DISTRIBUTION OF THE LARGEST OR THE SMALLEST CHARACTERISTIC ROOT UNDER NULL HYPOTHESIS CONCERNING COMPLEX MULTIVARIATE NORMAL POPULATIONS[1]

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1. Introduction. It has been pointed out by the author [1] that one can handle all the classical problems of point estimation and testing hypotheses concerning the parameters of complex multivariate normal populations much as one handles those for multivariate normal populations in real variates. In [1], [2], the author has derived an asymptotic formula for certain likelihood test-procedures and [2], has mentioned the maximum characteristic root statistic for testing the reality of a covariance matrix. The distribution of the characteristic roots under the null hypothesis established in those two papers can be written in a general form as

$$(1) \qquad c_1 \left\{ \prod_{j=1}^{q} \omega_j^m (1 - \omega_j)^n \right\} \left\{ \prod_{j=1}^{q-1} \prod_{k=j+1}^{q} (\omega_j - \omega_k)^2 \right\} d\omega_1 \cdots d\omega_q ,$$

where $c_1 = \prod_{j=1}^{q} \Gamma(n + m + q + j)/\{\Gamma(n + j)\Gamma(m + j)\Gamma(j)\}$ and $0 \leqq \omega_1 \leqq \omega_2 \leqq \cdots \leqq \omega_q \leqq 1$.

We may also note that when n is large, the joint distribution of $n\omega_j = f_j$ $(j = 1, 2, \cdots, q)$, $0 \leqq f_1 \leqq \cdots \leqq f_q < \infty$, can be written as

$$(2) \qquad c_2 \left(\prod_{j=1}^{q} f_j^m \right) \exp\left(-\sum_{j=1}^{q} f_j \right) \left\{ \prod_{j=1}^{q-1} \prod_{k=j+1}^{q} (f_j - f_k)^2 \right\} df_1 \cdots df_k ,$$

where $c_2 = 1/\{\prod_{j=1}^{q} [\Gamma(m + j)\Gamma(j)]\}$.

In this paper, we derive the distribution of ω_q (or f_q) and ω_1 (or f_1). The percentage points will be given and some applications will be discussed in another paper.

2. Distribution of ω_q or ω_1. For the distribution of ω_q, we shall require the following two lemmas:

Lemma 1.

$$\sum \int_{\mathfrak{D}} \prod_{j=1}^{s} [x_j^{m'_j} (1 - x_j)^{n'_j} dx_j] = \prod_{j=1}^{s} \left[\int_0^x x_j^{m_j} (1 - x_j)^{n_j} dx_j \right],$$

where $\mathfrak{D}: (0 \leqq x_1 \leqq \cdots \leqq x_s \leqq x)$, $(x \leqq 1)$; *and on the left hand side* $(m'_s, n'_s), \cdots, (m'_1, n'_1)$ *is any permutation of* $(m_s, n_s), \cdots, (m_1, n_1)$ *and the summation is taken over all such permutations.*

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For proof, one may refer to Roy ([3], (A.9.3), p. 203).

LEMMA 2.

$$\prod_{j=1}^{q-1}\prod_{k=j+1}^{q}(\omega_j-\omega_k)^2=\sum\begin{vmatrix}\omega_{j_1}^{2q-2} & \omega_{j_2}^{2q-3} & \cdots & \omega_{j_q}^{q-1}\\ \omega_{j_1}^{2q-3} & \omega_{j_2}^{2q-4} & \cdots & \omega_{j_q}^{q-2}\\ \cdot & \cdot & \cdots & \cdot\\ \omega_{j_1}^{q-1} & \omega_{j_2}^{q-2} & \cdots & \omega_{j_q}^{0}\end{vmatrix},$$

where $\sum$ *means summation over all permutations* $(j_1, j_2, \cdots, j_q)$ *of* $(1, 2, \cdots, q)$, *and* $|A|$ *means the determinant of* A.

PROOF. It is well known that a Vandermonde determinant

$$\begin{vmatrix}\omega_1^{q-1} & \omega_2^{q-1} & \cdots & \omega_q^{q-1}\\ \omega_1^{q-2} & \omega_2^{q-2} & \cdots & \omega_q^{q-2}\\ \cdot & \cdot & \cdots & \cdot\\ \omega_1 & \omega_2 & \cdots & \omega_q\\ 1 & 1 & \cdots & 1\end{vmatrix}^2=\left[\prod_{j=1}^{q-1}\prod_{k=j+1}^{q}(\omega_j-\omega_k)\right]^2=\alpha, \qquad \text{(say)}.$$

Then, the above expression can be written as

$$\alpha=\begin{vmatrix}\sum_{j=1}^{q}\omega_j^{2q-2} & \sum_{j=1}^{q}\omega_j^{2q-3} & \cdots & \sum_{j=1}^{q}\omega_j^{q-1}\\ \sum_{j=1}^{q}\omega_j^{2q-3} & \sum_{j=1}^{q}\omega_j^{2q-4} & \cdots & \sum_{j=1}^{q}\omega_j^{q-2}\\ \cdot & \cdot & \cdots & \cdot\\ \sum_{j=1}^{q}\omega_j^{q-1} & \sum_{j=1}^{q}\omega_j^{q-2} & \cdots & q\end{vmatrix}$$

$$=\sum_{j_1,j_2,\cdots,j_q}\begin{vmatrix}\omega_{j_1}^{2q-2} & \omega_{j_2}^{2q-3} & \cdots & \omega_{j_q}^{q-1}\\ \omega_{j_1}^{2q-3} & \omega_{j_2}^{2q-4} & \cdots & \omega_{j_q}^{q-2}\\ \cdot & \cdot & \cdots & \cdot\\ \omega_{j_1}^{q-1} & \omega_{j_2}^{q-2} & \cdots & 1\end{vmatrix}.$$

If in the right hand side, any two j_i and j_t are equal, then the value of the determinant is zero. Hence the summation over the right hand side over $(j_1, j_2, \cdots, j_q)$ reduces to the permutations of $(1, 2, \cdots, q)$, which establishes Lemma 2.

Now we shall prove the following theorem:

THEOREM 1. *If the joint distribution of* $\omega_1, \omega_2, \cdots, \omega_q$ *is given by* (1), *then*

$$\text{(3)} \qquad \Pr(\omega_q \leqq x) = c_1 \begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{q-1} \\ \beta_1 & \beta_2 & \cdots & \beta_q \\ \cdot & \cdot & \cdots & \cdot \\ \beta_{q-1} & \beta_q & \cdots & \beta_{2q-2} \end{vmatrix} = c_1 \left| (\beta_{i+j-2}) \right| ,$$

where c_1 *is defined in* (2), $\beta_{i+j-2} = \int_0^x \omega^{m+i+j-2}(1-\omega)^n \, d\omega$ *for* $i, j = 1, 2, \cdots q$ *and* (β_{i+j-2}) *is a* $q \times q$ *matrix.*

Proof. By definition, we have

$$\Pr(\omega_q \leqq x) = \Pr(0 \leqq \omega_1 \leqq \cdots \leqq \omega_q \leqq x)$$
$$= c_1 \int_{\mathfrak{D}} \prod_{j=1}^{q} [\omega_j^m (1-\omega_j)^n] \left[\prod_{j=1}^{q-1} \prod_{k=j+1}^{q} (\omega_j - \omega_k)^2 \right] \prod_{j=1}^{q} d\omega_j ,$$

where $\mathfrak{D}: (0 \leqq \omega_1 \leqq \omega_2 \leqq \cdots \leqq \omega_q \leqq x, x \leqq 1)$.

Using Lemma 2, the above expression can be written as

$$\text{(4)} \qquad \Pr(\omega_q \leqq x) = c_1 \sum \int_{\mathfrak{D}} \begin{vmatrix} \omega_{j_1}^{2q-2} & \omega_{j_2}^{2q-3} & \cdots & \omega_{j_q}^{q-1} \\ \omega_{j_1}^{2q-3} & \omega_{j_2}^{2q-4} & \cdots & \omega_{j_q}^{q-2} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_{j_1}^{q-1} & \omega_{j_2}^{q-2} & \cdots & \omega_{j_q}^{0} \end{vmatrix} \prod_{j=1}^{q} [\omega_j^m (1-\omega_j)^n \, d\omega_j],$$

where $\sum$ means summation over all permutations $(j_1, \cdots, j_q)$ of $(1, 2, \cdots, q)$. Now the determinant in the integral sign of (4), can be written as

$$\sum\nolimits_1 \operatorname{sign}(t_1, \cdots, t_q) \omega_{j_1}^{q-1+t_1} \omega_{j_2}^{q-2+t_2} \cdots \omega_{j_q}^{t_q} ,$$

where $(t_1, \cdots, t_q)$ is a permutation of $(0, 1, \cdots, q-1)$, sign $(t_1, \cdots, t_q)$ is positive if the permutation is even and negative if the permutation is odd, and $\sum_1$ means the summation over all such permutations. Then (4) becomes

$$\Pr(\omega_q \leqq x) = c_1 \sum \sum\nolimits_1 \int_{\mathfrak{D}} \operatorname{sign}(t_1, \cdots, t_q) \, (\omega_{j_1}^{q-1+t_1} \cdots \omega_{j_q}^{t_q})$$
$$\cdot \prod_{j=1}^{q} [\omega_j^m (1-\omega_j)^n \, d\omega_j].$$

First taking summation over $(j_1, j_2, \cdots, j_q)$, the permutation of $(1, 2, \cdots, q)$ and applying Lemma 1, we get

$$\Pr(\omega_q \leqq x) = c_1 \sum\nolimits_1 \operatorname{sign}(t_1, \cdots, t_q) \beta_{q-1+t_1} \beta_{q-2+t_2} \cdots \beta_{t_q} = c_1 |(\beta_{i+j-2})|,$$

which proves Theorem 1.

It may be noted here that

$$\Pr(\omega_1 \leqq x) = 1 - \Pr(\omega_1 \geqq x) = 1 - \Pr(x \leqq \omega_1 \leqq \cdots \leqq \omega_q \leqq 1).$$

Going back to the c.d.f. of $(\omega_1, \cdots, \omega_q)$ and using the transformation $\omega_j = 1 - z_j$ $(j = 1, 2, \cdots, q)$, we have

$$(5) \quad \Pr(\omega_1 \leqq x) = 1 - \Pr(x \leqq \omega_1 \leqq \cdots \leqq \omega_q \leqq 1) = 1 - c_1|(\delta_{i+j-2})|,$$

where $\delta_{i+j-2} = \int_0^{1-x} z^{n+i+j-2}(1 - z)^m \, dz$ and (δ_{i+j-2}) is a $q \times q$ matrix.

THEOREM 2. *If the distribution of* $f_1, \cdots, f_q$ *is given by* (2) *then*

$$(6) \qquad \Pr(f_q \leqq x) = c_2|(\gamma_{i+j-2})|,$$

where $\gamma_{i+j-2} = \int_0^x \omega^{m+i+j-2} \exp(-\omega) \, d\omega$, (γ_{i+j-2}) *is a* $q \times q$ *matrix and* c_2 *is defined in* (2).

Proof is similar to that of Theorem 1.

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REFERENCES

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