DISTRIBUTION OF THE ABSOLUTE MAXIMUM FOR CERTAIN BROWNIAN MOTIONS

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The processes referred to in the title are the Brownian motions on $[0, \infty)$ which have continuous paths up to a finite "lifetime" at which they are absorbed at 0. The absolute maximum of such a process $Y_x(t)$, $0 \leqq t < \zeta(x)$, where $Y_x(0) = x$ and $\zeta(x)$ is the lifetime, is defined by

$$M(x) = \max_{0 \leqq t < \zeta(x)} Y_x(t).$$

It is well known, and may be checked from [1], that the class of processes under consideration is defined by three parameters $p_1 > 0$, $p_2 \geqq 0$, $p_3 \geqq 0$, subject to $p_1 + p_2 + p_3 = 1$. The corresponding processes have as infinitesimal generators restrictions of $\frac{1}{2}\frac{d^2}{dx^2}$ to domains determined respectively by boundary conditions of the form

$$p_1 F(0) - p_2 \frac{d}{dx} F(0) + p_3 \frac{1}{2}\frac{d^2}{dx^2} F(0) = 0,$$

in which the derivatives are from the right.

(The author is indebted to the referee for correcting an oversight in the original formulation of the following theorem.)

Theorem. *The distribution function of $M(0)$ is given by*

$$P\{M(0) < y\} = (p_1 y)/(p_1 y + p_2).$$

That of $M(x)$ is

$$P\{M(x) < y\} = [(y - x)/y][(p_1 y)/(p_1 y + p_2)]; \qquad 0 \leqq x \leqq y,$$
$$= 0; \qquad y < x.$$

Proof. Since the distributions do not depend on p_3, we shall first carry out the proof in the case $p_3 = 0$, which is the elastic barrier case. Let $X_x(t)$, $X_x(0) = x$, be an ordinary Brownian motion, and let

$$t_0(t) = \frac{d}{dy} \int_0^t \chi_{(0,y)}(X_x(\tau))\, d\tau \big|_{y=0}$$

be its local time at 0. It is shown in [1] that $Y_x(t)$ may be defined by killing $|X_x(t)|$ at the time $\zeta(x) = \inf \{t > 0 : t_0(t) = (p_2/p_1)\rho\}$, where ρ is entirely independent of the process X_x and $P\{\rho > t\} = e^{-t}$. Let $f(y, t, w)$ denote the local time of $X_x(T(x) + t)$ at y, where $T(x) = \inf \{t > 0 : X_x(t) = 0\}$. In particular,

Received 16 September 1964.

$f(0, t, w) = t_0(t)$. For a proof of the existence of $f(y, t, w)$, we refer to [3]. According to the results of [2], if ρ is given, then $f(y, \zeta(x), w)$ is the diffusion in y, $0 \leqq y < \infty$, with initial value $f(0, \zeta(x), w) = (p_2/p_1)\rho$, infinitesimal generator $4z \frac{d^2}{dz^2}$, and an absorbing barrier at 0. Also $f(-y, \zeta(x), w)$, $0 \leqq y < \infty$, is another such diffusion, entirely independent of the first (when ρ is given). It is therefore clear that

$$M(0) = \max (\sup \{y > 0 : f(y, \zeta(x), w) > 0\}, \\ - \inf \{y < 0 : f(y, \zeta(x), w) > 0\})$$

and when ρ is given this becomes the maximum of the two independent first passage times to 0 for the two diffusion processes. The distribution of these passage times ([2], p. 60) is $\exp(-p_2\rho/2p_1y)$. Hence we have

$$\begin{aligned} P\{M(0) < y\} &= EP\{M(0) < y \mid \rho\} \\ &= E[\exp(-p_2\rho/p_1y)] \\ &= p_1y/(p_1y + p_2), \end{aligned}$$

as was to be shown.

The distribution of $M(x)$ now follows immediately since the process $X_x(t)$, $0 \leqq t \leqq T(x)$, is independent of $X_x(T(x) + t)$, $0 \leqq t < \infty$.

To extend these results to the case $p_3 > 0$, we use the representation of [1]. According to this, we may obtain $Y_x(t)$ by first constructing the process

$$Z_x(t) = |X_x(t_*^{-1}(t))|$$

where $t_*^{-1}(t)$ is the inverse function of $t_*(t) = t + (p_3/p_2)t_0(t)$, and then killing $Z_x(t)$ at the time $\zeta(x) = \inf \{t > 0 : t_0(t_*^{-1}(t)) = (p_2/p_1)\rho\}$. (If $p_2 = 0$, this construction is to be replaced by the process with $p_1 = 1$, except that an exponential wait at 0 parameter p_1/p_3 precedes $\zeta(x)$. This wait does not affect the distribution of $M(x)$.) It is now obvious that the change of time scale $t \leftrightarrow t_*^{-1}(t)$ transforms this process $Y_x(t)$ into the one with p_2/p_1 unchanged but $p_3 = 0$, and since the transformation does not change $M(x)$ the proof is complete.

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