NOTE ON MOOD'S TEST[1]

By C. F. Crouse

University of South Africa

0. Summary. Sukhatme (1958a), (1958b) proposed two test statistics for testing the equality of two variances, and showed that under certain conditions these tests are asymptotically distribution free. Contrary to statements in Sukhatme (1958a), (1958b) it is shown in this note that Mood's test shares this property. (At the time of establishing this result it was thought that it is known to the workers in the field. From a recent publication of Moses (1963) it appears that this may not generally be the case.)

1. Discussion. Let $X_1, X_2, \cdots, X_m; Y_1, Y_2, \cdots, Y_n$ be two independent samples from continuous distributions with cumulative distribution functions F and G. Let ξ and η be the corresponding medians. The hypothesis to be tested is

$$F(x - \xi) = G(x - \eta)$$

against the alternative that the two distributions have different variances. For $\xi = \eta$ Mood (1954) proposed as test statistic

$$M = \sum_i^n [r_i - (m + n + 1)/2]^2,$$

where r_i is the rank of Y_i in the combined sample of $(m + n)$ observations. For $\xi \neq \eta$ Sukhatme (1958a) considered Mood's test applied to the deviations of the observations from the sample medians, and based on general theorems regarding generalised U-statistics found that this test is not asymptotically distribution free, i.e. its asymptotic distribution is not independent of the original populations under the null hypothesis. In the derivation of this result Sukhatme (1958a) proceeds to write M in the form

$$M/N^3 = C_1U_N^{(1)} + C_2U_N^{(2)} + C_3U_N^{(3)} + P(1/N),$$

where $N = m + n$, C_1, C_2, C_3 are certain known constants, $P(1/N)$ is a third-degree polynomial in $1/N$, and

$$U_N^{(1)} = \binom{m}{2}^{-1}\binom{n}{1}^{-1}\sum_i^n\sum_{j\neq k}^m \psi(X_j, X_k, Y_i),$$

$$U_N^{(2)} = \binom{m}{1}^{-1}\binom{n}{2}^{-1}\sum_j^m\sum_{k\neq i}^n \psi(X_j, Y_k, Y_i),$$

$$U_N^{(3)} = \binom{m}{1}^{-1}\binom{n}{1}^{-1}\sum_i^m\sum_j^n \varphi(X_i, Y_j)$$

Received 6 February 1964.

[1] The contents of this note are contained in the author's Ph.D. thesis, University of London, 1960.

are three generalised U-statistics with ψ and φ defined in Sukhatme (1958a). In this form the generalised U-statistics $U_N^{(1)}$ and $U_N^{(2)}$ are misleading for they should be written in the form [cf. Sukhatme (1958a) for the definition of a generalised U-statistic]

$$U_N^{(1)} = \binom{m}{2}^{-1}\binom{n}{1}^{-1}\sum_{i}^{n}\sum_{j<k}^{m}\psi'(X_j, X_k, Y_i),$$

$$U_N^{(2)} = \binom{m}{1}^{-1}\binom{n}{2}^{-1}\sum_{j}^{m}\sum_{k<i}^{n}\psi''(X_j, Y_k, Y_i),$$

where

$$\psi'(X_j, X_k, Y_i) = \psi(X_j, X_k, Y_i) + \psi(X_k, X_j, Y_i),$$

$$\psi''(X_j, Y_k, Y_i) = \psi(X_j, Y_k, Y_i) + \psi(X_j, Y_i, Y_k).$$

This is the crucial point, for the underlying conditions that must be satisfied [cf. Sukhatme (1958a)] apply to ψ' and ψ'' and not, as performed by Sukhatme, to ψ. It can be shown that asymptotically $C_1 = \frac{1}{2}nm^2/N^3$, $C_2 = mn^2/N^3$, $C_3 = -mn/N^2$, and the condition on which Sukhatme found Mood's test to fail implies that

$$m\int f(x-\xi)g(x-\eta)[1-2F(x-\xi)]\,dx + n\int f(x-\xi)g(x-\eta)[1-2G(x-\eta)]\,dx$$

is identically zero, where f and g are the corresponding probability densities. This is achieved for f and g bounded and symmetric about ξ and η respectively, and this is exactly the same condition under which Sukhatme (1958a) found his test to be asymptotically distribution free.

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