∞ according to the metric ρ.

COROLLARY 1. *If the Conditions* A *and* B *hold, then a unique minimax estimate exists and the problem is regular C-uniform, where C is the class of estimates which lie in the range* (a, b) *of values of* $g(\theta)$*, except for sets of measure zero.*

PROOF. From Lemma 3, a unique minimax estimate exists as Condition B holds and this estimate is in the class C, i.e. lies in the interval (a, b). Also from Lemma 1, any minimax sequence in C is uniformly bounded and thus satisfies U_p. Thus from Theorem 1, any such minimax sequence in C converges to the unique minimax estimate according to metric ρ and the problem is regular C-uniform.

4. Approximations to minimax estimates. The minimax problem in the function space $\mathfrak{F}_\infty^{(p)}$ may be an intractable one and we consider here a method of approximating a minimax estimate in $\mathfrak{F}_\infty^{(p)}$ by minimax estimates in finite dimensional vector spaces. As shown before, the Banach space is separable and we suppose that it has a basis $x_1, x_2, x_3 \cdots$. We consider vector spaces V_n of estimates $\sum_{i=1}^n \alpha_i x_i$. In the vector space V_n we can find an estimate $\bar{u}_n = \sum_{i=1}^n \alpha_{n,i} x_i$ of $g(\theta)$ so that

$$\phi(\bar{u}_n) \leqq \operatorname{Inf}_{u \varepsilon V_n} \phi(u) + \epsilon_n \qquad \text{where} \quad \epsilon_n \to 0. \tag{40}$$

As $x_i \varepsilon L_{\theta_0}^{(p)}$ all linear functions $\sum \alpha_i x_i \varepsilon L_{\theta_0}^{(p)}$ so that $V_n \subset L_{\theta_0}^{(p)}$. In the Banach space $L_{\theta_0}^{(p)}$ with the basis $x_1, x_2, x_3 \cdots$ we can express any element u as the limit according to ρ_{θ_0} of a sequence $u_n = \sum_{i=1}^n a_{n,i}x_i$, i.e., $\|u - \sum_{i=1}^n a_{n,i}x_i\|_{\theta_0} \to 0$. We consider conditions under which $\phi(\bar{u}_n) \to \text{Inf}_{u \varepsilon \mathfrak{F}_\infty^{(p)}} \phi(u)$.

THEOREM 2. *If for every element $u \varepsilon \mathfrak{F}_\infty^{(p)}$ the sequence $u_n = \sum_{i=1}^n a_{n,i}x_i$ converges to u according to the metric ρ, for suitable constants $\{a_{n,i}\}$, then the sequence of estimates $\{\bar{u}_n\}$ is a minimax sequence.*

PROOF. Since $u_n = \sum_{i=1}^n a_{n,i}x_i$ converges to u according to the metric ρ, from (38) and (39) $u_n \varepsilon \mathfrak{F}_K^{(p)}$ for some K and thus from Lemma 5

$$\lim \phi(u_n) = \phi(u). \tag{41}$$

Thus $\phi(\bar{u}_n) \leqq \text{Inf}_{u \varepsilon V_n} \phi(u) + \epsilon_n \leqq \phi(u_n) + \epsilon_n$; i.e.

$$\lim \phi(\bar{u}_n) \leqq \phi(u). \tag{42}$$

Since this holds for all $u \varepsilon \mathfrak{F}_\infty^{(p)}$, we get $\lim \phi(\bar{u}_n) = \text{Inf}_{u \varepsilon \mathfrak{F}_\infty^{(p)}} \phi(u)$.

COROLLARY 1. *If C is a complete class of estimates and for some $\{a_{n,i}\}$, $u_n = \sum_{i=1}^n a_{n,i}x_i$ converge to u according to ρ, for $u \varepsilon \mathfrak{F}_\infty^{(p)} \cap C$, then $\{\bar{u}_n\}$ is a minimax sequence.*

PROOF. This immediately follows from Theorem 2, as we need consider only admissible estimates u, which are all in the complete class C and which contains a minimax estimate so that

$$\text{Inf}_{u \varepsilon \mathfrak{F}_\infty^{(p)}} \phi(u) = \text{Inf}_{u \varepsilon \mathfrak{F}_\infty^{(p)} \cap C} \phi(u). \tag{43}$$

REMARK. It should be noted that Theorem 2 does not imply that $\bar{u}_n$ converges to a minimax estimate $u \varepsilon \mathfrak{F}_\infty^{(p)}$ according to metric ρ or even according to metric ρ_{θ_0}. Thus $\{\bar{u}_n\}$ may not be a uniform minimax sequence.

We prove the following lemma

LEMMA 6. *If the right hand side of Condition* A *holds and $\{u_n\}$ is a sequence of elements of $L_{\theta_0}^{(p)}$, which satisfies the Condition* U_p *and converges according to the metric ρ_{θ_0}, then it also converges according to metric ρ.*

PROOF. Since $\{u_n\}$ converges according to ρ_{θ_0},

$$E\{|u_{n+r} - u_n|^p \mid \theta_0\} < \epsilon' \qquad \text{for} \quad r > 0. \tag{44}$$

From the right hand side of Condition A and Condition U_p, we get as in (35)

$$\begin{aligned} E\{|u_{n+r} - u_n|^p \mid \theta\} &< 2^p\epsilon + A_{M(\epsilon)} \int_{x \varepsilon M(\epsilon)} |u_{n+r} - u_n|^p f(x, \theta_0)\, d\mu(x) \\ &< 2^p\epsilon + A_{M(\epsilon)}\, \epsilon' = \epsilon'' \end{aligned} \tag{45}$$

i.e., $\text{Sup}_\theta E\{|u_{n+r} - u_n|^p \mid \theta\} < \epsilon''$ when n is large, hence the result.

Applying Lemma 6, we get the following

COROLLARY 2. *If the right hand side of inequality* A *and Condition* B *hold and the sequence $u_n = \sum_{i=1}^n a_{n,i}x_i$ boundedly converges* a.e. (μ) *and according to the metric ρ_{θ_0} for all $u \varepsilon \mathfrak{F}_\infty^{(p)} \cap C$, where C is the class of estimates for which $a \leqq u(x) \leqq b$ except for sets of measure zero, for suitable constants $\{a_{n,i}\}$, then $\{\bar{u}_n\}$ is a minimax sequence and the truncated estimates $\{\bar{u}_n'\}$ form a uniform minimax sequence.*

PROOF. As in Corollary 1, we need consider only the complete class C. Since

$u_n = \sum_{i=1}^{n} a_{n,i}x_i$ converges boundedly to u a.e. (μ), u_n lies in an interval $(-M, M)$ a.e. (μ) and thus from Lemma 1, it satisfies the condition U_p, as B holds. From Lemma 6, the sequence $\{u_n\}$ therefore converges to u according to the metric ρ, and from Theorem 2, Corollary 1, $\{\bar{u}_n\}$ is a minimax sequence. Again from Theorem 1, Corollary 1, $\{\bar{u}_n'\}$ is a uniform minimax sequence.

COROLLARY 3. *If the right hand side of inequality* A *and Condition* B *hold and if for some* $\{a_{n,i}\}$, *the sequence* $u_n = \sum_{i=1}^{n} a_{n,i}x_i$ *boundedly converges to* u a.e. (μ) *and according to metric* ρ_{θ_0}, *when* u *belongs to a set* V *of functions dense in* $\mathfrak{F}_\infty^{(p)} \cap C$ *according to the metric* ρ, *where* C *is defined in Corollary 2 above, then* $\{\bar{u}_n\}$ *is a minimax sequence and* $\{\bar{u}_n'\}$ *is a uniform minimax sequence.*

PROOF. Let $u \in \mathfrak{F}_\infty^{(p)} \cap C$. As V is dense in $\mathfrak{F}_\infty^{(p)} \cap C$ according to the metric ρ, we can find a sequence $\{v_n\}$ where $v_n \in V$, so that $\rho(u, v_n) \to 0$, and thus from (38) and (39) $v_n \in \mathfrak{F}_K^{(p)}$ for all n and some K. Thus from Lemma 5, there is a v_m so that $\phi(v_m) < \phi(u) + \epsilon$. As in (42) of Theorem 2 and Corollary 2,

$$\lim \phi(\bar{u}_n) \leqq \phi(v_m) < \phi(u) + \epsilon. \tag{46}$$

As this holds for all u and arbitrarily small ϵ,

$$\lim \phi(\bar{u}_n) = \text{Inf}_{u \in \mathfrak{F}_\infty^{(p)}} \phi(u).$$

Thus $\{\bar{u}_n\}$ is a minimax sequence and as before $\{\bar{u}_n'\}$ is a uniform minimax sequence.

REMARK. It is more convenient to deal with the Banach space $L_{\theta_0}^{(p)}$ than $\mathfrak{F}_\infty^{(p)}$ and since $\mathfrak{F}_\infty^{(p)} \subset L_{\theta_0}^{(p)}$, it is sufficient to verify the conditions stated in Theorem 2 and its corollaries hold when $\mathfrak{F}_\infty^{(p)}$ is replaced by $L_{\theta_0}^{(p)}$, for any θ_0.

5. Illustrations. We shall now consider two examples for illustration.

Example 1. Consider the problem of estimation of the mean of a normal distribution $N(m, 1)$ where $a < m < b$ from a sample of size n where the loss function is $(u - m)^2$. By taking $m' = m - \frac{1}{2}(a + b)$, the problem is equivalent to estimating the mean of a normal distribution $N(m', 1)$ where $m' < \frac{1}{2}(b - a)$ (see Hodges and Lehmann [3]). We may thus consider without loss of generality $a = -b$, and x measured in units of $n^{-\frac{1}{2}}$.

In this case $\bar{x} = z$ is a sufficient statistic so that the class of functions of $\bar{x}$ is a complete class. (See Lehmann and Scheffé [6]). The problem of estimation of m is also invariant with respect to the transformations $m \to -m$, $z \to -z$ so that we need consider invariant estimates only, since if there is a minimax estimate there is always an invariant minimax estimate. (Kiefer [5]). We thus consider the class of odd functions $u(-z) = -u(z)$, whose squares are integrable with respect to $\exp\{-(x - m)^2/2\}$. Such a class of function forms a separable Hilbert space which we denote by H_m. Obviously when $m > 0$, $H_m \subset H_0$.

We shall now show that the functions $z, z^3, z^5, \cdots$ form a basis of the Hilbert space H_0, and also of H_m. Suppose $g(z) \in H_0$ and

$$\int_{-\infty}^{\infty} z^{2r+1} g(z) e^{-\frac{1}{2}z^2}\, dz = 0 \qquad r = 0, 1, 2, 3 \cdots \tag{47}$$

then

$$\int_0^\infty \xi^r[g(\xi^{\frac{1}{2}}) - g(-\xi^{\frac{1}{2}})]e^{-\frac{1}{2}\xi}\, d\xi = 0$$

as $g(z)$ is an odd function,

$$\int_0^\infty \xi^r g(\xi^{\frac{1}{2}})e^{-\frac{1}{2}\xi}\, d\xi = 0.$$

Again $e^{-\theta\xi} = \sum [(-\theta)^s/s!]\xi^s$ is a uniformly convergent series for $0 \leqq \theta \leqq A < \infty$. Thus we have

$$\int_0^\infty \xi^r e^{-\theta\xi} g(\xi^{\frac{1}{2}})e^{-\frac{1}{2}\xi}\, d\xi = \sum \frac{(-\theta)^s}{s!}\int_0^\infty \xi^{r+s} g(\xi^{\frac{1}{2}})e^{-\frac{1}{2}\xi}\, d\xi = 0 \tag{48}$$

for all θ in $0 \leqq \theta \leqq A$.

From the uniqueness of Laplace transformation it follows that $g(\xi^{\frac{1}{2}}) = 0$ a.e. (Lehmann and Scheffé [7]). Also $z^{2n+1} \varepsilon H_m$ for all r, thus z, z^3, z^5, also form a basis of H_m.

Hence for every element $u(z) \varepsilon H_b$, we can find a polynomial $\sum_{r=0}^n a_{2r+1}z^{2r+1}$, so that

$$\int_{-\infty}^\infty \left[u(z) - \sum_{r=0}^n a_{2r+1}z^{2r+1}\right]^2 e^{-\frac{1}{2}(z-b)^2}\, dz < \epsilon. \tag{49}$$

Putting $z = -z$

$$\int_{-\infty}^\infty \left[u(z) - \sum_{r=0}^n a_{2r+1}\, z^{2r+1}\right]^2 e^{-\frac{1}{2}(z+b)^2}\, dz < \epsilon,$$

i.e.

$$\int_{-\infty}^\infty \left[u(z) - \sum_{r=1}^n a_{2r+1}\, z^{2r+1}\right]^2 \left(\sum_{r=0}^\infty \frac{b^{2r}z^{2r}}{(2r)!}\right) e^{-\frac{1}{2}z^2}\, dz < \epsilon\cdot e^{\frac{1}{2}b^2}$$

$$\int_{-\infty}^\infty \left[u(z) - \sum_{r=0}^n a_{2r+1}\, z^{2r+1}\right]^2 \left(\sum_{r=0}^\infty \frac{m^{2r}z^{2r}}{(2r)!}\right) e^{-\frac{1}{2}z^2}\, dz < \epsilon\cdot e^{\frac{1}{2}b^2} \tag{50}$$

$$\int_{-\infty}^\infty \left[u(z) - \sum_{r=0}^n a_{2r+1}\, z^{2r+1}\right]^2 e^{-\frac{1}{2}(z-m)^2}\, dz < \epsilon\cdot e^{\frac{1}{2}b^2} = \epsilon'$$

when $|m| < b$.

Let now $Q_1(z), Q_2(z), Q_3(z), \cdots$ be a system of orthonormal elements of H_b obtained from the basis z, z^3, $z^5 \cdots$ by orthogonalisation. We then have for any element $u(z) \varepsilon H_b$, $u(z) \sim \sum C_iQ_i(z)$, so that $u_n = \sum_{i=1}^n C_iQ_i(z)$ converges to $u(z)$ according to the metric ρ_b. It follows from (50) that $u_n(z)$ converges to $u(z)$ according to the metric ρ. Thus from Theorem 2 the estimates $\{\bar{u}_n\}$ in the vector spaces V_n satisfy

$$\lim \phi(\bar{u}_n) = \mathrm{Inf}_{u\varepsilon\mathfrak{F}_\infty^{(2)}}\, \phi(u). \tag{51}$$

Since Conditions A and B hold in this case, from Corollary 1, Theorem 1, the truncated estimates $\{\bar{u}_n'\}$ form a uniform minimax sequence.

Evaluation of $\alpha_{n,j}$. The constants $\{\alpha_{2k+1,j}\}$ in the minimax estimate of m in the vector space V_{2k+1} of estimates $\{\alpha_{2k+1,1}\bar{x} + \cdots + \alpha_{2k+1,2k+1}\bar{x}^{2k+1}\}$, when $-b < m < b$, may be evaluated for any k as a maximisation problem in $2k$ variables.

When $k = 0$, the risk function for the estimate $\alpha_{1,1}\bar{x}$ and for the a priori distribution G over $m^2(0 < m^2 < b^2)$ is

$$
\begin{aligned}
r(\alpha_{11}, G) &= \int_0^{b^2} E[(\alpha_{11}\bar{x} - m)^2 \mid m]\, dG(m^2) \\
&= \int_0^{b^2} \left[\alpha_{11}^2\left(\frac{1}{n} + m^2\right) - 2\alpha_{11}\, m + m^2\right] dG(m^2) \qquad (52)\\
&= \mu_1(\alpha_{11} - 1)^2 + \frac{\alpha_{11}^2}{n}
\end{aligned}
$$

where $\mu_r = \int_0^{b^2} m^{2r}\, dG(m^2)$.

For given G, $r(\alpha_{11}, G)$ is minimised for $\alpha_{11} = \mu_1/(\mu_1 + n^{-1})$ and this is maximised for G, when G is a one point distribution at $m^2 = b^2$. Thus the minimax estimate of m in V_1 is $b^2\bar{x}/(b^2 + n^{-1}) = \bar{x}(1 + 1/nb^2)^{-1}$.

When $k = 1$,

$$
\begin{aligned}
r(\alpha_{31}, \alpha_{33}, G) &= \int_0^{b^2} E\{(\alpha_{31}\,\bar{x} + \alpha_{33}\,\bar{x}^3 - m)^2 \mid m\}\, dG(m^2) \qquad (53)\\
&= C_{11}\,\alpha_{31}^2 + C_{33}\,\alpha_{33}^2 + 2C_{13}\,\alpha_{31}\,\alpha_{33} - 2\alpha_{31}\mu_1 - 2\alpha_{33}(\mu_2 + 3\mu_1/n) + \mu_1
\end{aligned}
$$

where

$$
C_{11} = \mu_1 + 1/n, \qquad C_{13} = \mu_2 + 6\mu_1/n + 3/n^2
$$

$$
C_{33} = \mu_3 + 15\mu_2/n + 45\mu_1/n^2 + 15/n^3.
$$

The values of α_{31} and α_{33} which minimise $r(\alpha_{31}, \alpha_{33}, G)$ for given G are solutions of equations

$$
C_{11}\hat{\alpha}_{31} + C_{13}\hat{\alpha}_{33} - \mu_1 = 0
$$

$$
C_{13}\hat{\alpha}_{31} + C_{33}\hat{\alpha}_{33} - (\mu_2 + 3\mu_1/n) = 0
$$

and the minimum value of the risk is

$$
\begin{aligned}
\mu_1 &= \hat{\alpha}_{31}\,\mu_1 - \hat{\alpha}_{33}(\mu_2 + 3\mu_1/n) \qquad (54)\\
&= \mu_1 - \frac{\mu_1^2}{C_{11}} - \frac{[C_{11}(\mu_2 + 3\mu_1/n) - C_{13}\,\mu_1]^2}{C_{11}[C_{11}\,C_{33} - C_{13}^2]}.
\end{aligned}
$$

We have to maximise the above risk for distributions G over $(0, b^2)$. Since μ_3 occurs only in C_{33}, and $C_{11} > 0$ and $C_{11}C_{33} - C_{13}^2 > 0$ holds, the maximum of the above risk for given μ_1 and μ_2 is obtained by putting the maximum possible value of μ_3 for given μ_1 and μ_2. As shown in [2] the maximum of μ_3 for given μ_1 and μ_2 is obtained for a two point distribution G in which one of the points is b^2, and has the value $b^6 - (b^4 - \mu_2)^2/(b^2 - \mu_1) + b^4\mu_1 - b^2\mu_2$.

We may now maximise the above for μ_1 and μ_2 satisfying the moment conditions $0 \leqq \mu_1 \leqq b^2$, $\mu_1^2 \leqq \mu_2 \leqq \mu_1 b^2$ or maximising for parameters λ and p, where $\mu_i = p\lambda^i + (1 - p)b^{2i}(0 \leqq \lambda \leqq b^2, 0 \leqq p \leqq 1)$. Similarly for any k, the problem of determination of $\alpha_{2k+1,1} \cdots \alpha_{2k+1,2k+1}$ reduces to the problem of maximisation of a function of $\mu_1 \cdots \mu_{2k+1}$, for parameters $\lambda_1 \cdots \lambda_k$, $p_1 \cdots p_k$ where $\mu_i = \sum_{j=1}^k p_j\lambda_j^i + (1 - \sum p_j)b^{2i}$, $(0 \leqq \lambda_j \leqq b^2, 0 \leqq p_j \leqq 1, \sum p_j \leqq 1)$.

The determination of the coefficients $\{\alpha_{2k+1,j}\}$ involves a large volume of computations and would be feasible for small values of k only. The method suggested here would therefore be useful as a perturbation method when a conventional estimate ($\bar{x}$ in this case) is given and a few additional terms $\bar{x}^3$, $\bar{x}^5$, etc. are needed to get a good enough approximation to the minimax estimate.

Example 2. Consider a set of distributions $\{f(x, \theta)\}$ where the range of x is $(0, 2\pi)$ for all θ with probability one. We assume that the right hand side of Condition A and Condition B are satisfied. As a concrete instance, we may consider the problem of estimating θ, when the distribution $f(x, \theta)$ is

$$(55)\quad f(x,\theta) = e^{-\theta x}\, x^m \Big/ \int_0^{2\pi} e^{-\theta x}\, x^m\, dx \quad \text{where } 0 \leqq \theta < L, \qquad 0 \leqq x \leqq 2\pi,$$

i.e., we have a truncated Γ-distribution with the parameter θ lying in a bounded range. The Conditions A and B are satisfied, when we consider the estimation of a continuous function $g(\theta)$, where $a < g(\theta) < b$, with the loss function $[u(x) - g(\theta)]^2$. As $g(\theta)$ lies in the range (a, b), we have a strictly complete class C of estimates $u(x)$ for which $a \leqq u(x) \leqq b$. Now as $u(x)$ is bounded it belongs to $L^{(2)}(0, 2\pi)$. The functions $\{\sin nx, \cos nx\}(n = 0, 1, 2, \cdots)$ form a C.O.N. system in $L^{(2)}(0, 2\pi)$ and we have the property that the Fourier Series of an absolutely continuous function is boundedly convergent. (Titchmarsh [9] p 408). Also for any $u(x) \varepsilon C$, we can find an absolutely continuous function $u_1(x)$ so that $u_1(x) \varepsilon C$ and $|u(x) - u_1(x)| < \epsilon$ except for a set B with Lebesgue measure less than δ', (arbitrarily small), thus

$$(56)\qquad \int [u(x) - u_1(x)]^2 f(x, \theta_0)\, dx < \epsilon^2 + 4(a^2 + b^2)\, \delta' = \epsilon'.$$

Since the Condition B holds, the class C also satisfies the Condition U_p from Lemma 1, and since the right hand side of Condition A holds, we get as in the proof of Lemma 6

$$(57)\qquad \text{Sup}_\theta \int [u(x) - u_1(x)]^2 f(x, \theta)\, dx < \epsilon''$$

which can be made arbitrarily small. Hence the class of absolutely continuous functions is dense in the class C, with respect to the metric ρ. The conditions of Corollary 3, Theorem 2 are satisfied and an approximate minimax estimate can be found by considering minimax estimates in V_{2n+1}, where V_{2n+1} is the vector space of functions $\sum_{r=0}^{n} (a_n \cos rx + b_n \sin rx)$.

6. Appendix.

Example. Consider the estimation problem, when we have normal distributions with mean m and variance $f(m)$, where $f(m) = 2$ when m is not an integer

(58) $$= (1 + km)^{-1} < 1 \qquad \text{when } m \text{ is an integer.}$$

The problem is to estimate $g(m)$, where

$$g(m) = m \qquad \text{when } m \text{ is not an integer}$$

$$= m - 1 \qquad \text{when } m \text{ is an integer.}$$

The estimate x of $g(m)$ has the risk $E\{(x - m)^2 \mid m\} = 2$, when m is not an integer and $f(m) + 1 < 2$, when m is an integer. According to a result of Karlin [4], for a set of normal distributions with mean m and variance unity, the estimate x is admissible and minimax and in fact unique minimax. Karlin's proof also holds if we omit integral values of m from its range so that x remains an admissible minimax estimate when m does not assume integral values. Since $g(m)$ differs from m only for integral values of m, and the corresponding risk function of the estimate x for integral values of m is less than 2; it follows that x is the unique admissible minimax estimate of $g(m)$ in this problem. We shall now show that by modifying the estimate x in the neighbourhood of an integer we can get an estimate whose maximum risk is less than $2 + \epsilon$, but whose risk for some integral values of m is less than $.01 + \epsilon$.

Consider the estimate $u_n(x)$ where

(60) $$\begin{aligned} u_n(x) &= x - 1 && \text{when } n - \delta_n < x < n + \delta_n \\ &= x && \text{otherwise'} \end{aligned}$$

where n is an integer and $\delta_n = 3[f(n)]^{\frac{1}{2}}$.

Then the risk of $u_n(x)$ at $m = n$, $r_{u_n}(m)$ is given by

$$r_{u_n}(m = n) = \frac{1}{[2\pi f(n)]^{\frac{1}{2}}} \int_{-\infty}^{\infty} [u_n(x) - (n - 1)]^2 \exp[-(x - n)^2/2f(n)]\, dx$$

(61) $$= \frac{f(n)}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \xi^2 e^{-\frac{1}{2}\xi^2}\, d\xi + \frac{1}{[2\pi f(n)]^{\frac{1}{2}}} \int_{|x-n|>\delta_n} [2\,|x - n| + 1]$$

$$\exp[-(x - n)^2/2f(n)]\, dx < f(n) + 2[f(n)]^{\frac{1}{2}} + .01 < .01 + \epsilon$$

when n is large.

When m is not an integer

$$
\begin{aligned}
r_{u_n}(m) &= \frac{1}{\pi^{\frac{1}{2}}} \int_{|x-n|<\delta_n} (x - 1 - m)^2 e^{-(x-m)^2/4}\, dx \\
&\qquad + \frac{1}{\pi^{\frac{1}{2}}} \int_{|x-n| \geq \delta_n} (x - m)^2 e^{-(x-m)^2/4}\, dx \qquad (62) \\
&= 2 + \frac{1}{\pi^{\frac{1}{2}}} \int_{|x-n|<\delta_n} e^{-(x-m)^2/4}\, dx - \frac{2}{\pi^{\frac{1}{2}}} \int_{|x-n|<\delta_n} (x - m) e^{-\frac{1}{2}(x-m)^2}\, dx \\
&< 2 + \epsilon \quad \text{when } n \text{ is large since } \delta_n \to 0.
\end{aligned}
$$

Also for large n, the distribution of x is concentrated at $x = n$ and thus the risk of u_n at a point $m = n + r$ differs from the risk of the estimate x at $m = n + r$ by a small quantity.

Thus the sequence of estimates $\{u_n(x)\}$ of $g(m)$ has a maximum risk tending to 2 as n tends to infinity and the risk of $u_n(x)$ at $m = n$ an integer is less than $01 + \epsilon$, when n is large.

The author believes that this phenomenon is not uncommon and a simpler and less pathological example may be given but mathematical difficulties stand in the way.

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REFERENCES

[1] Frank, P. and Kiefer, J. (1951). Almost subminimax and biased minimax procedures. *Ann. Math. Statist.* **22** 465–468.

[2] Harris, Bernard (1959). Determining bounds on integrals with applications to cataloging problems. *Ann. Math. Statist.* **30** 521–548.

[3] Hodges, J. L., Jr. and Lehmann, E. L. (1950). Some problems in minimax point estimation. *Ann. Math. Statist.* **21** 182–193.

[4] Karlin, S. (1958). Admissibility for estimation with quadratic loss function. *Ann. Math. Statist.* **29** 406–436.

[5] Kiefer, J. (1957). Invariance, minimax sequential estimation and continuous time processes. *Ann. Math. Statist.* **28** 573–601.

[6] Lehmann, E. L. and Scheffé, H. (1950). Completeness, similar regions and unbiased estimation: Part I, Sankhyā **10** 305–340.

[7] Lehmann, E. L. and Scheffé, H. (1955). Completeness, similar regions and unbiased estimation: Part II, Sankhyā **15** 220–236.

[8] Robbins, H. (1950). Asymptotically subminimax solutions of compound statistical decision problems. *Proc. Second Berkley Symp. Math. Statist. Prob.* 131–148. Univ. of California Press.

[9] Titchmarsh, E. C. (1939). *Theory of Functions*. Oxford Univ. Press.

[10] Wald, A. (1950). *Statistical Decision Functions*. Wiley, New York.

[11] Wolfowitz, J. (1951). On ϵ-complete classes of decision functions. *Ann. Math. Statist.* **22** 461–464.

[12] Zaanen, A. C. (1953). *Linear Analysis*. North Holland Publishing Co. Amsterdam.