ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Central Regional Meeting, Chicago, Illinois, December 27–30, 1964. Additional abstracts appeared in the December, 1964 issue.)

7. An Extension of Chernoff's Theorem with Applications to Stochastic Comparison of Tests (Preliminary Report). INNIS ABRAHAMSON, University of Chicago.

Let $X_1, X_2, \cdots$ be independent observations on a random vector X in a Hilbert space κ, and let $\bar{X}_n$ be the mean of the first n observations. For S a subset of κ, let $f(x) = \sup_{u \varepsilon S} (x, u)$. Under certain conditions on S and the moment-generating functions of the random variables (X, u), $u \varepsilon \kappa$, it is shown that for given $\epsilon > 0$ there exists ρ, $0 < \rho < 1$, such that $n^{-1} \log P\{f(\bar{X}_n) \geqq \epsilon\} \rightarrow \log \rho$. This theorem is used to compute exact slopes (see Bahadur, R. R. *Ann. Math. Statist.* **31** (1960) 276–295) of some sequences of statistics, e.g. the sequence $\{n\bar{X}_n'\Sigma^{-1}\bar{X}_n\}$ for testing $E(X) = 0$ and $\text{Cov}(X) = \Sigma$. The special case of the χ^2 goodness-of-fit test is treated in detail, including a new approach to the optimal spacing problem for testing parametric continuous distributions; the solution is found to agree with others reached by different means. Exact slopes are also obtained for some statistics arising in the non-parametric k-sample problem, e.g. the between-sample sum of squares and Kolmogorov-Smirnov type statistics.

8. Probability Integrals of the Multivariate F Distribution, with Tables for Special Cases. J. V. ARMITAGE AND P. R. KRISHNAIAH, Wright-Patterson Air Force Base.

Let $S = (s_{ij}) = Y'Y$ where $Y: m \times p$ is a matrix of m random vectors distributed independently and identically as a p-variate normal with mean vector $\boldsymbol{\mu}$ and with $\Sigma = (\sigma_{ij})$ as the covariance matrix. Also let s^2/σ^2, where $E(s^2) = \sigma^2$, be a chi-square variate with n degrees of freedom distributed independently of $s_{11}, s_{22}, \cdots, s_{pp}$. Then the joint distribution of $F_1, F_2, \cdots, F_p$, where $F_i = ns_{ii}\sigma^2/ms^2\sigma_{ii}$, $(i = 1, 2, \cdots, p)$, is known (see, P. R. Krishnaiah, *Multiple Comparison Tests in Multivariate Case*, ARL 64-124, Wright-Patterson AFB) to be a central or non-central p-variate F distribution with (m, n) degrees of freedom according as $\boldsymbol{\mu} = \mathbf{0}$ or $\boldsymbol{\mu} \neq \mathbf{0}$. In the present paper, the evaluation of the probability integrals associated with this distribution is discussed. The upper 10%, 5%, 2.5% and 1% points of this distribution are also tabulated for $p = 1(1)10$, $m = 1$, $n = 5(1)35$ and $\rho = 0.1(0.1)0.9$ when $\rho = \sigma_{ij}/[\sigma_{ii}\sigma_{jj}]^{\frac{1}{2}}$ for $i \neq j = 1, 2, \cdots, p$. Various applications of the multivariate F distribution are also discussed.

9. Some BAN Linear Estimates of the Location and Scale Parameters Based on Censored or Complete Samples. M. M. ALI AND L. K. CHAN, University of Western Ontario.

For symmetric distributions, satisfying certain conditions, the location and scale parameters are estimated by linear combinations of order statistics from (i) complete, (ii) singly censored or (iii) doubly censored samples. These estimates belong to the class of best asymptotically normal estimates. In obtaining these estimates the maximum likelihood equations are linearized in such a manner that the remainder terms are asymptotically negligible. Plackett (*Ann. Math. Statist.* **29** (1958) 131–142) has derived similar results for the doubly censored case; however, the asymptotic negligibility of the remainder terms are studied more closely in the present paper.

10. Tables for the Distribution of the Maximum of Correlated Chi-Square Variates with One Degree of Freedom. J. V. Armitage and P. R. Krishnaiah, Wright-Patterson Air Force Base. (By title.)

Let $x_1, x_2, \cdots, x_k$ be jointly distributed as a k-variate normal with zero means and with $\Sigma = (\sigma_{ij})$ as the covariance matrix where $\sigma_{ii} = 1$, $(i = 1, 2, \cdots, k)$ and $\sigma_{ij} = \rho$, $(i \neq j = 1, 2, \cdots, k)$. Also, let $V = \max(x_1, x_2, \cdots, x_k)$ where $X_i = x_i^2$ for $i = 1, 2, \cdots, k$. In the present paper, the evaluation of the probability integral $\int_0^c f(V)\, dV$ is discussed when $f(V)$ denotes the frequency function of V. Also, the values of this probability integral are tabulated for $k = 1(1)10$, $\rho = 0(0.0125)0.9$ and $c = 0.1(0.1)11.5$. Then, using cubic interpolation, upper 10%, 5%, 2.5% and 1% points of the distribution of V are computed for $k = 1(1)10$ and $\rho = 0(0.0125)0.9$. Various applications of these tables are also discussed.

11. Monotonicity Property of the Generalized SPRT (Preliminary Report). Donald R. Barr and Kenzo Seo, Colorado State University.

The fact that increasing the upper bound of a sequential probability ratio test (SPRT) and decreasing the lower one leads (under mild conditions) to a new test for which at least one of the error probabilities is increased has been proved by Wijsman (Wijsman, R. A. A monotonicity property of the SPRT, *Ann. Math. Statist.* **31** (1960), 677–684). In this paper, results similar to those of Wijsman are obtained for a sequential multiple decision procedure, which we call the "generalized SPRT." The generalized SPRT for deciding among k known distributions $f_1, f_2, \cdots, f_k$ is given in terms of a random walk on a space **X**, whose states are given in terms of the "generalized probability ratio" $(\sum f_i)^{-1}(f_1, f_2, \cdots, f_k)$. It is shown that the error probabilities and expected sample sizes for the procedure can be given by certain integral equations. Using these equations, a monotonicity property is obtained. In addition, an interesting identity is given, which in some sense characterizes the distribution of probability ratios, and a condition is introduced which determines, at least in part, an optimal shape for the absorbing barriers for the walk.

12. A Nonparametric Test for the Several Sample Location Problem. V. P. Bhapkar, University of North Carolina and University of Poona.

Let $\{x_{ij}, j = 1, 2, \cdots, n_i\}$ be a random sample from the ith population with continuous cdf. F_i, $i = 1, 2, \cdots, c$, and suppose that these samples are independent. We consider a nonparametric test of the hypothesis H_0: $F_1 = F_2 = \cdots = F_c$ against alternatives of the form $F_i(x) = F(x - \theta_i)$ with the θ_i's not all equal. The test is based on c-plets that can be formed by choosing one observation from each sample. The V-test (*Ann. Math. Statist.* **32** (1961) 1108) and the L-test (Abstract, *Ann. Math. Statist.* **34** (1963) 1624) offered earlier are based on the number of c-plets defined with respect to only the smallest and both the smallest and the largest observations in the c-plet, respectively. Even though they are fairly efficient for exponential and uniform distributions, they are not so efficient for the normal distribution. The W-test, being offered now, is based on c-plets with respect to all the positions in the c-plet. The asymptotic distribution of the W-statistic is shown to be χ^2 with $c - 1$ degrees of freedom under H_0 and it is shown that the test is asymptotically as efficient, in the Pitman sense, as the Kruskal-Wallis H-test.

13. Minimum Variance Bounds in the Non-Regular Case (Preliminary Report). Wallace R. Blischke, C-E-I-R, Inc.

For a number of distributions with location parameters, standard techniques for determining lower bounds on the variance of unbiased estimators (including some techniques specifically developed for the non-regular case) do not give meaningful results in certain non-trivial regions of the parameter space. For example, for the Pearson Type III distribution, $f(x) = [\beta\Gamma(\alpha)]^{-1}((x - a)/\beta)^{\alpha-1} \exp\{-((x - a)/\beta)\}$, $a < x < \infty$, where $\alpha, \beta > 0$ and $-\infty < a < \infty$, the Cramér-Rao lower bound as well as the bound constructed by Chapman and Robbins (*Ann. Math. Statist.* **22** (1951) 581–586) and the generalization of this result due to Fraser and Guttman (*Ann. Math. Statist.* **23** (1952) 629–632) all yield the trivial result Var $(T) \geqq 0$, T being an unbiased estimator of a, if $0 < \alpha < 2$. Furthermore, the result of Kiefer (*Ann. Math. Statist.* **23** (1952) 627–629) presents considerable analytical and computational difficulties in practice. A simple generalization of the Chapman-Robbins and Fraser-Guttman results, based on powers of the density, does yield a non-trivial bound for the location parameter in many non-regular cases. This bound is derived and investigated numerically for the Pearson Type III distribution. A comparison of all of the above bounds in the case of the exponential distribution is included.

14. **Characterizations of Some Distributions by Conditional Moments.** E. M. Bolger and W. L. Harkness, Bucknell University and Pennsylvania State University.

Let X_1 and X_2 be independent random variables with ch.f.'s φ_1 and φ_2, and let the ch.f. and d.f. of $Y = X_1 + X_2$ be g and G, respectively. Suppose that Y has finite second moment and that $E(X_i \mid Y) = \lambda_i y/\lambda$ a.e., $i = 1, 2$, and $V(X_1 \mid Y) = \lambda_1\lambda_2 u(y)/\lambda^2$ a.e., where λ_1 and λ_2 are positive constants, $\lambda = \lambda_1 + \lambda_2$, and $u(y)$ is non-negative. Then it is shown that (a) $\varphi_i(t) = [g(t)]^{\lambda_i/\lambda}$, $i = 1, 2$, and (b) $g(t)(d^2/dt^2) \ln g(t) = -\int u(y)e^{ity}\, dG(y)$. Conversely, if (a) and (b) hold, then the mean and variance of X_1, conditional on fixed values of Y, have the structural form given above a.e. Under the assumption that $u(y)$ is quadratic in y, say $u(y) = \alpha y^2 + \beta y + \gamma$, the differential equation for g, which in this case can be written in a much simpler form, can be solved in general. A discussion of the ch.f.'s which are solutions of the resulting equation is carried out in a manner similar to one given by Lukacs (*Ann. Math. Statist.* **23** 442–449).

15. **Confidence Limits for the Reliability of a Series System** (Preliminary Report). R. J. Buehler and A. H. El Mawaziny, University of Minnesota and Iowa State University.

Lentner and Buehler (*J. Amer. Statist. Assoc.* **58** (1963) 670) have shown how the theory of exponential families can be used to set exact confidence limits for the mean life or for the reliability (equals probability of successful operation up to a given mission time) of a series system of $k = 2$ dissimilar components whose life distributions are exponential. The present paper extends this theory to provide (a) exact distributions for any k, and (b) large sample approximations. Let $\alpha_1^{-1}, \cdots, \alpha_k^{-1}$ be component mean lifetimes, let $\alpha^{-1} = (\sum \alpha_i)^{-1}$ be the system mean lifetime, and let z_i be a gamma variate obtained by summing $(a_i + 1)$ observed lives of type i components. It is shown that when all samples are large, exact confidence limits for α can be approximated by formally ("fiducially") supposing α to be normally distributed with mean A^2/B and variance A^3/B^2 where $A = \sum a_i$ and $B = \sum a_i z_i$. Comparisons are made with alternative solutions.

16. **Difference Sets, Orthogonal Mappings and Orthogonal Latin Squares.** I. M. Chakravarti, University of North Carolina.

The concept of orthogonal mapping was developed by Bose, Chakravarti and Knuth (1960). It has been shown there, that the existence of m mutually orthogonal mappings of a group of order s, implies the existence of m mutually orthogonal Latin squares of order s.

This paper establishes a relationship between standardized difference sets defined by Bose and Bush (1952) and orthogonal mappings. Some methods of construction of such difference sets have been given by Masuyama (1957) and Butson (1962, 1963). This paper considers an alternative method of construction of such difference sets and as an illustration, gives a difference set of order (10, 10, 5).

17. Optimal Selection Based on Relative Rank (The "Secretary Problem"). Y. S. Chow, S. Moriguti, H. Robbins and S. M. Samuels, Purdue University, University of Tokyo, and Columbia University.

Let $X_1, \cdots, X_n$ be a random permutation of the integers $1, \cdots, n$; all $n!$ permutations being equally likely. Let Y_i be the relative rank of X_i among $X_1, \cdots, X_i$ (i.e. Y_i = one plus number of terms, $X_1 \cdots, X_{i-1}$, less than X_i). The Y_i's are observed sequentially and the object is to find the stopping rule τ (based on the Y_i's) which minimizes EX_τ. For each n, the minimizing rule and the minimal expectation are obtained, using backward induction, from a difference equation. It is shown that, as n increases, the minimal expectation increases monotonically to $\prod_{j=1}^{\infty} ((j+2)/j)^{1/(j+1)} = 3.87$. A different measure on the $n!$ permutations of $X_1, \cdots, X_n$ is exhibited for which $EX_\tau = (n+1)/2$ for every stopping rule, τ. This is an "optimal counter-strategy".

18. On the Asymptotic Theory of Fixed-Width Sequential Confidence Intervals for the Mean. Y. S. Chow and H. Robbins, Purdue University and Columbia University.

Let $x_1, x_2, \cdots$ be a sequence of independent observations from some population with unknown variance σ^2. We want to find a confidence interval of prescribed width $2d$ and prescribed coverage probability α for the unknown mean μ of the population. For $n \geqq 1$, define $\bar{x}_n = \sum_1^n x_i/n$, $I_n = [\bar{x}_n - d, \bar{x}_n + d]$ and choose $a > 0$ to satisfy $\Phi(a) - \Phi(-a) = \alpha$, where $\Phi(t)$ is the standard $N(0, 1)$ distribution function. Define $v_n = n^{-1}\sum_1^n (x_i - \bar{x}_i)^2 + n^{-1}$, $(n \geqq 1)$, and let N be the smallest $k \geqq 1$ such that $v_k \leqq d^2k/a_k^2$, where a_k is any sequence of positive constants tending to a. Then we have, generalizing results of Anscombe (Sequential estimation, *J. Roy. Statist. Soc. B* **15** 1–21) and Stein (Some problems in sequential estimation, *Econometrica* **17** 77–78), the Theorem: *As* $d \to 0$, $\lim d^2N/(a^2\sigma^2) = 1$ a.s. $\lim P[\mu \varepsilon I_N] = \alpha$, *and* $\lim d^2EN/(a^2\sigma^2) = 1$.

19. Moments of Randomly Stopped Sums. Y. S. Chow, H. Robbins and H. Teicher, Purdue University and Columbia University.

Let $\{y_n, n = 1, 2 \cdots\}$ denote a sequence of random variables (r.v.'s) on a probability space $(\Omega, \mathfrak{A}, P)$ satisfying $E\{y_{n+1} \mid \mathfrak{F}_n\} \overset{n}{\equiv} 0$, where $\mathfrak{F}_0 = (\phi, \Omega)$, $\mathfrak{F}_n$ is the σ-algebra generated by $y_1, \cdots, y_n$ and let t be a stopping rule, i.e. a positive integer-valued r.v. such that $\{t > n\} \varepsilon \mathfrak{F}_n$, $n \geqq 0$. Then if $S_n = \sum_{i=1}^n y_i$, S_t is a randomly stopped sum (sum of a random number of r.v.'s). The moments ES_t^j, $j = 2, 3, 4$ are evaluated under conditions which, in the special case of independent, identically distributed r.v.'s y_n, reduce to simple moment assumptions on y_1 and t. For example, if y_i are independent with $Ey_i \equiv 0$, $Ey_i^2 \equiv \sigma^2$ and t is a stopping rule with $Et < \infty$, then $ES_t^2 = \sigma^2 Et$. Cf. Wolfowitz [*Ann. Math. Statist.* **18** (1947)], Johnson [*Ann. Math. Statist.* **30** (1959)].

20. Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples. A. Clifford Cohen, University of Georgia.

This paper is concerned with the two-parameter Weibull distribution which is widely employed as a model in life testing. Maximum likelihood equations are derived for estimating the distribution parameters from (i) complete samples, (ii) singly censored samples and (iii) progressively (multiple) censored samples. Asymptotic variance-covariance matrices are given for each of these sample types. An illustrative example is included.

21. **Robust Estimation of Location.** E. L. CROW AND M. M. SIDDIQUI, National Bureau of Standards (Boulder Laboratories) and Colorado State University.

Let $F(x - \mu)$ be a one-dimensional family of df's where $F(x)$ is the prototype df and μ is a location parameter. We shall call this family a pencil of distributions. Suppose we have N independent observations, X_i, $i = 1, 2, \cdots, N$ on a random variable. Let F^* be a class of pencils. We shall say that the problem is of robust estimation of location relative to F^* if F^* contains at least two pencils and it is known that all X_i have come from some F in F^*. In this paper a class of pencils, F^*, is selected and two classes of estimators proposed and studied for their robustness properties relative to F^*. F^* includes only absolutely continuous df's whose densities are symmetric and unimodal.

22. **Some Potential Theoretic Concepts for Non-Markovian Chains.** DONALD A. DAWSON, McGill University.

In this paper generalizations of some of the concepts of discrete potential theory are developed for non-Markovian chains. In particular the harmonic functions, superharmonic functions, potentials and boundaries are considered. One case of special interest is the class of non-Markovian chains for which the Chapman-Kolmogorov equation is satisfied. In this case one gets a sequence of potential theories connected by homomorphisms and for each of these potential theories the Riesz decomposition theorem and other standard results may be obtained. For nth order Markovian chains these potential theories break up into a finite number of classes under isomorphism. In addition, several other classes of non-Markovian chains as well as particular examples are considered and their potential theoretic structures are investigated.

23. **A Multivariate Stochastic Approximation Procedure.** MARY L. EPLING, University of California Medical Center.

Let x be a real number and $\{y_i(x)\}$, $i = 1, \cdots, k$, be independent families of real-valued observable random variables with unknown expectations $E\{y_i(x)\} = M_i(x)$. A modification of the Robbins-Monro procedure is suggested for stochastically approximating a vector $\mathbf{z} = [z_1, \cdots, z_k]$ with the property that $M_i(z_i) = M_j(z_j)$ for $i, j = 1, \cdots, k$. Convergence and the asymptotic distribution of the new procedure are discussed, and comparisons of initial bias and asymptotic variance are made between it and the alternative of selecting a value R and obtaining independent Robbins-Monro approximations of z_i, such that $M_i(z_i) = R$, $i = 1, \cdots, k$. It is shown that the suggested procedure may retain desirable convergence properties when the families of random variables are time dependent.

24. **On a Partial Differential Equation of Epidemic Theory.** J. GANI, Michigan State University.

A basic equation in the stochastic theory of epidemics is concerned with the probability $p_{rs}(t)$ at time $t \geqq 0$ of the number r of uninfected susceptibles and s of infectives in a total population of size $n + a$, from which removal of infectives is permitted, and where $p_{na}(0) = 1$. If $\Pi(z, w, t) = \sum_{r,s} p_{rs}(t) z^r w^s (|z|, |w| \leqq 1)$, then it is known that $\partial\Pi/\partial t = w(w - z)(\partial^2\Pi/\partial z \partial w) + \rho(1 - w)(\partial\Pi/\partial w)$. While it is readily seen that each $p_{rs}(t)$ will be the sum of exponential terms of the form $e^{-i(\rho+j)t}$, no manageable way of deriving these expressions appears to have yet been found. A method is outlined for reducing the second order partial differential equation for Π to a set of first order equations which then allow the explicit solutions for the $p_{rs}(t)$ to be written, at least when n, a are small.

25. Locally Most Powerful Rank Tests for Censored Data. JOSEPH L. GASTWIRTH, Johns Hopkins University.

Let $x_1, x_2, \cdots, x_m$ and $y_1, \cdots, y_n$ denote two independent random samples from $F(x - \theta)$ and $F(x)$ respectively. If $(z_1, \cdots, z_N)$ denotes the entire combined sample, then the l.m.p.r.t is given in the basic paper of H. Chernoff and I. R. Savage, *Ann. Math. Statist.* **29** 972–994. If the combined sample is censored at the $[N_p]$th observation, then the l.m.p.r.t is found here by asymptotic methods. Specifically, if $J(u)$ is the standarized weight function corresponding to the l.m.p.r.t for the entire sample, then the weight function $K(u)$ corresponding to the l.m.p.r.t for the censored sample is given by the following formula:

$$K(u) = J(u), \quad 0 \leqq u \leqq p \quad \text{and} \quad K(u) = h \quad \text{if} \quad p < u \leqq 1.$$

The number h is chosen so that $\int_0^1 K(u)\,du = 0$. In particular, the version of the Wilcoxen test for data censored at the combined median is given ($h = \frac{1}{4}$).

26. Multiple Target Bombing Problems. DENNIS C. GILLILAND, Michigan State University.

Suppose that in n-space a target consisting of k nonoverlapping hyperspheres of radii r is to be attacked with a point bomb. Assume further that the bomb impact density is spherical normal, variance unity, about the aim point. The problem is to aim the bomb so as to maximize the probability of an impact within the target (a hit). It has been shown that if $k = 2$ and if the centers of the hyperspheres are within two units of each other, the midpoint of the line segment between the centers yields maximum probability of a hit (Gilliland, Dennis C. (1964) *Ann. Math. Statist.* **35** 441–442). This paper reports solutions for special arrangements of $k > 2$ hyperspheres. Also the following problem is considered. If the center of a spherical normal density, variance unity, is fixed, how should k hyperspheres of radii r be placed so as not to overlap and so as to maximize the probability measure of their union? For 2-space this has been referred to as a cookie cutter problem. Given a sheet of dough with thickness proportional to the circular normal density, how should one punch out k circular cookies so as to get the greatest amount of dough? This cookie cutter problem is solved for $k \leqq 4$ cookies and small r.

27. On the Non-Minimax Property of a Multivariate Likelihood Ratio Test (Preliminary Report). N. GIRI, Université de Montréal.

Let $X = (X_1 \cdots X_p)'$ be normally distributed with unknown mean $\xi = (\xi_1 \cdots \xi_p)'$ and unknown nonsingular covariance matrix Σ. In this paper it is shown that the likelihood ratio test of the hypothesis $H_0: \xi_1 = \cdots = \xi_{p'} = 0$ against the alternative $H_1: \xi_1 = \cdots = \xi_q = 0$ when $p \geqq p' > q$ is not minimax.

28. Efficiency Comparisons of Certain Multivariate Analysis of Variance Test Procedures (Preliminary Report). R. GNANADESIKAN, E. LAUH, M. SNYDER AND Y. YAO, Bell Telephone Laboratories and University of Chicago.

Five functions of eigenvalues of the matrix, $S_h S_e^{-1}$, are considered where S_h and S_e are the so-called matrices due to the hypothesis and due to error respectively. Statistical tests, for the familiar null hypothesis of no effects in multivariate analysis of variance, may be based on these functions. Three of these are the classical ones of (i) the likelihood ratio test, (ii) the trace test, and (iii) the largest root.

The major concern of the paper is to compare the efficiencies of these statistics as measured by the sensitivity (power of test) to values of certain natural distance measures in the parameter space. The comparisons are of two kinds. First, for the bivariate case, comparisons are made using computer generated random samples and using a variety of distance measures. These Monte Carlo comparisons are made for small sample sizes and for choices of values for the "degrees-of-freedom" that are of interest in practice. An interesting finding is that the two non-classical test statistics appear to be more sensitive than the three classical ones to departures from the null hypothesis as measured by several natural distance measures.

The second kind of comparison is analytical and for general dimensionality, but for large samples, using the concept of Bahadur efficiency following the work of L. J. Gleser (1963), Stanford University, Technical Report No. 8.

29. On the Operating Characteristics of Multiple Decision Procedure. S. S. Gupta and M. N. Waknis, University of Purdue.

Let $\Pi_1, \Pi_2, \cdots, \Pi_k$ be k gamma populations with scale parameters $\theta_1, \theta_2, \cdots \theta_k$, respectively, each having the same shape parameter (degrees of freedom). For these populations, this paper studies the performance characteristics of a multiple decision (selection and ranking) procedure for selecting a subset containing the population with the smallest value of θ. For certain specified configurations, the probabilities of selection of the ith ranked population have been evaluated for all $i \leqq k$. The expected proportion and the expected sum of ranks of the populations in the selected subset have also been computed. These expressions give the various related criteria of the performance of the procedure. Again, using a specific loss function which is linear in $\eta_j = \theta_{[j]}/\theta_{[1]}$ ($\theta_{[j]}$ is the jth ranked value of θ, $\theta_{[1]}$ being the smallest), the risk function associated with the procedure has been computed and some results on its behavior have been obtained.

30. Estimation of a Spectral Density by Fitting Moments. M. J. Hinich, Carnegie Institute of Technology.

Let $X(t)$ denote a weakly stationary time series with $E\{X(t)\} = 0$ and with a spectral density function $S(f) = Af^{-\alpha} \exp(-\beta f^{-n})$ where A is a known constant and n is a known integer, but α and β are unknown parameters. (Bretschneider, C. L. (1963) "One Dimensional Gravity Wave Spectrum," *Ocean Wave Spectrum*. 41–60. Prentice-Hall, N. J.) Consistent estimators of α and β are derived as functions of the number of zero-axis crossings and the number of peaks and valleys of $X(t)$ during a finite sampling period. Formulas for the variances of the estimators are derived as a function of the parameters. The estimators given are easy to compute from the data.

31. Use of the Range to Control Percentages in Both Tails of a Normal Population. C. H. Kapadia, D. B. Owen and J. N. K. Rao, Southern Methodist University and Graduate Research Center of the Southwest.

Sample ranges, mean ranges, and sample means from a normally distributed population are used to set up two-sided tolerance limits and two-sided sampling plans which control the percentages defective in both tails of the distribution. The approach used by Owen [Control of Percentages in Both Tails of the Normal Distribution—*Technometrics* **6** No. 4 is extended by using Patnaik's [*Biometrika* **37** pp. 78–87] Chi-approximation to the distribution of the range. The exact distributions needed for the tolerance limits and sampling plans are also derived, and the accuracy of the approximation is investigated.

32. **Exact Distribution for a Chi-Square Test.** S. K. Katti, Florida State University.

Given a one-way table, there are occasions when one tests to see if all the p cells in the table are equally likely by computing the chi-square statistic and obtaining the critical region through the use of the tables of chi-square distribution with $p - 1$ degrees of freedom. Since the tabulated chi-square distribution is only an approximation to the distribution of the computed statistic, exact distribution has been obtained using the multinomial distribution of the frequencies in the p cells and critical regions tabulated.

33. **Locally Unbiased Type M Test.** Marakatha Krishnan, State University of New York, Buffalo, and Indian Statistical Institute, Madras.

In certain tests of hypotheses concerning more than one parameter, it is found that the criterion of unbiasedness cannot be satisfied. In such situations, it is proposed to make the test locally unbiased in a modified sense, termed Type M. Here, the power surface in the parametric space does not always lie above the level of significance of the test. However, the Mean Value of the Power, in a small neighborhood of the null point, exceeds the significance level. This paper demonstrates that the uniformly most powerful unbiased test (based on the sample sum of squares) for a normal variance is "Type M unbiased" against the wider class of alternatives permitting the means of the observations to differ, and similarly that the UMPU test (based on the sample variance ratio) for a population variance ratio is "Type M unbiased" against the same widened class of alternatives.

34. **A Nonparametric Estimate of a Multivariate Density Function.** D. O. Loftsgaarden and C. P. Quesenberry, Montana State College.

Let $x_1, \cdots, x_n$ be n independent observations on a p-dimensional random variable $X = (X_1, \cdots X_p)$ with absolutely continuous distribution function $F(x_1, \cdots, x_p)$. The problem considered is the estimation of the probability density function $f(x_1, \cdots, x_p)$ at a point $z = (z_1, \cdots, z_p)$ where f is positive and continuous. A consistent nonparametric density function estimator is introduced. For each n a neighborhood about the point z is chosen. The neighborhood chosen depends not only on n, but also on the observations $x_1, \cdots, x_n$. It is thus random, which distinguishes the approach used here from approaches used in other papers on nonparametric density estimation.

35. **Tables for the V_{NM} Two-Sample Test.** Urs Maag and M. A. Stephens, University of Toronto and McGill University. (By title.)

Let $F_N(x)$ and $G_M(x)$ be the sample distribution functions of two independent random samples (sizes N and M respectively from the unknown continuous distributions $F(x)$ and $G(x)$. Formulas for the distribution of the statistic $V_{N,M} = \sup_x [F_N(x) - G_M(x)] - \inf_x [F_N(x) - G_M(x)]$ were reported by Gnedenko (*Math. Nachr.* **12** (1954) 29–63) for the case $M = N$. Kuiper (*Nederl. Akad. van Wetensch. Proc. Ser. A* **63** (1960) 38–47) gave an asymptotic expansion also for equal sample sizes. Kuiper suggested the use of this statistic for testing the hypothesis $H_0 : F(x) = G(x)$, especially when the observations are points on a circle. In this paper Gnedenko's formula is simplified, evaluated for N up to 100 and compared with Kuiper's result. Furthermore the distribution is generated by enumeration for all pairs $2 < M, N < 10$.

36. A Partition of Fibonacci Numbers and a Related Two-Coin Tossing Game. S. G. Mohanty, McMaster University.

Let $N(n; u_1, \cdots, u_r)$ defined for integers $n, u_1, \cdots, u_r$ satisfy the following difference equations: For $m > 0$, $r > 0$ and $j = 1, \cdots, r$, $N(rm + j; u_1, \cdots, u_r) = N(rm + j - 1; u_1, \cdots, u_r) + N(r(m - 1) + j; u_1, \cdots, u_{j-1}, u_j - 1, u_{j+1}, \cdots, u_r)$, with boundary conditions $N(n; u_1, \cdots, u_r) = 0$ when $n = 0, -1, -2, \cdots$ and for $n = 1, \cdots, r$, $N(n; u_1, \cdots, u_r) = 1$ (when $u_1 = \cdots = u_r = 0$) $= 0$ (otherwise). Then $N(rm + j; u_1, \cdots, u_r)$ is equal to $\prod_{i=1}^{j} \binom{m - \sum_{k=1}^{r} u_k + u_i}{u_i} \prod_{i=j+1}^{r} \binom{m - \sum_{k=1}^{r} u_k + u_i - 1}{u_i}$. It follows from the difference equations that the sequence $N(n)$, $n = 1, 2, \cdots$ where $N(n) = \sum_{u_1} \sum_{u_2} N(n; u_1, u_2)$ is the sequence of Fibonacci numbers. Given two coins 1 and 2 with probabilities p_1 and p_2 of obtaining a head in a single trial, a game is defined with the following rules: (1) The $(k + 1)$st trial is made with coin 1 or 2 according as the number of heads x_k among the first k trials is odd or even (including zero); (2) Stop tossing the coins when for the first k, $x_k + n \leqq 2k$, $n = 1, 2, \cdots$. It is seen that the number of sequences each containing exactly u_i tails with coin i $(i = 1, 2)$ such that for the first k, $x_k + n = 2k$, is $N(n + 1; u_1, u_2)$. Using this result, the distribution of the duration of the game is obtained. The problem of estimation of p_i is also investigated.

37. Distribution-Free Slippage Tests—A Correction. Peter Nemenyi, COFO.

Ranks and other transforms used in distribution-free k-sample tests have equal correlations $(\rho = -1/(N - 1))$, and it was shown in papers presented at the June and December 1961 meetings that the covariance matrix, and hence the asymptotic joint distribution, of the statistics $(\bar{y}_i - \bar{y}_j)/(1 - \rho)$, (and hence of any linear combinations of these), does not depend on ρ. Distribution-free rank- median- and other tests, using standard multiple comparison tables, were based on this. The slippage statistics $\bar{y}_i - \bar{y}$ are contrasts in the $(\bar{y}_i - \bar{y}_j)$, and their variances are equal to $\sigma^2/n_i \cdot (N - n_i)/(N - 1)$. Normal-theory statistics $(\bar{x}_i - \bar{x})/\sigma$ based on uncorrelated x's have variances equal to $(k - 1)/k$. Hence statistics of the form max $(\bar{y}_i - \bar{y})/D$ and max $(\bar{y}_i - \bar{y})/D$ based on the correlated transforms can be used, asymptotically, with the tables of Nair or Halperin, resp., where the denominator D is equal to $[k/(k - 1) \cdot (N - n)/(N - 1) \cdot \sigma^2/n]^{\frac{1}{2}} = [k\sigma^2/(N - 1)]^{\frac{1}{2}}$ in the case of equal sample sizes. CC denotes a Continuity Correction, used in applications to improve and in the present work to study the fit of the asymptotic formula, which is found to be very good in the cases where exact values are available. In earlier versions the factor $k/(k - 1)$ was overlooked and the fit was very bad, "continuity corrections" of 4.0 and more being required to achieve perfect fit in some cases. (See Distribution-Free Multiple Comparisons, 1963).

38. An Extension of Lindley's Theorem to the Single Server Queue with Semi-Markovian Input and Service Time Processes. Marcel F. Neuts and Mahabanoo N. Tata, Purdue University.

Consider a queueing process with first-come, first-served discipline. The interarrival times minus the service times form an additive process defined on a finite Markov chain. This occurs in particular when the input and service processes are semi-Markovian. It is shown, by an extension of Lindley's argument, that the waiting time of the nth customer tends in distribution to a limit. This limit is independent of the initial conditions and the necessary and sufficient condition is found under which the limit is the distribution of a random variable. In this case the limiting distribution is obtained as the solution of a system of integral equations of Wiener-Hopf type.

39. Maximum Likelihood Estimation Under the Incorrect Family of Densities. HEEBOK PARK, Purdue University.

Let $\{g(x \mid \theta); \theta = (\theta_1, \theta_2) \varepsilon \Theta\}$ and $\{f(x \mid \varphi); \varphi = (\varphi_1, \varphi_2) \varepsilon \Phi\}$ denote two parametric families of densities with parameter space which are open subsets of 2-dimensional Euclidean space. These two families might be alternative models for observed data. Assume that a one-to-one correspondence $\theta \rightarrow \varphi = \varphi(\theta)$ has been established for interpretation. The asymptotic distribution theory of maximum likelihood estimation based on trimmed sample (including the complete sample as a special case) is developed when the data are sampled from $g(\cdot \mid \theta)$, but the estimate were computed with the assumption that they are $f(\cdot \mid \varphi)$. Under some regularity conditions slightly different from those are necessary to obtain the correct-model likelihood theory, it is shown that such a wrong-model maximum likelihood estimate is typically a consistent and asymptotically normal estimate not of θ but of some function $\varphi'(\theta)$ of θ.

40. Some Non-Markovian Processes Arising in Queueing Theory. N. U. PRABHU, Michigan State University.

In a single server queueing system, let $Y(t)$ be the time which the customer being served (if any) at time t has already spent in the system, and set $Y(t) = 0$ if the counter is unoccupied. The stochastic process $Y(t)$ is in general non-Markovian. In the special case $GI/M/1$, this process was studied by the author [*Acta Math. Acad. Sci. Hung.* **15**]. In the present paper further investigations are carried out on $Y(t)$ and also on the corresponding discrete process.

41. Alternative Estimators in p.p.s. Sampling for Several Characteristics. J. N. K. RAO, Graduate Research Center of the Southwest.

In large-scale sample surveys, it is often necessary to estimate totals or means of several characteristics y_i, where the first-stage units are selected with probability proportional to size (p.p.s.) of a supplementary variate x. The usual estimators in sampling with or without replacement are efficient if the correlations between y_i and x are fairly high and positive. However, often times, some of the characteristics y_i may not be strongly related to x and the correlations are very small. In such situations, the usual estimators, though unbiased, lead to large variances. Alternative estimators are proposed here which lead to smaller variances when the correlations are small. These estimators are biased; but the bias is negligible if the correlations are small. Using a super-population in which y and x are unrelated, it is shown that the average variance of the alternative estimators is smaller than the average variance of the usual estimators. Also, efficiency comparisons between the alternative estimators are made. The alternative estimators may be used for *only* those characteristics for which the correlations with x are small.

42. Mixtures of Probability Measures and Regular Conditional Probabilities. ROBERT H. RODINE, State University of New York, Buffalo.

The notation of abstract number 18 (*Ann. Math. Statist.* **35** 931) will be used throughout the following, together with the following definition. If $(Y, \mathfrak{J}, \nu)$ is a probability space and if for each y in Y there is a probability measure μ_y on a measurable space $(X, \mathcal{S})$, then the family $\{\mu_y : y \varepsilon Y\}$ will be called equiperfect almost everywhere $[\nu]$ (e.p.a.e. $[\nu]$) if for every $\mathcal{S}$-measurable, real-valued function f on X there is a linear Borel set $B(f)$ contained in $f(X)$ and a set $N(f)$ in $\mathfrak{J}$ such that $\nu(N(f)) = 0$ and $\mu_y(f^{-1}(B(f))) = 1$ for every y not in $N(f)$. THEOREM. *The mixture measure* μ *is perfect if and only if the family* $\{\mu_y : y \varepsilon Y\}$ *is* e.p.a.e. [ν]. COROLLARY. *Let* $(X, \mathcal{S}, \mu)$ *be a probability space and* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be sub-sigma-algebras of* $\mathcal{S}$ *for which there exists a* r.c.p. $\mu(\cdot, \cdot \mid \mathcal{S}_1, \mathcal{S}_2)$. *The restriction of* μ *to* $\mathcal{S}_1$, $\mu \mid \mathcal{S}_1$, *is perfect if and only if the family* $\{\mu(\cdot, x \mid \mathcal{S}_1, \mathcal{S}_2): x \varepsilon X\}$ *is* e.p.a.e. [$\mu \mid \mathcal{S}_2$]. LEMMA. *Let* $(X, \mathcal{S}, \mu)$ *be a probability space and* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be sub-sigma-algebras of* $\mathcal{S}$, *with* $\mathcal{S}_1$ *countably generated. If* $\mu \mid \mathcal{S}_1$ *is perfect, then there is a* r.c.p. $\mu(\cdot, \cdot \mid \mathcal{S}_1, \mathcal{S}_2)$. THEOREM. *Let* $(X, \mathcal{S}, \mu)$ *be a probability space and* $\mathcal{R}$ *any sub-sigma-algebra of* $\mathcal{S}$. *The probability measure* μ *is perfect if and only if* (i) *there is a* r.c.p. $\mu(\cdot, \cdot \mid \mathcal{J}, \mathcal{R})$, *for every countably generated sub-sigma-algebra* $\mathcal{J}$ *of* $\mathcal{S}$, *and* (ii) *the family* $\{\mu(\cdot, x \mid \mathcal{S}, \mathcal{R}): x \varepsilon X\}$ *is* e.p.a.e. [$\mu \mid \mathcal{R}$]. The Lemma is a sharpening of a result of Jiřina regarding the existence of r.c.p.'s and the second theorem is a generalization of the result announced in the abstract mentioned above.

43. On the Non Existence of PBIBD with the Parameters $v - 28$, $n_1 = 12$, $n_2 = 18$, $p_{11}^2 = 4$, $b < v$. ESTHER SEIDEN, Michigan State University.

A. J. Hoffman showed ("On the uniqueness of the triangular association scheme," *Ann. Math Statist.* **31** 492–497) that for $n = 8$ the triangular association scheme is not unique. L. C. Chang obtained the same result ("The uniqueness and nonuniqueness of the triangular association scheme," Science Record, **3**, New Series, 604–613). He also showed ("Association Schemes of partially balanced designs with parameters $v = 28$, $n_1 = 12$, $n_r = 15$ and $p_{11}^2 = 4$) that there are exactly three counter examples for $n = 8$. It seems therefore of interest to construct designs with $n = 8$ which would not be constructed with the triangular association scheme. It is shown that no such designs exist provided that $b < v$.

44. On the Median Test for the c-Sample p-Variate Nonparametric Location Problem. PRANAB KUMAR SEN, University of California, Berkeley. (By title.) (Introduced by J. L. Hodges, Jr.)

The univariate two-sample non-parametric median test by Mood (1950) has been extended here to the case of c independent samples ($c > 2$), each sample observation being a p-variate one ($p \geqq 2$). This test strictly distribution-free under the null hypothesis that the cp-variate cdf's are all identical. The exact null distribution of the test criterion has been obtained, and approximated by a chi-square distribution with $(c - 1)p$ degrees of freedom when the sample sizes are large. The consistency of the test has been established for any divergence of the c population median-vectors of the cp-variate cdf's (and hence against any translation type of alternatives), and the asymptotic non-null distribution of the test criterion has also been deduced for a sequence of translation-type of alternatives. The asymptotic power-efficiency of this test with respect to the multivariate extension of the rank-sum test (Chatterjee and Sen, 1964, unpublished) and Hotelling's T^2 test has also been studied.

45. A Characterization of Pareto's Distribution and $(k + 1)x^k/\theta^{(k+1)}$. M. S. SRIVASTAVA, University of Toronto. (By title.)

The paper derives theorems characterizing the following two distributions: (1) $p(x, \theta) = (k + 1)x^k/\theta^{k+1}$, $0 \leqq x \leqq \theta, 0 < \theta < \infty$, $k \geqq 0$, (2) Pareto's distribution:

$$q(x, \theta) = (k - 1)\theta^{k-1}/x^k, \; x \geqq \theta, \; 0 < \theta < \infty, \; k > 1.$$

THEOREM 1. *Let* F *be an absolutely continuous distribution function of the random variable* X, $F(0) = 0$, $F(\theta) = 1$. *Let* $X_1 < X_2 < \cdots < X_n$ *denote the order statistics of a random sample of size n from this distribution. The statistic* X_n *and* $(X_1 + \cdots + X_n)/X_n$, *or equiv-*

alently $(X_1 + \cdots + X_{n-1})/X_n$ *are independent if and only if the* pdf *of* X *is* $p(x_2\theta) = (k+1)x^k/\theta^{k+1}$, $0 \leqq x \leqq \theta$, $0 < \theta < \infty$, $k \geqq 0$.

THEOREM 2. *Let* F *be an absolutely continuous distribution function of the random variable* X, $F(\theta) = 0$. *Let* $X_1 < X_2 < \cdots < X_n$ *denote the order statistics of a random sample of size* n *from this distribution. The statistic* X_1 *and* $(X_1 + \cdots + X_n)/X_1$, *or equivalently* $(X_2 + \cdots + X_n)/X_1$ *are independent if and only if* X *follows Pareto's distribution.*

46. Non-Linear Predictors Depending on the Last Observation. HOWARD J. WEINER, Stanford University.

We consider the non-linear predictor of X_R, $R \geqq 0$, given the observable time series $\{X_t, t \varepsilon [-T, 0]\}$, denoted by X_R^*. Let $\phi_{X_{t_1}, \cdots, X_{t_n}}(u_1, \cdots, u_n)$ denote the n-dimensional characteristic function of $X_{t_1}, \cdots, X_{t_n}$. For a covariance stationary time series with zero means, if $X_R^* = c_{0R}X_0$, then $\phi_{X_R, X_t}(u, v) = F[u \exp(-\beta R), v, t]$, $t \varepsilon [-T, 0]$, and some $\beta > 0$, and $c_{0R} = \exp(-\beta R)$. A partial converse is obtainable. Let the time series have finite kth moments, and for every n, time points $t_1 \leqq t_2 \leqq \cdots \leqq t_n \varepsilon [-T, 0]$, and $R \geqq 0$, the $n+1$ dimensional characteristic function of $X_{t_1}, \cdots, X_{t_n}, X_R$ depends on $t_1, \cdots, t_n, R$ only through the differences $t_i - t_{i-1}$, $i = 2, \cdots, n$, and $R - t_n$. Then $(X_R^k)^* = c_{0R}X_0^k + d_{0R}$ if and only if

$$i^{-k}\partial^k/\partial u^k(\phi_{X_{t_1}, \cdots, X_{t_n}, X_R}(u_1, \cdots, u_n, u))\,|_{u=0} = \exp(-\beta(R - t_n)) \cdot f(u_1, \cdots, u_n) + h(u_1, \cdots, u_n),$$

where $\beta > 0$. $h(u_1, \cdots, u_n)$, c_{0R}, and d_{0R} may be specified exactly. This result is shown to be a "strongest" theorem of its type and is also extended to processes with stationary independent increments and finite kth moments. (Preliminary report.)

47. The Goodness-of-Fit Statistic V_N: Distribution and Significance Points. MICHAEL A. STEPHENS, McGill University.

Kuiper (*Koninkl. Nederl. Akad. van Wetenschappen*, Series A, **63** (pp. 38–47) has suggested $V_N = \sup_x (F_N(x) - F(x)) - \inf_x (F_N(x) - F(x))$ as a suitable statistic for testing whether a random sample, size N, sample distribution function $F_N(x)$, comes from a given continuous distribution $F(x)$. It is useful for observations on a circle, for which the Kolmogorov statistic $D_N = \sup_x |F_N(x) - F(x)|$ is unsuitable, since D_N takes different values for different choices of origin. V_N may be used also for observations on a line (see Pearson, *Biometrika* **50** pp. 315–326). In this paper the exact distribution of V_N, on the null hypothesis, is found for both the upper and lower tails. Kuiper has given an approximation to the distribution for large N. The results are used to calculate tables of upper and lower significance points, for $N = 2(1)\ 12(2)\ 20(10)\ 80(20)\ 100$, for use with the goodness-of-fit test. Critical values of $N^{\frac{1}{2}}V_N$ are included to facilitate interpolation for significance points for larger N.

(Abstract of a paper to be presented at the Western Regional Meeting, Berkeley, California, July 19–21, 1965. Additional abstracts will appear in future issues.)

1. The Sequential Compound Decision Problem with $m \times n$ Loss Matrix. J. VAN RYZIN, Argonne National Laboratory.

Let X_k, $k = 1, 2, \cdots$, be a sequence of independent random variables where X_k is distributed as P_{θ_k}, θ_k in $\Omega = \{1, \cdots, m\}$. After observing $X_1, \cdots, X_k$ we make one of n

decisions d_k in $\mathfrak{D} = \{1, \cdots, n\}$, $k = 1, 2, \cdots$. The loss matrix is the same for all k. Let $T = \{t_k(X_1, \cdots, X_k); k = 1, 2, \cdots\}$ be a sequential decision procedure for the problem and define $R(T, \theta_k)$ as the risk in the kth component problem. With $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots)$, let $R_N(T, \boldsymbol{\theta}) = N^{-1}\sum_{k=1}^{N} R(T, \theta_k)$ and $\phi_N(\boldsymbol{\theta}) = N^{-1}\sum_{k=1}^{N} R(t(X_k, \theta_1, \cdots, \theta_N), \theta_k)$, where $t(X_k, \theta_1, \cdots, \theta_N)$ is a procedure which in the kth component problem is Bayes against the N-stage empirical distribution, ξ_N, of the θ's on Ω. A sequential procedure $T^* = \{t(X_k, \hat{\xi}_{k-1}); k = 1, 2, \cdots\}$ is obtained by substituting estimates $\hat{\xi}_{k-1} = \hat{\xi}_{k-1}(X_1, \cdots, X_{k-1})$ for ξ_{k-1}, $k = 1, 2, \cdots$, $(\hat{\xi}_0 = \xi_0 = m^{-1}(1, \cdots, 1))$ in a procedure whose kth component rule is a Bayes solution with respect to ξ_{k-1}, the $(k - 1)$st stage empirical distribution of the θ's on Ω. Then, for a wide class of such estimates $\hat{\xi}_{k-1}$ we have $R_N(T^*, \boldsymbol{\theta}) - \phi_N(\boldsymbol{\theta}) \leqq cN^{-\frac{1}{2}}$ uniformly in the sequence $\boldsymbol{\theta}$. In particular, if $m = n = 2$ this theorem generalizes the results of Samuel (*Ann. Math. Statist.* **34** 1079–1094) to arbitrary pairs of distributions $\{P_1, P_2, P_1 \neq P_2\}$ and simultaneously strengthens the convergence rate therein. Higher order convergence rates in the case $m = n = 2$ are also obtained under more stringent conditions.

(Abstracts of papers not connected with any meeting of the Institute.)

1. Geometrical Properties of Solutions of Linear Inequalities (Preliminary Report). THOMAS M. COVER AND BRADLEY EFRON, Stanford University.

The set of solutions to a system of N homogeneous linear inequalities in d unknowns is a d-dimensional polyhedral convex cone given by the intersection of N half-spaces. For a suitably nondegenerate set of linear inequalities, it is known (Schläfli, Gesammelte Mathematische Abhandlungen I, Verlag Birkhauser, Basel, Switzerland, (1950) pp. 209–212) that precisely $C(N, d) = 2\sum_{k=0}^{d-1} \binom{N-1}{k}$ of the 2^N assignments of the sense of the N inequalities are consistent (in the sense that the corresponding solution cones are nonempty). This result remains true for a random system of linear inequalities, subject to simple weak conditions on the distribution of the coefficients. In this paper the random solution cone to such a system of linear inequalities is described in the sense that simple binomial expressions are given for the expected number and expected volume of the k-faces of the solution cone for $k = 0, 1, 2, \cdots, d$. The expected number and expected volume of k-faces of the dual cone to the solution cone spanned by the N random d-dimensional vectors of coefficients are also determined. The character of these results is summed up by the asymptotic behavior when the number N of linear inequalities is much greater than the number d of unknowns. In this case, the solution cone has the same expected number of faces of each dimension as a $(d - 1)$-cube, and the dual cone has an expected number of faces of a generalized $(d - 1)$-dimensional octahedron.

2. Distribution-Free Statistics for Some Linear Models with One Observation Per Cell. KJELL DOKSUM, University of California, Berkeley.

For the observations $X_{i\alpha}$, $(i = 1, \cdots, r; \alpha = 1, \cdots, n)$, assume the model $X_{i\alpha} = \gamma + \xi_i + \mu_\alpha + Y_{i\alpha}(\sum \xi_i = \sum \mu_\alpha = 0$, and the Y's are independent with common continuous distribution $F)$. For testing $H: \xi_1 = \cdots = \xi_r$, the Kruskal-Wallis statistic is applicable when each $\mu_\alpha = 0$, and can be based on the quantities $U_i - U_j$, where $U_i =$ no. of triples (k, α, β) such that $k \neq i$ and $(X_{i\alpha} - X_{k\beta}) > 0$. Two extensions to the case, $\mu_\alpha \neq 0$ for some α, are $\binom{n}{2}T_{ij} =$ no. of pairs (α, β) with $\alpha < \beta$ and $(X_{i\alpha} - X_{j\alpha} + X_{i\beta} - X_{j\beta}) > 0$, and $S_{ij} = T_{i\cdot} - T_{j\cdot}$. The asymptotic $\frac{1}{2}r(r - 1)$ variate normal distributions of $\{T_{ij}, i < j\}$ and $\{S_{ij}, i < j\}$ (properly normalized) are given for alternatives such that $\xi_i - \xi_j = a_{ij}n^{-\frac{1}{2}}$, and it is shown that the ARE (asymptotic relative efficiency) of T_{ij} to $X_{i\cdot} - X_{j\cdot}$ is $12\,\sigma^2[\int g^2]^2 = e$, where σ^2 is the variance and g the density of $Y_{11} - Y_{12}$, while the ARE of S_{ij} to $X_{i\cdot} - X_{j\cdot}$ is $e[r/24\{\frac{1}{12} + (r - 2)[\lambda(F) - \frac{1}{4}]\}]$ (which is $\geqq e$), where $\lambda(F)$ is as in Lehmann, *Ann. Math. Statist.* **35** 726–734. For H, statistics of the form $\sum[\sum a_{ij}(T_{ij} - \frac{1}{2})]^2$ and $\sum (\sum a_{ij}S_{ij})^2$ are distribution-free wrt the class of continuous F, have the ARE's of T_{ij} and S_{ij} (as given above) to the classical $\mathfrak{F}$-statistic, and are asymptotically chi-square variables.

3. Admissibility and Bayes Estimation in Sampling Finite Populations. V. P. Godambe and V. M. Joshi, Institute of Science, Bombay.

The concept of linear estimation for a finite population was generally defined by Godambe (1955). Since then the following results have been established concerning the class of *linear* unbiased estimates. [i] Non-existence of a uniformly least variance estimate (Godambe 1955). [ii] Non-existence of a uniformly least variance estimates in subclasses [Koop (1957), Prabhu Ajagaonkear (1962)]. [iii] Admissibility of a certain estimate in common use [Godambe (1960), Roy and Chakravarti (1960)]. [iv] The Bayesness of the preceding estimate [Godambe (1955), Hájek (1959)]. In this paper, (A): results [i], [ii], [iv] are extended to the entire class of unbiased estimates removing the restriction of linearity, (B): for fixed sample size designs, a class of estimates of which the estimate in [iii] is a member, is shown to be admissible in the entire class of estimates, removing the restriction of unbiasedness in (A), (C): the usual arithmetic mean is proved to be admissible, in the class of all estimates, whatever the sampling design.

4. Limiting Diffusion Approximations for the Many Server Queue and the Repairman Problem. Donald L. Inglehart, Cornell University.

Consider a many server queueing process in which the customers arrive according to a Poisson process and form a single line. We assume that the service times are independent exponentially distributed random variables. The service discipline requires that no server be idle if there is a customer waiting, but it is otherwise arbitrary. We let $X_n(t)$ denote the number of customers waiting or being served at time t. Next we choose a value $\rho\,(0 < \rho < 1)$ and set the parameter of the Poisson arrival process equal to $n\rho$. We keep the mean of the service distribution fixed at one. Then the sequence of processes $Y_n(t) = \{X_n(t) - n\rho\}/(n\rho)^{\frac{1}{2}}(n = 1, 2, \cdots)$ is weakly convergent (Markov sense) to the Ornstein-Uhlenbeck diffusion process as $n \rightarrow \infty$. This weak convergence gives us approximations, when the number of servers is large, for most of the distribution functions of interest in queueing theory. Similar results are obtained for the repairman (or machine interference) problem. The limits in this case are taken as the number of machines becomes large. These results are all obtained by straightforward applications of the paper by Stone [Limit theorems for random walks, birth and death processes, and diffusion processes, *Illinois J. Math.* **4** (1963)].

5. Sampling Distributions under Missing Values. A. M. Mathai, McGill University.

In this article some of the problems connected with statistical analysis of missing observations are discussed. When some of the observations are missing, in many cases, the resulting sample size cannot be treated as a constant. So the sampling distributions of some statistics under a generating system for the sample size are considered to a great extent in this article. For convenience, the generating system is assumed to make the resulting sample size a truncated binomial or a truncated Poisson distribution. Generalizations to multivariate populations and distribution of the sample mean vector, when the data is completely fragmentary, are also discussed. For simplicity the parent population is assumed to be normal. Best approximate procedures for the distributions are also considered.

6. Some Useful Results in Analysis of Data. A. M. MATHAI, McGill University.

In experimental design problems usually there are missing observations. The usual procedure of statistical analysis of missing values is to estimate the missing values and complete the analysis. In experiments involving plants or animals, some of the experimental units may die out before completing the experiment. The force of mortality may be acting in such a way that the resulting frequencies may be random variables. In this article statistical analysis of multiway classifications, when the cell frequencies are random variables, is considered. Different test criteria under the modified assumptions are also discussed.

7. Approximate Analysis for a Two-Way Layout. A. M. MATHAI, McGill University.

Analysis of two-way classifications with multiple observations per cell becomes difficult because of the problem of solving a system of complicated normal equations. In this article, an efficient approximate method of analysis, which is sufficient on many problems of testing hypothesis, is discussed. The conditions in the experimental design model are conveniently used to reduce a singular system of linear equations to a non-singular system. A suitably selected "norm" of a matrix is minimized to achieve faster convergence of a matrix series. The estimating equations for the treatment effects β_i, $i = 1, 2, \cdots t$, which may be reduced to the form $(I - A)\hat{\beta} = Q$ where $\hat{\beta}$, Q are vectors and A is a matrix, are reduced to the form $(I - B)\hat{\beta} = Q$. Where any "norm" of B is less than one and is a minimum under the given situations. So $\hat{\beta} = (I + B + B^2 + \cdots)Q$ and this equation gives successive approximations.

8. Operating Characteristic and Expected Sample Size of a Sequential Probability Ratio Test. M. RAGHAVACHARI, University of California, Berkeley.

For a class of sequential probability ratio tests used to test the simple hypothesis H: $\theta = \theta_1$ against the simple alternative K: $\theta = \theta_2$ with $\theta_2 > \theta_1$, where θ is the unknown parameter of the negative exponential distribution, exact expressions for the expected sample size of the operating characteristic are obtained. (Similar results were also obtained by L. Weiss in an ONR report.) In this paper the nature of the expected sample size function $E_\theta(N)$ and the effect of Wald's approximations to the stopping bounds have been studied. It is also shown that $\sup_\theta E_\theta(N)$ may be attained outside $[\theta_1, \theta_2]$ even for a symmetrical procedure, i.e., one with error probabilities equal, thus disposing of a problem raised in E. L. Lehmann (1959) *Testing Statistical Hypotheses*, Wiley, New York, p. 102.