INEQUALITIES FOR THE rth ABSOLUTE MOMENT OF A SUM OF RANDOM VARIABLES, $1 \leqq r \leqq 2$

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1. Introduction and summary. Let $X_1, X_2, \cdots, X_n$ be a sequence of random variables (r.v.'s) and put $S_m = \sum_{\nu=1}^m X_\nu$, $1 \leqq m \leqq n$. It is well-known that

$$(1) \qquad \begin{aligned} E|S_n|^r &\leqq n^{r-1} \sum_{\nu=1}^n E|X_\nu|^r && r > 1, \\ E|S_n|^r &\leqq \sum_{\nu=1}^n E|X_\nu|^r, && r \leqq 1. \end{aligned}$$

However, if the r.v.'s satisfy the relations

$$(2) \qquad E(X_{m+1} \mid S_m) = 0 \text{ a.s.} \qquad 1 \leqq m \leqq n-1,$$

it is possible to improve the first inequality considerably. The case $r > 2$ with independent r.v.'s will be treated elsewhere by one of the authors, von Bahr. If $r = 2$, we have, under (2),

$$(3) \qquad ES_n^2 = \sum_{\nu=1}^n EX_\nu^2.$$

In the case $1 \leqq r \leqq 2$, we will show that under (2)

$$(4) \qquad E|S_n|^r \leqq C(r, n) \sum_{\nu=1}^n E|X_\nu|^r,$$

where $C(r, n)$ is a bounded function of r and n. In Theorem 2 we show that (4) is true with $C(r, n) = 2$. If the distribution of each X_{m+1} conditioned by S_m is symmetric about zero, one can put $C(r, n) = 1$ (Theorem 1). Further, if the r.v.'s satisfy the following conditions

$$(5) \qquad E(X_i \mid R_{mi}) = 0 \text{ a.s.} \qquad 1 \leqq i \leqq m+1 \leqq n,$$

where $$R_{mi} = \sum_{\nu=1, \nu \neq i}^{m+1} X_\nu$$

it is possible to put $C(r, n) = 2 - n^{-1}$.

The conditions (2) and (5) are fulfilled if the r.v.'s are independent and have zero means. In this case, however, it is possible to make $C(r, n)$ dependent on r, so that $C(r, n) \to 1$ as $r \to 2$. It is possible to show by an example, that (4) is not generally true with $C(r, n) = 1$ even in this case.

If $1 \leqq r < s \leqq 2$ and $E|X_\nu|^s < \infty$, $1 \leqq \nu \leqq n$, it is generally better not to use (4) directly, but to use it together with $E|S_n|^r \leqq (E|S_n|^s)^{r/s}$, so that $E|S_n|^r \leqq (C(s, n) \sum_{\nu=1}^n E|X_\nu|^s)^{r/s}$.

The case $r < 1$ is by (1) trivial.

2. Symmetric conditional distributions. We start by stating without proof a special case of an inequality due to Clarkson [2]:

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$$(6) \qquad |x + y|^r + |x - y|^r \leqq 2(|x|^r + |y|^r), \qquad 1 \leqq r \leqq 2,$$

where x and y are real or complex quantities. In the real case this inequality will also follow as a special case of Theorem 1, which we shall prove by different means in Section 3.

We say that the distribution of a r.v. Y conditioned by a r.v. X is symmetric (about zero), if for every $a \geqq 0$, $P(Y > a \mid X) = P(Y < -a \mid X)$ a.s.

LEMMA 1. *Let X and Y be two* r.v.'s *with $E|X|^r < \infty$ and $E|Y|^r < \infty$. If the distribution of Y conditioned by X is symmetric, then $E|X + Y|^r \leqq E|X|^r + E|Y|^r$, $1 \leqq r \leqq 2$.*

The lemma follows from

$$E|X + Y|^r = E\{E(|X + Y|^r \mid X)\} = E\{E(|X - Y|^r \mid X)\} = E|X - Y|^r$$

by taking expectations in (6).

The following theorem is now easily proved by induction.

THEOREM 1. *Let $X_1, X_2, \cdots X_n$ be a sequence of* r.v.'s *with $E|X_\nu|^r < \infty$, $1 \leqq \nu \leqq n$. If the distribution of each X_{m+1} conditioned by S_m is symmetric, $1 \leqq m \leqq n - 1$, then*

$$E|S_n|^r \leqq \sum_{\nu=1}^n E|X_\nu|^r, \qquad 1 \leqq r \leqq 2.$$

3. Zero conditional expectations. In order to be able to prove a corresponding inequality under the condition (2), we shall express absolute moments by means of characteristic functions (Lemma 2), state an inequality due to Loève (Lemma 3) and finally examine the effect of symmetrization (Lemma 4).

We shall use the formula

$$(7) \qquad |x|^r = K(r)\int_{-\infty}^{\infty} (1 - \cos xt)/|t|^{r+1}\, dt, \qquad 0 < r < 2,$$

where x is real and

$$(8) \qquad K(r) = \left(\int_{-\infty}^{\infty} (1 - \cos u)/|u|^{r+1}\, du\right)^{-1} = (\Gamma(r + 1)/\pi) \sin r\pi/2.$$

LEMMA 2. *Let X be a* r.v. *with the distribution function* (d.f.) *$F(x)$ and the characteristic function* (ch.f.) *$f(t)$, where $f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$. If $E|X|^r < \infty$, then*

$$E|X|^r = K(r)\int_{-\infty}^{\infty} (1 - Rf(t))/|t|^{r+1}\, dt, \qquad 0 < r < 2,$$

where R stands for the real part.

PROOF. From (7) we have

$$E|X|^r = \int_{-\infty}^{\infty} |x|^r\, dF(x) = K(r)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1 - \cos xt)/|t|^{r+1}\, dt\, dF(x)$$
$$= K(r)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1 - \cos xt)\, dF(x)\, dt/|t|^{r+1} = K(r)\int_{-\infty}^{\infty} (1 - Rf(t))/|t|^{r+1}\, dt.$$

Since the integrand is non-negative we may invert the order of integration.

This integral representation of the absolute moment enables us to give another proof of Theorem 1 without using the inequality (6).

PROOF OF THEOREM 1: The theorem is true if $n = 1$. We fix m, $1 \leqq m \leqq n - 1$ and let $f_m(t)$ be the ch.f. of X_{m+1} conditioned by $S_m : f_m(t) = E(\exp (itX_{m+1}) \mid S_m)$.

According to the assumptions, $f_m(t)$ is real. The ch.f. of S_{m+1} conditioned by S_m is $\exp(itS_m)f_m(t)$, and consequently we have from Lemma 2 a.s.

$$E(|S_{m+1}|^r \mid S_m) = K(r)\int_{-\infty}^{\infty} [1 - R(\exp(itS_m)f_m(t))]/|t|^{r+1}\, dt;$$

but

$$1 - R(\exp(itS_m)f_m(t)) = 1 - f_m(t)\cos tS_m = (1 - \cos tS_m)$$
$$+ (1 - f_m(t)) - (1 - \cos tS_m)(1 - f_m(t)) \leqq (1 - \cos tS_m) + (1 - f_m(t)),$$

so that

$$E(|S_{m+1}|^r \mid S_m) \leqq K(r)\int_{-\infty}^{\infty} \frac{1 - \cos tS_m}{|t|^{r+1}}\, dt + K(r)\int_{-\infty}^{\infty} \frac{1 - f_m(t)}{|t|^{r+1}}\, dt$$
$$= |S_m|^r + E(|X_{m+1}|^r \mid S_m).$$

Taking expectations, we get $E|S_{m+1}|^r \leqq E|S_m|^r + E|X_{m+1}|^r$, and the theorem follows by induction.

REMARK: Applying Theorem 1 to two independent r.v.'s X and Y, where $X = x$ with probability 1 and $Y = \pm y$ each with probability $\frac{1}{2}$, we get the inequality (6).

LEMMA 3. *If X and Y are two* r.v.'s *with $E|X|^r < \infty$, $E|Y|^r < \infty$ and $E(Y \mid X) = 0$* a.s., *then $E|X + Y|^r \geqq E|X|^r$, $r \geqq 1$.*

For a proof see Loève [3], p. 263.

We will denote a r.v. by X', if it is independent of and has the same d.f. as X. If $f(t)$ is the ch.f. of X, $X - X'$ has the ch.f. $|f(t)|^2$, which is real and non-negative. From Lemma 2 and the identity $2(1 - Rf(t)) = (1 - |f(t)|^2) + |1 - f(t)|^2$, we obtain

$$(9)\quad E|X|^r = \tfrac{1}{2}E|X - X'|^r + \tfrac{1}{2}K(r)\int_{-\infty}^{\infty}(|1 - f(t)|^2/|t|^{r+1})\, dt, \quad 0 < r < 2.$$

From this formula and Lemma 3, we get the following lemma.

LEMMA 4. *If X is a* r.v. *with $EX = 0$ and $E|X|^r < \infty$ then*

$$(10)\qquad \tfrac{1}{2}E|X - X'|^r \leqq E|X|^r \leqq E|X - X'|^r, \qquad 1 \leqq r \leqq 2.$$

We are now ready to prove the inequality (4) in two cases.

THEOREM 2. *Let $X_1, X_2, \cdots X_n$ be a sequence of* r.v.'s *satisfying* (2). *If $E|X_\nu|^r < \infty$, $1 \leqq \nu \leqq n$, then $E|S_n|^r \leqq 2\sum_{\nu=1}^{n} E|X_\nu|^r$, $1 \leqq r \leqq 2$.*

PROOF. The theorem is true if $n = 1$. We fix m, $1 \leqq m \leqq n - 1$ and introduce a r.v. X'_{m+1}, which, conditioned by S_m, is independent of and has the same conditional d.f. as X_{m+1}. By Lemma 3, Lemma 1 and the left hand inequality of Lemma 4 we obtain $E|S_{m+1}|^r = E|S_m + X_{m+1}|^r \leqq E|S_m + X_{m+1} - X'_{m+1}|^r \leqq E|S_m|^r + E|X_{m+1} - X'_{m+1}|^r \leqq E|S_m|^r + 2E|X_{m+1}|^r$ and the theorem follows by induction.

THEOREM 3. *Let $X_1, \cdots X_n$ be a sequence of* r.v.'s *satisfying* (5). *If $E|X_\nu|^r < \infty$, $1 \leqq \nu \leqq n$, then*

$$E|S_n|^r \leqq (2 - n^{-1})\sum_{\nu=1}^{n} E|X_\nu|^r, \qquad 1 \leqq r \leqq 2.$$

PROOF: The theorem is true if $n = 1$. Supposing it is true if $n = m$, we fix an i, $1 \leqq i \leqq m + 1$, and put $S_{m+1} = R_{mi} + X_i$, where $R_{mi} = \sum_{\nu=1,\nu\neq i}^{m+1} X_\nu$. Introducing a r.v. X_i', which conditioned by R_{mi} is independent of and has the same conditional d.f. as X_i, we obtain from Lemma 3, Lemma 1, the assumption and Lemma 4:

$$E|S_{m+1}|^r = E|R_{mi} + X_i|^r \leqq E|R_{mi} + X_i - X_i'|^r$$
$$\leqq E|R_{mi}|^r + E|X_i - X_i'|^r \leqq (2 - m^{-1}) \sum_{\nu=1,\nu\neq i}^{m+1} E|X_\nu|^r + 2E|X_i|^r.$$

Suming over i from 1 to $m + 1$, we obtain

$$(m + 1)E|S_{m+1}|^r \leqq (2m + 1) \sum_{\nu=1}^{m+1} E|X_\nu|^r,$$

i.e.

$$E|S_{m+1}|^r \leqq (2 - (m + 1)^{-1}) \sum_{\nu=1}^{m+1} E|X_\nu|^r$$

and the theorem follows by induction.

4. Independence. We will now suppose that the r.v.'s X_ν are independent and have zero means. In this case it is possible to state an inequality of the type (4), which continuously passes over to (3) as $r \to 2$. The method is to estimate the last term in (9).

LEMMA 5. *If $f(t)$ is the* ch.f. *of a* r.v. X, *where $EX = 0$ and $E|X|^r = \beta_r < \infty$, then*

$$|1 - f(t)| \leqq [3.38/(2.6)^r]\beta_r|t|^r, \qquad -\infty < t < \infty, \quad 1 \leqq r \leqq 2.$$

PROOF. By simple calculations one obtains

$$|1 - e^{itx} + itx| \leqq 1.3|tx|, \qquad -\infty < tx < \infty,$$
$$|1 - e^{itx} + itx| \leqq 0.5(tx)^2, \qquad -\infty < tx < \infty.$$

Multiplying the $(2 - r)$th power of the first inequality by the $(r - 1)$th power of the second, we have $|1 - e^{itx} + itx| \leqq [3.38/(2.6)^r]|tx|^r$. From $1 - f(t) = \int_{-\infty}^{\infty} (1 - e^{itx} + itx)\, dF(x)$ the stated inequality follows.

We shall now estimate the integral $J = \int_{-\infty}^{\infty} |1 - f(t)|^2/|t|^{r+1}\, dt$, where $f(t)$ satisfies the conditions of Lemma 5. Let a be a positive parameter. Then

$$J = 2\left[\int_0^a + \int_a^\infty\right] \leqq 2\left(\frac{3.38}{(2.6)^r}\beta_r\right)^2 \int_0^a \frac{t^{2r}}{t^{r+1}}\, dt + 2\int_a^\infty \frac{4}{t^{r+1}}\, dt$$
$$= \frac{2}{r}\left[\left(\frac{3.38}{(2.6)^r}\beta_r\right)^2 a^r + \frac{4}{a^r}\right].$$

We choose a so that this expression is a minimum, and obtain

$$J \leqq [27.04/(r2.6)^r]\beta_r. \tag{11}$$

We now combine (9) and (11) into the following lemma.

LEMMA 6. *If X is a* r.v. *with $EX = 0$ and $E|X|^r < \infty$, where r satisfies the following inequalities*

$$(12) \qquad D(r) = [13.52/(\pi 2.6)^r]\Gamma(r) \sin r\pi/2 < 1 \quad \text{and} \quad 1 \leqq r \leqq 2,$$

then $E|X|^r \leqq [2(1 - D(r))]^{-1}E|X - X'|^r$.

This inequality is sharper than the right hand side of (10) if $D(r) < \frac{1}{2}$. Now, $D(r)$ decreases monotonically to zero as r varies from 1 to 2, and $D(r) < \frac{1}{2}$ when $r > 1.6$. For $r = 1.8$ we have $[2(1 - D(r))]^{-1} \approx 0.643$.

THEOREM 4. *Let $X_1, X_2, \cdots X_n$ be a sequence of independent* r.v.'s *with $EX_\nu = 0$ and $E|X_\nu|^r < \infty$, $1 \leqq \nu \leqq n$. If r satisfies* (12), *then* $E|S_n|^r \leqq [1 - D(r)]^{-1} \cdot \sum_{\nu=1}^n E|X_\nu|^r$.

PROOF. We introduce a new sequence of independent r.v.'s $X_1', X_2', \cdots X_n'$ which are independent of the original ones and where X_ν' has the same distribution as X_ν, $1 \leqq \nu \leqq n$. Putting $S_n' = \sum_{\nu=1}^n X_\nu'$, we get by Lemma 6, Theorem 1 and Lemma 4

$$E|S_n|^r \leqq [2(1 - D(r))]^{-1}E|S_n - S_n'|^r \leqq [2(1 - D(r))]^{-1} \sum_{\nu=1}^n E|X_\nu - X_\nu'|^r$$
$$\leqq [1 - D(r)]^{-1} \sum_{\nu=1}^n E|X_\nu|^r.$$

5. Application. Let $X_1, X_2, \cdots X_n$ be a sequence of independent r.v.'s with the same d.f. and with $EX = 0$, $E|X_\nu|^r = \beta_r < \infty$, $1 \leqq \nu \leqq n$.

D. Brillinger [1] has shown that in this case

$$P(|\bar{X}_n| > a) = o(n^{1-r}), n \to \infty, \qquad 1 \leqq r < 2,$$

where $\bar{X}_n = n^{-1} \sum_{\nu=1}^n X_\nu$. From Theorem 3, Theorem 4 and Markov's inequality we easily deduce the following inequality, which is slightly weaker than Brillinger's, but on the other side only contains explicit quantities:

$$P(|\bar{X}_n| > a) \leqq M(r, n)\beta_r a^{-r} n^{1-r}, \qquad 1 \leqq r \leqq 2,$$

where $M(r, n) = \min \{2 - n^{-1}, [1 - D(r)]^{-1}\}$ if (12) is satisfied and $M(r, n) = 2 - n^{-1}$ otherwise.

REFERENCES

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