SOME ASPECTS OF THE RANDOM SEQUENCE

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1. Summary. This is primarily a review of combinatorial problems connected with ballot problems, runs, records, and amalgamation, but numerous new results and applications occur throughout the paper. The early history of the classical ballot problem is clarified, and many recent generalizations and applications are noted. For runs and records, the main emphasis is on the derivation of the null-hypothesis distributions, with only passing reference to asymptotics and non-null distributions. An appendix lists recent work on the Kolmogorov and Smirnov statistics. There are 183 references.

2. Ballot problems.

2.1 *The classical ballot problems.* Suppose two candidates A_1 and A_2 obtain respectively a_1 and a_2 votes in an election. ($a_1 \geqq a_2$, $a_1 + a_2 = n$). If the votes are counted in a random order, what is the probability that the winning candidate holds the lead throughout the counting? This is the classical *ballot problem* (problème du scrutin), priority in the posing and solution of which has often been ascribed (e.g. by Dvoretsky and Motzkin (1947), Feller (1957b), Takács (1962a)) to Bertrand (1887). However, Whitworth had posed and solved the problem in 1878 and included it in the fourth edition of his *Choice and Chance* (1886). His method (which remained unchanged in the fifth edition (1901), which has recently been reprinted) consists of setting up a recurrence relation and verifying that it is satisfied by the proposed solution. Application of the method of recurrence relations can often be simplified by the use of generating functions; another powerful tool in this area is the *reflection principle* usually ascribed to André (1887), which is essentially Kelvin's method of images in an enumerative context. For a thorough discussion of this principle and many of its applications, see Feller (1957b) (in Chapter 3). An excellent historical survey of the subsequent development of the Ballot Problem is given in Dvoretsky and Motzkin (1947).

There are two versions of the basic problem, according as we do or do not allow A_2's partial total to equal A_1's during the counting; we call these the weak-sense and strict-sense versions of the problem respectively. When m votes have been counted, suppose the two partial totals are $A_1(m)$ and $A_2(m)$ ($A_1(0) = A_2(0) = 0$, $A_1(n) = a_1$, $A_2(n) = a_2$). Then the events of interest are

$$E_W = \{A_1(m) \geqq A_2(m), m = 1, 2, \cdots, n\},$$

$$E_S = \{A_1(m) > A_2(m), m = 1, 2, \cdots, n\}.$$

Obviously, there is a very close relationship between these two events; thus, E_S

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holds if, and only if, $A_1(1) = 1$ and E_W holds for the remaining $(a_1 - 1) + a_2$ votes. Whitworth's result is

$$P\{E_W \mid a_1, a_2\} = (a_1 + 1 - a_2)/(a_1 + 1) \tag{1}$$

from which it follows that $P\{E_S \mid a_1, a_2\} = (a_1 - a_2)/(a_1 + a_2)$. We shall consider several generalizations of these original problems; first, however, as an indication of the ubiquitous nature of our topic, we show that a result of Wine and Freund (1957) enumerating the "decision patterns" of a set of n means is equivalent to Whitworth's ballot problem (with $a_1 = a_2 = n$). Given a set of n real numbers $x_1, \cdots, x_n$, a *decision pattern* is a system of subsets of adjacent elements subject to the rule that no subset in the system may be completely contained in any other. (The subset $(x_i, x_{i+1}, \cdots, x_j)$ is in the system if the means x_i and x_j are not judged to be significantly different according to some test procedure, but the pairs (x_i, x_{j+1}) and (x_{i-1}, x_j) are both significant.) A number may lie in more than one subset; a subset may contain only one number. Corresponding to any such decision pattern, we construct an arrangement of $2n$ votes (n each for A_1 and A_2) as follows: The rth vote for A_2 is to fall between the sth and $s + 1$th votes for A_1 if and only if x_s is the largest number such that x_r, x_s are in the same subset. (We use the obvious convention if $s = n$.) Obviously, s will always be greater than or equal to r, so the arrangements of the votes will satisfy E_W. It is readily seen that the correspondence is one-to-one; and since the total number of arrangements of the votes is $\binom{2n}{n}$, the number of decision patterns must be $(n + 1)^{-1} \binom{2n}{n}$ which is the result obtained by Wine and Freund.

2.2 *Barbier's generalization.* In 1887 Barbier considered the event

$$E_S(\mu) = \{A_1(1) = 1, A_1(m) > \mu A_2(m), m = 1, 2, \cdots, n\}$$

for arbitrary $\mu \geqq 0$, and published the result

$$P\{E_S(\mu) \mid a_1, a_2\} = (a_1 - \mu a_2)/(a_1 + a_2). \tag{2}$$

In fact (2) is true only if μ is integral, as was shown much later by Aeppli (1924). The corresponding weak-sense result is

$$P\{E_W(\mu) \mid a_1, a_2\} = (a_1 + 1 - \mu a_2)/(a_1 + 1). \tag{3}$$

Dvoretsky and Motzkin (1947) gave a simple proof of (2) and (3) (for integral μ) depending on the observation that if $v_1, v_2, \cdots, v_n$ is any one arrangement of the votes, then all n cyclic permutations of $v_1, \cdots, v_n$ are equally likely; amongst these it is shown that exactly $a_1 - \mu a_2$ of these permutations are favorable to $E_S(\mu)$, and (2) follows directly. Recently Takács (1962a) has extended this method by showing that if $k_1, \cdots, k_n$ are any non-negative integers summing to k, and if $X_1, \cdots, X_n$ is a random cyclic permutation of $k_1, \cdots, k_n$, then

$$P\{X_1 + X_2 + \cdots + X_r < r, r = 1, 2, \cdots, n\} = 1 - (k/n). \tag{4}$$

The result (2) follows by taking the k's to be a_1 0's and a_2 $(\mu + 1)$'s. Dwass (unpublished; referred to in Dwass (1962)) obtained the same result.

Using the evident recurrence relation

$$N(a_1, a_2) = N(a_1 - 1, a_2) + N(a_1, a_2 - 1)$$

for the number $N(a_1, a_2)$ of permutations of the votes that are favorable to $E_S(\mu)$, Takács (1962b) has obtained the general formula

$$\binom{a_1 + a_2}{a_1} P\{E_S(\mu) \mid a_1, a_2\} = \sum_{j=0}^{a_2} C_j \binom{a_1 + a_2 - 1 - j}{a_2 - j} \tag{5}$$

where the constants C_j are determined by

$$C_0 = 1, \quad \sum_{j=0}^{b} C_j \binom{[\mu b] + b - 1 - j}{b - j} = 0, \qquad b = 1, 2, \cdots. \tag{6}$$

When μ is an integer, $C_j = -\mu (j = 1, 2, \cdots)$ and (5) reduces to (2). Hence, Takács obtains the distribution of $\sup(T_n/n)$, where T_n is the number of successes in the first n of an infinite sequence of Bernoulli trials, the probability of success being p at each trial. The result is

$$P\{T_n/n < 1/1 + \mu, n = 1, 2, \cdots\} = (1 - p) \textstyle\sum_{j=0}^{\infty} C_j p^j$$

where the C_j's are as in (6). In the case $p = \frac{1}{2}$ and μ rational, this problem was solved previously by Newman (1960).

Considering the special case $a_1 = k\alpha$, $a_2 = k\beta$, $\mu = \alpha/\beta$ (with α prime to β), Grossman (1950) obtained the result

$$P\{E_W(\alpha/\beta) \mid k\alpha, k\beta\} = \sum F_i^r F_j^s \cdots / r!s! \cdots \tag{7}$$

where the sum is over all numbers satisfying $ri + sj + \cdots = k$, and where

$$F_r = \frac{1}{r\alpha + r\beta} \binom{r\alpha + r\beta}{r\alpha}.$$

By considering cyclic permutations, Bizley (1954) supplied a proof of (7), gave the corresponding strict-sense result, and showed that the number of arrangements in which $A_1(m) > (\alpha/\beta) A_2(m)$ throughout with equality exactly t times is the coefficient of x^k in $[1 - \exp - (xF_1 + x^2 F_2 + \cdots)]^t$. These results have obvious interpretations in terms of enumerations of paths on the rectangular lattice; Moser and Zayachkowski (1963), have shown that Bizley's formulae can be modified to cover the case where diagonal steps on the lattice are allowed. See also Good (1958). Gobel (1963) has obtained two recurrences for the number of paths lying completely below an arbitrary (specified) boundary. See also Switzer (1964).

Here is another variation on the classical ballot problem. Consider any (fixed) sequence of votes containing a_1 votes for A_1 and a_2 votes for $A_2 (a_1 + a_2 = n)$. We compare this sequence with another arbitrary sequence of n votes, and ask in how many ways the second sequence may be chosen (different from the first) such that on the first occasion on which the two sequences disagree, it is because a vote for A_1 in the first sequence has been replaced by a vote for A_2 in the second sequence (and not the reverse). In terms of paths on the rectangular lattice, we ask for the number of n-step paths from the origin which do not agree completely with a given standard path, and first deviate from this path in the upward direction rather than downwards. Let us call these *upper paths* (relative to the standard path). This problem has an elegant solution, as follows. Write the standard sequence as a string of 1's and 0's (1 for A_1, 0 for A_2). This gives the binary representation of some number N, which is exactly the number desired. Further, if

$$N = 2^{N_1} + 2^{N_2} + \cdots + 2^{N_{a_1}} (N_1 > N_2 > \cdots > N_{a_1}),$$

then the number of upper paths that end at the point (b_1, b_2) is

$$(8) \qquad N_{(b_1)} = \binom{N_1}{b_1} + \binom{N_2}{b_1 - 1} + \cdots + \binom{N_{a_1}}{b_1 - a_1 + 1}.$$

Expressions of this type arose in the work of Dubins and Savage (1963) (which suggested the present paragraphs); substituting a_1 for b_1 in (8) gives what Kruskal (1963) calls the a_1-*canonical* representation of the integer $N_{(a_1)}$ (except that he omits the final term if $N_{a_1} = 0$: then the representation is unique).

Narayana (1959) has shown that the number of pairs of paths from the origin to the point (a_1, a_2) such that the first never falls below the second is

$$\frac{1}{a_1 + a_2 + 2} \binom{a_1 + a_2}{a_1} \binom{a_1 + a_2 + 2}{a_1 + 1}.$$

Narayana's discussion is in terms of a partial ordering of the r-partitions of a number n; one partition $(t) = (t_1, \cdots, t_r)$ is said to dominate another $(t') = (t_1', \cdots, t_r')$ if $s_i \equiv t_1 + \cdots + t_i \geqq t_1' + \cdots + t_i'$ for $i = 1, \cdots, r$. Corresponding to any such partition, we can construct a ballot sequence with $a_1 = n - r$, $a_2 = r - 1$ (and an equivalent walk on the rectangular lattice) by placing the A_2 votes in positions $s_1, \cdots, s_{r-1}$. Then it is easily seen that (t) dominates (t') if and only if the (t)-walk never rises above the (t')-walk.

2.3 *Generalization to several candidates.* Suppose now that k candidates A_1, $A_2, \cdots, A_k$ obtain respectively $a_1 \geqq a_2 \geqq \cdots \geqq a_k > 0$ votes. Write $n = a_1 + a_2 + \cdots + a_k$. Then the events of interest are

$$E_W = \{A_1(m) \geqq A_2(m) \geqq \cdots \geqq A_k(m), m = 1, 2, \cdots, n\}$$

$$E_S = \{A_r(m) > A_{r+1}(m) \text{ whenever } A_r(m) > 0,$$

$$r = 1, \cdots, k - 1, m = 1, 2, \cdots, n\}.$$

Thus, for E_S two candidates are allowed to tie only in the initial stage, before either receives a vote. There is a 1-1 correspondence between arrangements satisfying E_W (also known as *lattice permutations*) and standard Young tableaux (see e.g. Littlewood (1950)) with the row-specification $(a) = (a_1, a_2, \cdots, a_k)$;

for example, the lattice permutation 12132112 (where 1 denotes a vote for A_1, etc.; here $a_1 = 4$, $a_2 = 3$, $a_3 = 1$) corresponds to the standard tableau

$$\begin{matrix} 1367 \\ 258 \\ 4 \end{matrix}$$

(in which the position of the integer 6, for example, denotes that the sixth vote to be counted was for candidate A_1, and was his third vote). Evidently by interchanging the roles of rows and columns in the Young tableaux we obtain a 1-1 correspondence with lattice permutations for the specification $(a') = (a_1', a_2', \cdots, a_{k'}')$ conjugate to (a). In the above example $k' = 4$, $(a') = (3, 2, 2, 1)$, and the corresponding lattice permutation is 11212343. Thus the number of lattice permutations is the same for the specification (a) as for the conjugate specification (a'); let this number be $N_W(a)$ $(= N_W(a'))$.

We remark that this number arises in the theory of the symmetric group S_n; (see e.g. Littlewood (1950)); it is the simple character $\chi_{(1^n)}^{(a)}$ of the identity element in the irreducible representation of S_n corresponding to the partition (a). It is also the coefficient of $x_1^{b_1}, x_2^{b_2}, \cdots, x_k^{b_k}$ in the expansion of

$$(\sum x)^n \Delta(x) = (x_1 + \cdots + x_k)^n \prod_{i<j} (x_i - x_j)$$

where $b_i = a_i - i + k$, and it has several explicit representations. In 1900 Frobenius found

$$N_W(a) = n!\Delta(c)\Delta(c')/\prod_i c_i!\prod_j c_j'!\prod_{ij}(c_i + c_j' + 1);$$

here $c_i = a_i - i$, $c_j' = a_i' - i$ $(i = 1, 2, \cdots, l)$ where l is the number of elements in the leading diagonal of the graph of the partition (a); in the above example $l = 2$, $(c_1, c_2) = (3, 1)$, $(c_1', c_2') = (2, 0)$. The concept of a lattice permutation is due to MacMahon (1915); he found

$$N_W(a) = n!\Delta(b)/\prod_i b_i! = n! \det |1/(a_i - i + j)!| \tag{9}$$

(det = determinant). This form was found also by Young (1927). Hence, MacMahon's result for the weak-sense ballot problem

$$P\{E_W \mid (a)\} = \prod_{i<j}\{1 - (a_i - i + k)^{-1}\}.$$

MacMahon obtained his formula as the proper solution of the recurrence relation

$$\begin{aligned} N_W(a_1, a_2, \cdots, a_k) &= N_W(a_1 - 1, a_2, \cdots, a_k) \\ &\quad + N_W(a_1, a_2 - 1, \cdots, a_k) + \cdots + N_W(a_1, a_2, \cdots, a_k - 1) \end{aligned} \tag{10}$$

which is obtained by considering the possibilities for the last vote counted. He remarks on the asymmetry of the formula (9) as between the conjugate partitions (a) and (a'); the situation was clarified by Frame, Robinson, and Thrall (1954) (see also Robinson (1961)) who showed that

$$N_W(a) = n!/\prod_{ij} h_{ij}$$

where the *hook-length* h_{ij} corresponding to the i, j cell in the Young tableau is defined to be one more than the sum of the number of cells to the right of that cell and the number of positions below that cell; i.e. $h_{ij} = a_i + a_j' - i - j + 1$. In this form, the result is obviously invariant under conjugation.

Narayana (1955), (1959) has found the number of lattice paths from the origin to the point $(a_1, \cdots a_k)$ where at each step, every coordinate must increase by at least one unit, and the event E_W holds (here n is the number of steps and $a_1 \geqq \cdots \geqq a_k \geqq n$). The number is

$$\det \left| \binom{a_i - 1}{n + i - j - 1} \right| .$$

An open problem is the determination of the probability that E_W (or E_S) holds when the votes are counted in batches of fixed size.

2.4 *The strict-sense many-candidate ballot problem.* The strict-sense problem seems to have attracted little attention before it was posed by Grossman in 1950; in 1952 Grossman solved the case $k = 3$, and Thrall (1952) established generally that

$$P\{E_S \mid a_1, \cdots, a_k\} = \prod_{i<j} \left(\frac{a_i - a_j}{a_i + a_j} \right) \tag{11}$$

(see also Srinivasan (1963)). Both authors proceeded by showing that this form satisfies the recurrence (10) (read N_S for N_W) and the appropriate boundary conditions. An alternative formula can be obtained by using the reflection principle as follows.

Let $\varphi = (\varphi(1), \varphi(2), \cdots, \varphi(k))$ denote a permutation of the integers 1, 2, $\cdots, k$, with φ_0 denoting the identity permutation. Let $N(\varphi)$ denote the number of arrangements in which the first vote is for $A_{\varphi(1)}$, and now ignoring subsequent votes for $A_{\varphi(1)}$ the first succeeding vote for any of the remaining candidates is for $A_{\varphi(2)}$, and so on; thus the candidates' partial totals first increase from zero in the order specified by φ. Let $N'(\varphi)$ denote the number of arrangements in $N(\varphi)$ that are unfavorable to E_S. Then if N_S is the total number of arrangements favorable to E_S, we have

$$N(\varphi) = N'(\varphi) \quad \text{except when} \quad \varphi = \varphi_0 \tag{12}$$

$$N(\varphi_0) = N'(\varphi_0) + N_S . \tag{13}$$

The total number of arrangements is $N = (\sum a_j)!/\prod a_j!$, and we have

$$N(\varphi) = N \prod_{i=1}^{k} \frac{a_{\varphi(i)}}{a_{\varphi(i)} + a_{\varphi(i+1)} + \cdots + a_{\varphi(k)}} . \tag{14}$$

Now fix φ and consider an arrangement in $N'(\varphi)$. Since E_S fails to hold, there must be a smallest integer $m_0 \geqq 1$ such that at $m = m_0$, two nonzero partial totals are equal; and the corresponding pair of candidates must be adjacent in the permutation φ. Thus we can define r_0 uniquely by

$$A_{(r_0)}(m_0) = A_{(r_0+1)}(m_0).$$

Now if in the first m_0 votes we interchange votes for $A_{(r_0)}$ with votes for $A_{(r_0+1)}$, we shall obtain one of the arrangements in $N'(\psi)$, where ψ is the permutation obtained from φ by interchanging $\varphi(r_0)$ and $\varphi(r_0 + 1)$; denote this relationship by $\psi = [r_0]\varphi$. Also this relationship between arrangements is biunique. Let $N_r'(\varphi)$ be the number of arrangements in $N'(\varphi)$ for which $r_0 = r$; then

$$N'(\varphi) = \textstyle\sum_r N_r'(\varphi) \tag{15}$$

and the above argument has shown that

$$N_r'(\varphi) = N_r'([r]\varphi).$$

Thus if $T(\varphi)$ denotes the number of inversions in φ, we have for all φ and r

$$0 = (-1)^{T(\varphi)}N_r'(\varphi) + (-1)^{T([r]\varphi)}N_r'([r]\varphi).$$

Summing over r and φ and using (15), then adding N_S to both sides and using (12) and (13) we obtain

$$N_S = \textstyle\sum_\varphi (-1)^{T(\varphi)}N(\varphi) \tag{16}$$

where $N(\varphi)$ is given by (14).

If we define the indicators

$$\begin{aligned} X_{ij} &= +1 \quad \text{if } A_i \text{ receives a vote before } A_j \text{ does} \quad 1 \leqq i < j \leqq k \\ &= -1 \quad \text{if not} \end{aligned}$$

then we have (with E for expectation)

$$\begin{aligned} P\{E_S \mid a_1, \cdots, a_k\} &= \textstyle\prod_{i<j} E(X_{ij}) && \text{(from (11))} \\ &= E(\textstyle\prod_{i<j} X_{ij}) && \text{(from (16))}. \end{aligned}$$

The simplicity of these results strongly suggests the possibility of direct probabilistic proofs, but none has been found. The obvious conjectures regarding independence amongst the X_{ij}'s are incorrect, as are the natural conjectures generalizing (2) and (3) to the k-candidate case.

2.5 *Considerations involving amount of lead.* Another way in which the classical ballot problems can be generalized is by consideration of events of the following type:

$$E(\alpha; \mu) = \{A_1(m) + \alpha > \mu A_2(m), m = 1, 2, \cdots, n\}$$

where we shall suppose α and μ to be non-negative integers. Evidently $E(0; \mu) = E_S(\mu)$, $E(1; \mu) = E_W(\mu)$. The result

$$P\{E(\alpha; 1) \mid a_1, a_2\} = 1 - \binom{a_1 + a_2}{a_1 + \alpha}\binom{a_1 + a_2}{a_1}^{-1} \tag{17}$$

was given by Whitworth (1878) and is easily proved using the reflection principle. Evidently $P\{E(\alpha, \mu)\}$ vanishes unless $a_1 + \alpha > \mu a_2$; for $\alpha \geqq 0$ we have

$$P\{E(\alpha;\,\mu) \mid a_1, a_2\} = [a_1/(a_1 + a_2)]P\{E(\alpha + 1;\,\mu) \mid a_1 - 1, a_2\} + [a_2/(a_1 + a_2)]P\{E(\alpha - \mu;\,\mu) \mid a_1, a_2 - 1\},$$

whence we can establish

$$P\{E(\alpha;\,\mu) \mid a_1, a_2\} = \sum_j (-1)^j \binom{\alpha - j\mu - 1}{j} \frac{a_1^{(j\mu-\alpha)} a_2^{(j)}}{(a_1 + a_2)^{(j\mu+j+1-\alpha)}} (a_1 + \alpha - \mu a_2)$$

(where $x^{(r)} = x(x - 1) \cdots (x - r + 1)$). For the case $a_1 = \mu a_2$, Korolyuk (1955a), Kemperman (1957), and Blackman (1958) give different formulae. In this case, enumeration of the permutations satisfying $E(\alpha;\,\mu)$ or alternatively $\alpha > A_1(m) - \mu A_2(m) > -\beta$, $m = 1, 2, \cdots\ n$ yields the null-hypothesis distributions of the Smirnov and Kolmogorov statistics. Following up this topic in detail would take us rather far from our main theme: we refer to Darling (1957) for the development up to 1957 and to Feller (1957b) (Chapter 14) for a discussion of the associated random walk problems. We have included an appendix listing work in this area since Darling's paper.

There is an elegant generalization of (17) to the case of several candidates; writing

$$\text{(18)}\quad \begin{aligned} E((\alpha)) &= E(\alpha_1, \alpha_2, \cdots, \alpha_k) \\ &= \{A_1(m) + \alpha_1 > A_2(m) + \alpha_2 > \cdots > A_k(m) + \alpha_k,\ m = 1, \cdots, n\} \end{aligned}$$

with $\alpha_1 > \alpha_2 > \cdots > \alpha_k$, we have

$$\text{(19)}\qquad P\{E((\alpha)) \mid a_1, \cdots, a_k\} = \det \left| \frac{a_i\,!}{(a_i + \alpha_i - \alpha_j)!} \right| .$$

A rapid proof of this result can be obtained from a much more general theorem of Karlin and McGregor (1959), which concerns the probability that the sample paths of k simultaneous independent Markov processes do not intersect one another. We consider k simultaneous independent Poisson processes $X_i(t)$, with common density λ, with $X_i(0) = \alpha_i$ $(i = 1, 2, \cdots, k)$. By the theorem of Karlin and McGregor, the probability that by time t these processes have reached the positions $a_i + \alpha_i$ respectively without being equal at any time in $(0, t)$ is

$$\det |P\{\alpha_j \to a_i + \alpha_i\}| = \det \left| \frac{(\lambda t)^{a_i+\alpha_i-\alpha_j} e^{-\lambda t}}{(a_i + \alpha_i - \alpha_j)!} \right|$$

The unconditional probability is the product of the leading diagonal elements of this determinant, so the quotient (19) is the probability that there are no equalities in $(0, t)$, conditional on the given end-points. But once the end-points are fixed, all arrangements amongst the several processes of the n points of increase within $(0, t)$ are equally likely; so (19) gives the solution to our ballot problem.

If all the α's in (18) are equal, and if (as before) we accept the convention that the event $E((\alpha))$ occurs if the strict inequalities $A_i(m) > A_{i+1}(m)$ hold whenever $A_i(m) > 0$, we retrieve the strict-sense ballot problem of the previous section. The mixed case, where some but not all α's are equal, seems difficult; we can easily obtain relations of the type exemplified by

$$P\{E(\alpha, \beta, \beta, \gamma) \mid a_1, a_2, a_3, a_4\}$$
$$= \sum_{b_1, b_4} \binom{n}{b_1 + b_4}^{-1} \binom{a_1}{b_1} \binom{a_4}{b_4} P\{E(\alpha, \beta, \gamma) \mid b_1, 0, b_4\}$$
$$\times \frac{a_2}{n - b_1 - b_4} P\{E(\alpha + b_1, \beta + 1, \beta, \gamma + b_4) \mid a_1 - b_1, a_2 - 1, a_3, a_4 - b_4\}$$

in which the probabilities on the right can be obtained from (19), but these expressions do not seem to reduce in general.

H. T. David (1958) has used an ingenious reflection argument to find the probability of the event

$$A_1(m) - A_2(m) > \alpha, \qquad A_2(m) - A_3(m) > \alpha, \qquad A_3(m) - A_1(m) > \alpha$$

with $a_1 = a_2 = a_3$. There seems to be little possibility of extending this method to cases involving more candidates.

2.6 *Some other problems.* We refer to Feller (1957b) (Chapter 3) for a detailed discussion of a variety of problems concerning returns to equality and first passage epochs; the derivations are remarkably simplified by employing the basic ballot results. We shall not attempt to cover this topic in any detail, but will merely indicate some recent work and point to some open problems.

One of the simplest results in Feller is that if $a_1 = a_2 = a$ and if we count the occasions on which either $A_1(m) > A_2(m)$ or $A_1(m) = A_2(m)$ and $A_1(m - 1) > A_2(m - 1)$ $(m = 1, 2, \cdots, 2a)$, then this number (N', say) is uniformly distributed on the set $0, 2, 4, \cdots, 2a$. Engelberg (1963) and Gobel (1963) have extended this result to the case $a_i \neq a_2$; supposing $a_1 > a_2$, the result is

$$P\{N' = 2j \mid a_1, a_2\}$$
$$= \binom{a_1 + a_2}{a_1}^{-1} \sum_{i=j}^{b} \frac{1}{i + 1} \binom{2i}{i} \frac{a_1 - a_2}{a_1 + a_2 - 2_i} \binom{a_1 + a_2 - 2_i}{a_2 - i}.$$

Gobel investigates some asymptotic properties of this result; Engelberg studies also the distributions of the number of occasions on which $A_1(m) > \mu A_2(m)$ (resp. $A_1(m) \geqq \mu A_2(m)$) for μ integral. Takács (1963) has found the distribution of the number of occasions on which $A_1(m) > \mu A_2(m) - c$ (c is a non-negative integer); for $\mu = 1$ and all c, corresponding results have been obtained by Riordan (1963).

Feller (1957b) shows that in the symmetric random walk, the probability that by the nth step there have been exactly r returns to the origin is $2^{r-2n} \binom{2n-r}{n}$; this is also the probability that there is a return at the $2n$th step, this being at least the rth return. If R_n is the number of returns, then as n increases, the limiting distribution of $R_n/(2n)^{\frac{1}{2}}$ is a standard half-normal. Barton (1957) demonstrated an analogous result for the asymmetric walk; if the probability of success at any trial is $\lambda/(\lambda + \mu)$ (λ, μ relatively prime) and if now R_n is the number of times the ratio of successes to failures has equalled λ/μ by the nth trial, then $R_n[\lambda\mu/n(\lambda + \mu)]^{\frac{1}{2}}$ has the half-normal limit. With $\lambda = 1$, it is not difficult to establish the exact result

$$P\{A_1(m) = \mu A_2(m) \text{ at least } r \text{ times} \mid a_1 = k\mu, a_2 = k\} = (1 + \mu)^r \binom{k+k\mu-r}{k\mu} \binom{k+k\mu}{k\mu}^{-1}.$$

Another problem that yields to the reflection principle is that of finding the number of times the lead changes hands during the counting. Smirnov (1939) and Dwass (1961) give some asymptotic results; Mihalevic (1952) and Feller (1957a) give some exact formulae. The corresponding problems with more than two candidates are not solved, and seem to be difficult.

2.7 *Applications to queue theory*. Consider the simple queue ($M/M/1$) with Poisson input with density λ, and exponential service times with mean μ. Suppose a customer arrives at time $t_0 = 0$ and finds the server idle. His arrival thus initiates a busy period (b.p.); suppose that during this b.p. the successive services end at times $s_1, s_2, \cdots$, while further customers arrive at times $t_1, t_2, \cdots$. Then the b.p. will last for exactly n customers, and will end in $(s, s + ds)$ if

$$t_1 < s_1, \quad t_2 < s_2, \cdots, t_{n-1} < s_{n-1}, \quad s < s_n < s + ds, \quad s_n < t_n .$$

The probability of this event is

$$P\{\text{exactly } n - 1 \text{ arrivals in } (0, s)\} Q_n \, dP\{s_n = s\}$$

where Q_n is the conditional probability that $t_i < s_i$, $i = 1, \cdots, n - 1$, given that $s_n = s$ and that there are exactly $n - 1$ arrivals in $(0, s)$. But, given this conditioning, the times $t_1, \cdots, t_{n-1}, s_1, \cdots, s_{n-1}$ are equally likely to occur in any order; so by Whitworth's result (1) we have $Q_n = 1/n$, and the probability is found explicitly.

Combinatorial arguments such as this have arisen only recently. Champernowne (1956) used the result (1) for the queue $M/M/1$. Pyke (1959) studied the queues $D/M/1$ and $M/D/1$ using an argument similar to the above; Tanner (1961) used this method explicitly for $M/D/1$. Takács has used his result (4), and some variations, extensively to derive expressions for (i) the distribution of the length of a b.p. for the queues $M/G/1$ and $G/M/1$ (1961), (ii) the probability that a customer finds the server idle for the queue $M/G/1$ (1962c), (iii) the distribution of the number served in a b.p. for the queue $M/M/1$ with batch arrivals or batch service (1962b), (iv) the distributions of queue size, waiting time, number served in a b.p. and length of a b.p. for the queue $G/M/1$ (1962d), and (v) for the queue $M/G/1$, the probability that the server is idle, the distribution of an initial b.p., and the distribution of the occupation time, i.e. the total service time of all those customers who arrive in some given time-interval (1962e).

3. Amalgamation, records and Simon Newcomb's problems.

3.1 *Introduction.* Suppose $X_1, \cdots, X_n$ is a sequence of symmetrically dependent random variables whose joint distribution function is absolutely continuous, or more generally satisfies $P\{E\} = 0$ where E is the event

$$E = \bigcup_{(a),(b)} \{\bar{X}_{(a)} = \bar{X}_{(b)}\}$$

(where $(a),b)$ denote disjoint subsets of $\{1, \cdots, n\}$ and

$$\bar{X}_{(a)} = \sum_{i\varepsilon(a)} X_i / \sum_{i\varepsilon(a)} 1.$$

Thus in particular the X's could be independent and identically distributed, or could be a random permutation of n general fixed numbers. We denote this assumption by H_0. In the Simon Newcomb group of problems we study properties of the sequence $S_2, \cdots S_n$ where $S_i = \text{sign}\,(X_i - X_{i-1})$; in the Records group the concern is with the derived sequence $S_2', \cdots, S_n'$ where $S_i' = \text{sign}\,(X_i - \max\,(X_1, \cdots, X_{i-1}))$; and the Amalgamation group relates to what amounts to the Records in the sequence of cumulative means $\bar{X}_1, \cdots, \bar{X}_n$ where $i\bar{X} = X_1 + \cdots + X_i$.

It is easily seen that the Records and Simon Newcomb's problems are purely combinatorial (i.e. they do not depend on the particular distribution assumed); this is true also (rather surprisingly) in the Amalgamation group of problems.

We will not do more than briefly mention in this review the contributions to the theory of these statistics under alternative hypotheses, as these are mostly asymptotic and of relatively minor combinatorial interest.

3.2 *Simon Newcomb's problem.* If we consider a typical sequence of signs S_i such as

$$+ + + - - + + - + + - - - + + - \cdots$$

one aspect of interest is the number, m say, of $+$'s, or, equivalently, the number of *ascending sequences* in the sequence $X_1, \cdots X_n$ where, if $X_{r-1} > X_r > X_{r+1}$, X_r comprises an ascending sequence of length one. These sequences are called strings in the theory of sorting and their study is basic to the distribution theory arising from "string merging" algorithms.

The distribution of m under H_0 was obtained by MacMahon (1908) and later tabulated by Wallis and Moore (1943). It has the form

$$F_n(m) = (1/n!) \sum_{j=0}^{m} \binom{n}{j} (-1)^j (m + 1 - j)^n.$$

This, incidentally, is a very curious distribution function; for integer values of m it is identically equal to the d.f. of one less than the sum of n independent random variables each uniformly distributed in $(0, 1)$—as obtained by Laplace (1820). Paradoxically, the first n cumulants of this d.f. are the same as those of a sum of $n + 1$ such uniform variables (as noticed, e.g. by Barton and David (1962)).

This suggests that there might be some means (awaiting discovery) of expressing m in terms of the sum of n independent and identically distributed random variables. Such an analysis might well shed considerable light on the more complex distributional problems noticed later.

It should be noted that tabulation can easily be performed by means of the recurrence

$$(n + 1)f_{n+1}(m) = (m + 1)f_n(m) + (n + 1 - m)f_n(m - 1)$$

(where $f_n(m)$ is the density function), which was also given by MacMahon.

Stuart (1952) examined the power of m as a test against trend and found it a very poor test for this type of alternative (see Cox and Stuart (1955) and also Levene (1952) for a comparative discussion).

The more general case where some of the $\{X_i\}$ are equal to each other has been considered by Riordan (1958) but we have not seen any statistical exploitation of his results (or those of MacMahon (1915) which are rather less explicitly cognate to the problem).

3.3 *Runs*. Another problem of interest is the number s, say, of runs of $+$'s and $-$'s. This is equivalent to the number of extrema (maxima and minima) also called turning points (peaks and troughs) in the X sequence. Bienaymé (1874) gave, without proof, the mean and variance of s and stated the normal limit. The first four cumulants of s (obtained from Kendall (1946)) are

$$K_1 = \tfrac{2}{3}(n + 1) - 1, \qquad K_2 = (8/45)(n + 1) - \tfrac{1}{2},$$

$$K_3 = (-16/945)(n + 1), \qquad K_4 = (-352/4725)(n + 1) + \tfrac{1}{4}.$$

Bienaymé (1875) was well aware of the statistical importance of s and this paper stimulated Bertrand (1875) to produce a simple reason for the leading term in K_1. A little later André (1879), (1881), (1883a) studied the probability of a sequence with the maximum number $(n - 1)$ of runs, which he called alternating sequences, and subsequently, (1883b), (1884) he studied the connexion with Bienaymé's problem and produced the recurrence

$$(n + 1)f_{n+1}(s) = sf_n(s) + 2f_n(s - 1) + (n - s + 1)f_n(s - 2)$$

for the density function of s. It was this recurrence that Gleissberg (1954) and Moore and Wallis (1941) used for computing tables of $f_n(s)$. André later (1894), (1895) investigated the number of sequences with one less than the maximum number of runs, (quasi-alternating sequences) and Netto (1901) gathered the results thus far in his book. Generating functions for $f_n(s)$ were obtained by Kermack and McKendrick (1937a, b) and Barton and David (1962). Levene and Wolfowitz (1944) justified the normal limiting form of the distribution of s using a general Central Limit theorem, Wolfowitz (1943), (1944). In fact, (see Barton and David (1962)) it may be shown that the rth cumulant is of the form

$$K_r = (-1)^r[a_r(n + 1) + b_r], \qquad 1 \leqq r \leqq (n - 1)/2$$

where a_r and b_r are the rth Taylor Coefficients in $\frac{1}{2} \log (1 - e^{-2\theta}) - \log \cosh^{-1} e^{\theta}$ and $-2 \log (1 + e^{-\theta})$ respectively, so that Frechet and Shohat's Second Limit theorem may be applied.

This form of cumulant, linear in n, is suggestive also of an analysis of the distribution of s in terms of a sum of independent symmetrically distributed random variables. The power of s as a test against trend has been studied by Stuart (1952), (1954) and Levene (1952), the general conclusion being that it is very poor in respect of such alternatives (see also Cox and Stuart (1955)). Moran and Chown (1951) noticed that s is effectively a serial correlation coefficient (of lag one) applied to the series $s_1, \cdots s_{n-1}$ and obtained its expected values under normal serial correlation. Against such alternatives it has a relatively high efficiency. Grant (1952) found the mean of s in a series smoothed by a moving average. This is of interest as s, under the alternative here, is also distributed independently of the structure of the elements averaged and the distribution of s under the alternative hypothesis is also an unsolved combinatorial problem.

Kermack and McKendrick (1937a) were initially interested in the sequential version of this problem; namely the distribution of the number of elements in the first s runs of an indefinitely long sequence. This is equivalent to solving the problem of the distribution of s for fixed n by virtue of the usual principle:

$$\begin{aligned} &P\{s \text{ or less runs in the first } n\} \\ &\qquad = P\{n \text{ or more elements in first } s \text{ runs}\} \end{aligned} \tag{20}$$

and Kermack and McKendrick found it necessary to consider the problem of the number of runs for a given n as a basis. They also considered the number of runs in a circular arrangement of n elements.

Various modifications, number of peaks, number of runs of +'s, and so on are dealt with in detail in Barton and David (1962).

The problems of the distributions of the number of pairs of consecutive elements and of the lengths of runs of consecutive elements will not be considered in detail here. (Two elements form a consecutive pair if all other elements are either greater than both or less than both.) Apart from Euler's initial contribution giving the chance of no pairs, and so no runs, in an ordered random subsequence of a given length, and a later proof that the chance of no consecutive pairs in the whole random sequence tended to e^{-2} as n approached infinity (Aiyar & Fortey, 1894), the basic combinatorial analysis and Poisson limits are due to Wolfowitz (1942) and Barton & David (1962).

3.4 *Lengths of runs.* The problem of lengths of run was first considered, and solved in principle, by MacMahon (1908), (1915). He showed by an elementary combinatorial argument, in respect of ascending runs, that the probability that a random arrangement of n elements gave a sequence of ascending sequences of lengths $r_1, r_2, \cdots, r_m$ respectively (so that $r_1 + r_2 + \cdots + r_m = n$) was

$$f_n(r_1, \cdots, r_m; m) = \begin{vmatrix} \frac{1}{r_1!} & \frac{1}{r_1'!} & \frac{1}{r_1''!} & \cdots & \frac{1}{n!} \\ 1 & \frac{1}{r_2!} & \frac{1}{r_2'!} & \cdots & \\ 0 & 1 & \frac{1}{r_3!} & \cdots & \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{r_m!} \end{vmatrix}$$

where $r_j' = r_j + r_{j+1}$, $r_j'' = r_j + r_{j+1} + r_{j+2}$, etc.

We remark that this result can be obtained from (19) above; we identify $r_1, \cdots, r_m$ with $a_m, \cdots, a_1$ respectively, and put $\alpha_r = (a_1 + a_2 + \cdots + a_r)$. Then the event $E((\alpha))$ occurs if and only if the first vote for A_i comes before the last vote for A_{i+1}, $i = 1, 2, \cdots, m - 1$. The inequalities implied here are exactly those needed for the run-specification $r_1, \cdots, r_m$.

MacMahon pointed out that if we have a sequence of runs up and down of lengths $\omega_1, \omega_2, \cdots \omega_s$, this is the same as a set of ascending sequences of lengths $\omega_1 + 1, 1^{\omega_2 - 1}, \omega_3 + 1, 1^{\omega_4 - 1}, \cdots$ (if the initial run is *up*) and of lengths $1^{\omega_1 - 1}$, $\omega_2 + 1, 1^{\omega_3 - 1}, \omega_4 + 1, \cdots$ (if *down*) so that the probability of $(\omega_1, \omega_2, \cdots, \omega_s)$ is also given by this determinant. However, for n above about 10 this is not a computational possibility (by hand, at least).

Later the emphasis became more statistical and interest focussed on the relative frequence of runs (up or down) of a given length. Besson (1920) obtained the mean number of runs of length p for various values of p and Fisher (1926) obtained a general form for the leading term of this mean. This is the probability that an arbitrary run chosen from an infinite sequence is of length p signs, which has the form

$$f(p) = 3[1/(p + 1)! - 2/(p + 2)! + 1/(p + 3)!], \quad p = 1, 2, \cdots.$$

In papers on sorting Goetz (1961), (1963) has focussed attention on the distribution of the length of the kth ascending run and the rate at which it tends (as $k \to \infty$) to have its limiting distribution. The limiting distribution is of course, for runs of r elements

$$f(r) = 2[1/r! - 2/(r + 1)! + 1/(r + 2)!], \quad r = 1, 2, \cdots,$$

with mean 2 (see Moore (1957)) and it is possible to show that the p.g.f. of the kth run $\pi_k(t)$ say, obeys the recurrence

$$1 - \pi_{k+1}(t) = (1 - t)^{-1}(1 - \pi_k(t)) - \pi_1(t)\mu_k$$

where $t\pi_1(t) = 1 + (t - 1)e^t$, $\mu_k = \pi_k'(1)$. Hence we can establish

$$\sum \mu_k \theta^k = \theta(e - e^\theta)(e^\theta - \theta e)^{-1}. \tag{21}$$

Levene and Wolfowitz (1944) made a systematic study and obtained the covariance matrix of the numbers of runs of lengths $p = 1, 2, 3, \cdots$ and also that of runs of p or more. They did not use the slightly more general combinatorial proofs possible (cf. Barton and David (1962)). Using the asymptotic normal theory of Wolfowitz (1943), (1944) they proposed the corresponding chi-squared statistic as a goodness of fit test for H_0.

The distribution of "phase-length", that is the distance between successive peaks or the total length of two successive runs, was studied by Kermack and McKendrick (1937b) (see also Palmer (1957)) and the relative frequency of given phase-length in long series was found.

The asymptotic theory of run-length statistics was developed by Levene (1952) under quite a wide class of alternative hypotheses but detailed computations were given only for trend alternatives and no comparative study made. The mean run length in long series is of course equal to $\lim_{n\to\infty}$ $[(\text{mean number of runs in } n)/n]^{-1}$. This ergodic device also gives mean phase length and was used by Kendall (1945) to find the mean phase length in long series under general serial correlation.

Concise expressions for the distribution of the run lengths in a finite sequence (analogous to MacMahon's distribution for ascending sequences) have not yet been found.

Long runs have always been a salient feature of observed sequences and Fisher's paper (1926) was stimulated by the need to assess the significance of long runs in meteorological series. It is possible, for small n, to use MacMahon's distribution of run lengths to determine this numerically. Olmstead (1946) produced a recurrence relation which slightly reduces the labour of this and gave tables, essentially, of the distribution for $n \leqq 12$. This served to confirm the accuracy of approximation of the asymptotic theory of Wolfowitz (1943), (1944) (who showed that the extreme value distribution for independent variables extended to this case) so far as the upper tail of the distribution was concerned. Barton and David (1959) who were systematically investigating the application of Bonferroni's inequalities to combinatorial distributions (a device of H. A. David (1956)) showed that the integer valued upper "percentage points" for the longest could be found exactly by these inequalities for most values of n. (Where the upper $100\alpha\%$ point is the smallest integer such that the probability that the longest sum exceeds it is not greater than α).

The chance that the longest ascending run does not exceed k was shown by Barton and David (1962) (using Feller's theory of recurrent events) to be the coefficient of u^n in the reciprocal of

$$\textstyle\sum_{j=0}^{\infty} \{[u^{kj}/(kj)!] - [u^{kj+1}/(kj+1)!]\}.$$

The corresponding generating function for the longest run up or down has not yet been found.

The power of the longest run test has not yet been investigated explicitly so far as we are aware though Levene's (1952) results are plainly relevant.

Tables of the integer valued functions: $n!$ times probability for most of the distributions discussed above are collected together in Barton, David and Kendall (1964).

3.5 *Records*. In the sequence $X_1, \cdots X_n$ we may distinguish those elements which exceed (or are less than) all preceding values as upper (or lower) *records*, following Chandler (1952). It will be convenient to consider X_1 as the first upper and lower record. We may further qualify these as forward records when we wish to distinguish them from the backward records (i.e. the records in the series read in reverse order). The distributional problems related to records are recognisably combinatorial, depending only on the $n!$ permutations of the original sequence, although the two main papers in this field, Chandler (1952) and Foster and Stuart (1954), derived their results under slightly more restrictive assumptions (cf. Barton and David (1962)).

The joint distribution of the numbers, u and v say, of upper and lower (forward) records respectively was found by Foster and Stuart. The distribution of either (say u) has density function $f_n(u) = S_n^u/n!$ where the $\{S_n^u\}$ are Stirling's numbers of the first kind (in modulus).

The rth factorial cumulant of u is

$$K_{[r]} = (r-1)!(-1)^r \sum_{t=2}^{n} t^{-r}$$

so that the standard cumulants above the second order tend to zero and u is asymptotically normally distributed. However, both mean and variance are $0(\log n)$ and the asymptotic distribution is a very poor approximation for $n \leqq 1000$ say, which is the only region of much statistical importance. It follows that approximations to the Stirling numbers are of some importance both for this distribution and that of the next section on amalgamation (which has the same form) and we shall briefly discuss this problem at the end of the next section.

From symmetry it is plain the distributions of numbers of lower records and of backward upper and lower records (u' and v' say) have the same form.

Foster and Stuart were concerned with number of records as a test of H_0. They also considered the "round trip" statistic $D = (u - u') - (v - v')$. Although, as shown in Barton and Mallows (1961) the joint distribution of numbers of forward and backward upper (or lower) records has the same form as that for upper and lower forward records, the joint distribution of (u, u', v, v') and so of D, is as yet unknown. It is clearly symmetric but the form of the variance of D suggests it is likely to be very complex. The distribution is of some interest though, since any asymptotic normality (assumed but not proved by Foster and Stuart (1954)) will be approached as slowly as in the case of u and so will not be of use for approximative purposes.

Little is known of the power of tests based on u or D. Foster and Stuart conducted a sampling experiment to examine the power under a trending alternative (see also Foster and Teichroew (1955)) and concluded that whilst they were better than s and m they were far from good.

Chandler (1952) was concerned with the intervals between upper records or, equivalently, with the ordinals of upper records. The distribution of the ordinal of the rth record may be found by the argument noticed above ((20)), since this is just the sequential form of the problem of the number of records and it is not surprising that its distribution is also simply expressible in terms of the Stirling numbers. Chandler showed the surprising result that the average interval between the rth and the $r + 1$th records was $0(\log n)^{r+1}$ so that in an indefinitely long series the average interval is infinite. The joint distribution of the ordinals is given in Barton and David (1962).

3.6 *Amalgamation.* For the purposes of discussing the amalgamation problem we shall consider the additional restriction that no average (arithmetic mean) of any set of the $\{X_r\}$ is equal to the average of any other set.

The amalgamation process may be described as a process of successively averaging adjacent elements in descending order. More precisely, at the ith stage, $i = 1, 2, \cdots$ (and $i = 1$ initial stage) of the process the series of initial elements will have been partially amalgamated into groups of adjacent elements. We compare the means of any two adjacent groups, $\bar{X}_a$, $\bar{X}_{a+1}$, say; if $\bar{X}_a > \bar{X}_{a+1}$ we amalgamate these groups (i.e. form one group from them) whereas if $\bar{X}_a < \bar{X}_{a+1}$ no action is taken. This operation is repeated until a resultant series of groups is obtained whose means are in increasing order. Miles (1959) showed that the resultant is unique, irrespective of the order of amalgamation. The sequence of n numbers is then replaced by a resultant set of, ℓ say, group-averages, with $1 \leqq \ell \leqq n$.

Sparre Andersen (1953) showed that the distribution of ℓ under H_0 was independent of the particular set of $\{X_r\}$ and that the density function of ℓ was $f_n(\ell) = S_n^{\ell}/n!$ that is, just that of the distribution of number of upper records. Spitzer (1956) gave a different proof of this, later improved by Wendel (1958). (See also Baxter (1961), Chacko (1963), and Brunk (1964)).

In a statistical context, Bartholomew (1959), (1961) found that, in minimizing $\sum_{r=1}^{n} (X_r - \mu_r)^2$ with respect to $\mu_1, \cdots, \mu_n$ subject to $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_n$ according to an algorithm of Van Eeden, it was necessary to successively average pairs of successive X's which were in descending order. He conjectured and Miles (1959) proved that in whatever succession this was done the resultant set was the same and that ℓ was distributed as above. Miles showed that if the averaging were done with a set of n weights $\omega_1, \omega_2, \cdots, \omega_n$ in random order the distribution of ℓ over the $(n!)^2$ permutations of $[\omega_r][X_r]$ the result was the same as for equal weights. Barton and Mallows (1961) showed the result held over the $n!$ permutations of $\{(\omega_r, X_r)\}$, the statistical rationale being extended by Barton (1961). They also gave reasons for the same distribution arising as in the records distribution (as noticed above). Recently Brunk (1964) has made a deeper analysis and has laid bare the essential mathematical connexion in a series of theorems with wide statistical application.

The power of ℓ as against a trend alternative has been discussed asymptotically by Brunk (1961), and comparison made with other tests.

As remarked above, and discussed in more detail in Barton and Mallows (1961) and Barton, David and Merrington (1963), for series of typical length in statistical contexts, $20 \leqq n \leqq 500$ say, some approximation for S_n^{ℓ} is required for the use of l as a statistical test. The extreme upper tail may be well approximated by Moser and Wyman's (1958) method and the extreme lower tail by the approximations in Barton and Mallows (1961). Broadly these are for $\ell = 0(n)$ and $\ell = 0(1)$ whereas, of course, what is required is $\ell - \log n = 0(\log n)^{\frac{1}{2}}$; the normal approximation is insufficient since $\log n$ is still quite small for $n \leqq 500$.

Barton and David (1964) have developed the following series

$$(u!/v!)S_{v+1}^{u+1} \sim f_u(\lambda) + (1/2!v)P_2(D)f_u(\lambda) + (1/4!v^2)P_4(D)f_u(\lambda) + \cdots$$

where $\lambda = \log v$, $f_u(\lambda) = \sum_{r=0}^{u} \binom{u}{r}\lambda^r g_{u-r}$, $g_r = \left[\frac{d^r}{dx^r}\frac{1}{\Gamma(x+1)}\right]$, $r = 1, 2, \cdots$ (tabulated by Bourguet (1883)) and $P_2(D)$ is a polynomial in the operator D which has the property

$$D^m f_u(\lambda) = u^{(m)} f_{u-m}(\lambda), \qquad m = 1, 2, \cdots .$$

The polynomials P_{2r} are given by

$$P_{2r} = \sum \frac{(2r)!}{\pi_1 ! \dots \pi_v !}\left(\frac{\beta_{p_1+1}}{p_1(p_1+1)}\right)^{\pi_1} \cdots \left(\frac{\beta_{p_v+1}}{p_v(p_v+1)}\right)^{\pi_\lambda}$$

where the summation is over all partitions $(p_1^{\pi_1} \cdots p_v^{\pi_v})$ of r, and where $\beta_r(x) = B_r(-x) - B_r(0)$ in terms of the generalized Bernoulli polynomials of order 1.

This looks formidable, but for fixed u, it gives S_{v+1}^{u+1} as a sequence of polynomials in λ multiplied by ascending powers of v^{-1}. It promises considerable accuracy in the relevant range.

Some of the problems considered in this paper are closely connected with the problem of efficient sorting on computers; we refer to Gottlieb (1963) for a description of the main sorting methods. We hope to discuss some of the combinatorial problems arising in this area at another time.

APPENDIX

Recent work on the Kolmogorov-Smirnov Statistics (since Darling's 1957 Review Paper)

Blackman (1958) and Kemperman (1959) gave exact formulae for $D_{n,np}$ and Hodges (1957) studied the asymptotic behavior; he claimed that Korolyuk's results (1955b) were incorrect. Vincze (1961) and Reimann and Vincze (1960) studied the joint distribution of D_{mn} and R_{mn}, the index of the observations at which a deviation D_{mn} is attained. We remark that these two-sample results imply easily some one-sample results in sampling from a finite population.

For the 1-sample, 1-sided case Birnbaum and Pyke (1957) studied $(R_n, F(R_n))$, and Kuiper (1959) gave a simple proof that $F(R_n)$ is uniform. Dwass (1958) used Sparre Andersen's results to show this, and also that the measure of the set $[F: F_n > F]$ is uniformly distributed. The asymptotic behavior of the 1-sample statistics has been studied by Kemperman (1959) (for D_n^+), and Darling (1960) (for D_n, and also for the number of zeroes of $F_n - F - a/n^{\frac{1}{2}}$ and the sum of the vertical sections of F_n exceeding $F - a/n^{\frac{1}{2}}$).

Carvalho (1959) gave a new derivation (by reflection) of the distribution of D_n; Chapman (1958), Pyke (1959), Dempster (1959) and Dwass (1959) all found the probability that $F_n \leqq a + \gamma F(x)$. (The result for $a = 0$ was given by Daniels in 1945.) Pyke (1959) suggested a modified version of the D_n^+ statistic, namely max $(i/n + 1 - F(X_{(i)}))$; Brunk (1962) considered the modified $D^+ + D^-$. Similar work was done by Kuiper (1960) and Watson (1961), (1962). H. T. David (1961) considered augmenting D_n with supplementary statistics; Ishii (1959) and Tang (1962) studied Rényi's modifications. Whittle (1961) obtained some new exact results for D_n^+; Schmid (1958) considered discontinuous populations; and Becker, Codding and Cron (1962) and Judah Rosenblatt (1962) studied the power of the tests.

Turning to the several-sample problem, Ozols (1956) considered max $(D_{1,2}^+ D_{2,3}^+)$ Chang and Fisz (1957) considered the Kolmogorov test, Fisz (1960) gave a simple way of achieving some exact results, and Gihman (1957) considered the statistic sup $\sum (F_i - F)^2$. H. T. David (1958) found the distribution of max $(D_{1,2}^+, D_{2,3}^+, D_{3,1}^+)$ by an ingenious reflection argument. Birnbaum and Hall (1960) tabulated the distribution of max $(D_{1,2}, D_{1,3}, D_{2,3})$. Kiefer (1959), Dwass (1960) and Fisz (1960) each considered several different statistics.

The bivariate versions of D_n and D_n^+ are not distribution free, as was shown by Simpson (1951). Kiefer and Wolfowitz (1958) found the asymptotic distribution of D_n and D_n^+, and Vincze (1960) studied the two-sample statistics. The titles of Kiefer (1961) and Blum, Kiefer, and Rosenblatt (1961) are self-explanatory. Finally, we mention that Durbin (1962) has found some novel uses for the distance statistics.

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