NOTES

NOTE ON DECISION PROCEDURES FOR FINITE DECISION PROBLEMS UNDER COMPLETE IGNORANCE

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1. Summary. The decision procedures suggested in [1] for finite matrix games are shown to extend successfully to closed and bounded convex S-games, and, with some loss of desirable properties, to the general decision situation.

2. Introduction. In their paper "Decision Procedures for Finite Decision Problems Under Complete Ignorance", Atkinson, Church, and Harris, [1], postulate eight desirable criteria for decision procedures on finite matrix games $A = (u_{ij})$, u_{ij} representing here the loss to the statistician when he takes action i against state of nature j, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$. If A is given the usual S-game representation in n-space, ([2], p. 47), and $Q(A)$ denotes the optimum rules under some decision procedure, then the eight criteria are

(1) $Q(A)$ is always non-empty.

(2) Permuting the columns of the matrix A induces the same permutation on the coordinates of each point in $Q(A)$.

(3) Every point in $Q(A)$ is admissible.

(4) $Q(A)$ is convex.

(5) If $A = (u_{ij})$ and $A' = (\lambda u_{ij} + c_j)$, $\lambda > 0$, then $Q(A') = \{\lambda x + (c_1, c_2, \cdots, c_n) : x \,\varepsilon\, Q(A)\}$.

(6) Deleting a column of A which is a convex combination of the other columns affects $Q(A)$ by deleting the corresponding coordinate of each point.

(7) If A_1 and A_2 have the same admissible points, then $Q(A_1) = Q(A_2)$.

(8) If $A^N \to A$ (entry-wise), and $s^N \,\varepsilon\, Q(A^N)$ for all N, then $s^N \to s$ implies $x \,\varepsilon\, Q(A)$.

A class of decision procedures, which might be called "iterated minimax regret rules", is shown to satisfy all eight criteria. These rules are described as follows: Let $\{\epsilon_h\}$ be a sequence of positive numbers tending to zero, and let Q_1 be the convex hull of the row vectors of A (Q_1 is the "S-figure" representing the game A in n-space). Define

$$v_1(j) = \min_{x \varepsilon Q_1} x(j),$$

where $x = (x(1), \cdots, x(n))$, and

$$z_1 = \min_{x \varepsilon Q_1} d(v_1, x),$$

where $v_1 = (v_1(1), v_1(2), \cdots, v_1(n))$, and $d(\cdot, \cdot)$ is the distance function

$$d(v_1, x) = \max_{1 \leqq j \leqq n} |v_1(j) - x(j)|.$$

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Now let Q_2 be the closed convex set

$$Q_2 = \{x \varepsilon Q_1, d(v_1, x) \leqq z_1 + \epsilon_1 z_1\}.$$

Iterating this procedure yields sequences of sets Q_h, vectors v_h, and numbers z_h, described by

$$v_h(j) = \min_{x \varepsilon Q_h} x(j), \qquad j = 1, 2, \cdots, n,$$

$$z_h = \min_{x \varepsilon Q_h} d(v_h, x)$$

and

$$Q_{h+1} = \{x \varepsilon Q_h, d(v_h, x) \leqq z_h + \epsilon_h z_1\}.$$

It is shown that the sets Q_h decrease to a single point s, the iterated minimax regret point, and that the eight criteria are satisfied for every choice of the ϵ_h.

The purpose of this note is two-fold: First of all it is shown that the iterated minimax regret procedures continue to satisfy the eight criteria when applied to arbitrary finite-dimensional closed and bounded S-games (that is, games where the statistician is allowed an infinite number of strategies, or equivalently, where the figure Q_1 is an arbitrary closed and bounded convex set). Secondly, counter-examples are given to show that the procedure does not satisfy Criterion 1 when there are an infinite number of states of nature, although Criteria 2–8 continue to hold in the general decision situation.

3. Iterated minimax regret for arbitrary S-games. Let A be an S-game described by Q_1, a closed and bounded convex figure in n-dimensional space. Criteria 1–7 are still meaningful when applied to such games, (certain obvious changes should be made in the statements of 2, 5, and 6). Criterion 8 is also meaningful, when convergence of games is interpreted as convergence of the corresponding S-figure under the metric

$$d(R, S) = \max \{\max_{r \varepsilon R} \min_{s \varepsilon S} d(r, s), \max_{s \varepsilon S} \min_{r \varepsilon R} d(r, s)\}.$$

LEMMA 0. *The iterated minimax regret procedures satisfy Criteria* 1–7 *when applied to the class of closed and bounded convex S-games. For each such game the closed convex sets Q_h decrease to a single point s, the numbers z_h decrease to* 0, *and the vectors v_h increase component-wise to s.*

PROOF. The proofs in [1] of the statements above do not depend on Q_1 being polyhedral (that is, the set of decisions available to the statistician being finite) and hence can be applied here without change.

It remains to verify Criterion 8. Fix a sequence $\{\epsilon_h\}$ of positive numbers tending to zero. Let $\{Q_1^N\}$ be the S-figures representing a sequence of closed and bounded convex S-games in n-space, with corresponding superscripts for the various elements of the iterated minimax decision procedure, Q_h^N, v_h^N, z_h^N, etc.

LEMMA 1. $\lim_{N\to\infty} d(Q_1^N, Q_1) = 0$ *implies that for every value of h*

(i) $\lim_{N\to\infty} v_h^N(j) = v_h(j)$ *for* $j = 1, 2, \cdots, n$,

(ii) $\lim_{N\to\infty} z_h^N = z_h$, *and*

(iii) $\lim_{N\to\infty} d(Q_h^N, Q_h) = 0$.

PROOF. It is sufficient to verify the lemma for $h = 1$, the general result following by iteration.

(i) For each j, $1 \leqq j \leqq n$, there exists a vector x in Q_1 such that $x(j) = \min_{y \varepsilon Q_1} y(j) \equiv v_1(j)$. If $d(Q_1^N, Q_1) < \delta_0$ for all sufficiently large N, then for each such N there exists x^N in Q_1^N such that $d(x^N, x) < \delta_0$, and thus

$$v_1^N(j) \equiv \min_{y \varepsilon Q_1^N} y(j)$$
$$\leqq x^N(j) \leqq x(j) + \delta_0 = v_1(j) + \delta_0 .$$

Letting N go to infinity gives

$$\limsup_N v_1^N(j) \leqq v_1(j) + \delta_0 \qquad \text{for all} \quad \delta_0 > 0,$$

while, by a symmetric argument,

$$\liminf_N v_1^N(j) \geqq v_1(j) - \delta_0 \qquad \text{for all} \quad \delta_0 > 0,$$

or, equivalently, $\lim_N v_1^N(j) = v_1(j)$.

(ii) Choose x in Q_1 such that $d(v_1, x) = z_1$. Let δ_0 and the sequence x^N be defined as above. Then $z_1^N \leqq d(v_1^N, x^N) \leqq d(v_1^N, v_1) + d(v_1, x) + d(x, x^N)$. Letting N approach infinity and applying part (i),

$$\limsup_N z_1^N \leqq z_1 + \delta_0 \qquad \text{for all} \quad \delta_0 > 0.$$

A symmetric argument gives

$$\liminf_N z_1^N \geqq z_1 - \delta_0 \qquad \text{for all} \quad \delta_0 > 0,$$

or, equivalently, $\lim_N z_1^N = z_1$.

(iii) If $z_1 = 0$ the result follows from (i) and (ii). By Criterion 5 it may therefore be assumed, without loss of generality, that $z_1 = 1$.

Assume that $\lim_N d(Q_1^N, Q_1) = 0$, but $\limsup_N d(Q_2^N, Q_2) = \delta_0 > 0$. There then exists an infinite sequence of positive integers, $\{N'\} \equiv I'$, and a sequence of vectors $\{w^{N'}\}$, such that either

(a) $w^{N'} \varepsilon Q_2$, $\min_{x \varepsilon Q_2^{N'}} d(w^{N'}, x) > \frac{3}{4}\delta_0$ for all $N' \varepsilon I'$

or

(b) $w^{N'} \varepsilon Q_2^{N'}$, $\min_{x \varepsilon Q_2} d(w^{N'}, x) > \frac{3}{4}\delta_0$ for all $N' \varepsilon I'$.

The two cases will be treated separately.

Case (a). The infinite sequence of points $\{w^{N'}\}$ in Q_2 has at least one accumulation point w in Q_2. Assume first that $d(v_1, w) < z_1 + \epsilon_1$. Since $d(Q_1^N, Q_1)$ is going to zero, there exists a sequence of points $\{y^N\}$, $y^N \varepsilon Q_1^N$, such that $\lim_N d(w, y^N) = 0$. By the definition of w,

$$(*) \qquad \min_{y \varepsilon Q_2^{N'}} d(w, y) \geqq \min_{y \varepsilon Q_2^{N'}} d(w^{N'}, y) - d(w^{N'}, w) > \tfrac{1}{2}\delta_0$$

for some infinite subsequence of N' in I', and therefore $y^N \varepsilon Q_1^N - Q_2^N$ infinitely often. However (i) and (ii) imply that for N sufficiently large,

$$d(v_1, v_1^N) < \tfrac{1}{4}[z_1 + \epsilon_1 - d(v_1, w)]$$

and

$$|z_1 - z_1^N| < \tfrac{1}{4}[z_1 + \epsilon_1 - d(v_1, w)],$$

and therefore

$$\begin{aligned}\inf_{y \varepsilon Q_1^N - Q_2^N} d(w, y) &\geqq \inf_{y \varepsilon C^N} d(w, y)\\ &\geqq \inf_{y \varepsilon C} d(w, y) - \tfrac{1}{4}[z_1 + \epsilon_1 - d(v_1, w)]\\ &\geqq \tfrac{1}{2}[z_1 + \epsilon_1 - d(v_1, w)\end{aligned}$$

where $C^N \equiv \{x: d(v_1^N, x) > z_1^N + \epsilon_1\}$ and $C \equiv \{x: d(v_1, x) > z_1 + \epsilon_1\}$. Thus $y^N \not\varepsilon\, Q_1^N - Q_2^N$ for all sufficiently large N, a contradiction, and $d(v_1, w)$ must equal $z_1 + \epsilon_1$.

Define the vector u in Q_2 by

$$u = \lambda s + (1 - \lambda)w, \qquad \lambda = \tfrac{1}{4}\delta_0(z_1 + \epsilon_1)^{-1},$$

where s is any vector in Q_1 such that $d(v_1, s) = z_1$. Then

$$\begin{aligned}d(v_1, u) &\leqq \lambda\, d(v_1, s) + (1 - \lambda)\, d(v_1, w)\\ &= z_1 + \epsilon_1 - \tfrac{1}{4}\delta_0(\epsilon_1/(z_1 + \epsilon_1)) < z_1 + \epsilon_1,\end{aligned}$$

while

$$\begin{aligned}d(u, w) &= \lambda\, d(s, w)\\ &\leqq \lambda(z_1 + \epsilon_1) \qquad [\text{Since both } s \text{ and } w\, \varepsilon\, Q_2]\\ &= \tfrac{1}{4}\delta_0.\end{aligned}$$

The last inequality and a previously described property of w, $(^*)$, imply that for an infinite subsequence of I',

$$\min_{x \varepsilon Q_2^{N'}} d(u, x) \geqq \min_{x \varepsilon Q_2^{N'}} d(w, x) - \tfrac{1}{4}\delta_0 > \tfrac{1}{4}\delta_0.$$

The first argument may now be repeated with w replaced by u and δ_0 replaced by $\frac{1}{2}\delta_0$, yielding $d(v_1, u) = z_1 + \epsilon_1$, a contradiction.

CASE (b). Let w be an accumulation point of the infinite bounded sequence $\{w^{N'}\}$, with some subsequence $\{w^{N''}\} \equiv I''$ tending to w. By parts (i) and (ii),

$$\begin{aligned}d(v_1, w) &\leqq \limsup_{N'' \varepsilon I''} d(v_1, v_1^{N''}) + \limsup_{N'' \varepsilon I''} d(v_1^{N''}, w^{N''})\\ &\qquad + \limsup_{N'' \varepsilon I''} d(w^{N''}, w)\\ &\leqq z_1 + \epsilon_1.\end{aligned}$$

By the definition of w,

$$\begin{aligned}\min_{x \varepsilon Q_2} d(w, x) &\geqq \min_{x \varepsilon Q_2} d(w^{N''}, x) - d(w, w^{N''})\\ &\geqq \tfrac{1}{2}\delta_0 \quad \text{for sufficiently large} \quad N'' \varepsilon I''.\end{aligned}$$

Thus $w \not\varepsilon\, Q_2$, which implies $w \not\varepsilon\, Q_1$, since $d(v_1, w) \leqq z_1 + \epsilon_1$. Because of the closure of Q_1, $\min_{x \varepsilon Q_1} d(w, x) = \delta_1 > 0$. Then for all N'' such that $d(w^{N''}, w) < \frac{1}{2}\delta_1$,

$$d(Q_1^{N''}, Q_1) \geqq \min_{x \varepsilon Q_1} d(w^{N''}, x)$$

$$\geqq \min_{x \varepsilon Q_1} d(w, x) - d(w, w^{N''}) \geqq \tfrac{1}{2}\delta_1,$$

a contradiction. This verifies Lemma 1.

THEOREM 1. *The iterated minimax regret procedures satisfy Criteria* 1–8 *when applied to the class of closed and bounded convex S-games.*

PROOF. Suppose that $\lim_N d(Q_1^N, Q_1) = 0$, and $\lim_N d(s^N, x) = 0$, where s^N is the iterated minimax regret point for the game Q_1^N. For each s^N there exists x^N in Q_1 such that $d(s^N, x^N) \leqq d(Q_1^N, Q_1)$. Taking limits in $d(x, x^N) \leqq d(x, s^N) + d(s^N, x^N)$ gives $\lim_N d(x, x^N) = 0$, and so $x \varepsilon Q_1$ by closure.

Assume that $x \varepsilon Q_h$. The relationship

$$d(v_h, x) \leqq d(v_h, v_h^N) + d(v_h^N, s^N) + d(s^N, x)$$

implies, by Lemma 1, that $d(v_h, x) \leqq z_h + \epsilon_h$, and hence $x \varepsilon Q_{h+1}$. By induction, $x \varepsilon \bigcap_{h=1}^{\infty} Q_h$ and is therefore the iterated minimax regret point for Q_1.

4. Extensions and counter-examples. The examples constructed in this section will be representable as closed, bounded, and convex S-games in a countably-infinite dimensional vector space. That is, the statistician's decision consists of the choice of a vector in Q_1, a closed, bounded, and convex set in the l_∞ space over some infinite subset I of the integers. Nature's decision is the choice of a coordinate. The distance function in $l_\infty(I)$ is $d(x, y) = \sup_{j \varepsilon I} |x(j) - y(j)|$ and the iterated minimax regret decision rules are well-defined (with infimums replacing minimums in the definitions of the v_h and z_h).

First, let $I = I^+$, the non-negative integers, and define the vectors b_p by $b_p = (0, 0, \cdots, 0, 1 + (1/2^{p-1}), 1 + (1/2^{p-1}), \cdots)$, where there are p 0's.

LEMMA 2. *Let $C\{b_k\}$ be the convex hull of the vectors b_k, $k = 1, 2, 3, \cdots$, and Q_1 the closure in $l_\infty(I^+)$ of $C\{b_k\}$. Then the iterated minimax regret rule with $\epsilon_h = 2^{-h}$ yields no decisions.* (*That is,* $\bigcap_{h=1}^{\infty} Q_h = \phi$.)

PROOF. It is easily seen that $v_h = 0 \equiv (0, 0, 0, \cdots)$ and $z_h = 1$ for all h. Suppose that $x \varepsilon \bigcap_{h=1}^{\infty} Q_h$, implying that $d(0, x) = 1$. By the definition of Q_1, there exists a sequence $\{x_\alpha\} \subset C\{b_k\}$ such that $d(x, x_\alpha) \rightarrow 0$. Let $x(j)$ be the first non-zero coordinate of x, so $x_\alpha(j) \rightarrow x(j)$. If $x_\alpha = \sum_{k=1}^{k_\alpha} \lambda_{\alpha k} b_k$, $\lambda_{\alpha k} \geqq 0$, $k = 1, 2, \cdots, k_\alpha$, $\sum_{k=1}^{k_\alpha} \lambda_{\alpha k} = 1$, then, defining $\lambda_{\alpha k} = 0$ for $k > k_\alpha$,

$$x_\alpha(j) = \sum_{k=1}^{j-1} \lambda_{\alpha k}[1 + (1/2^{k-1)})] \rightarrow x(j),$$

(since $b_k(j) = 0$ for $k \geqq j$). This implies

$$p \equiv \liminf_\alpha \sum_{k=1}^{j-1} \lambda_{\alpha k} \geqq \tfrac{1}{2}x(j) > 0.$$

But for $i > k_\alpha$,

$$\begin{aligned} x_\alpha(i) &= \textstyle\sum_{k=1}^{k_\alpha} \lambda_{\alpha k} b_k(i) \\ &= 1 + \textstyle\sum_{k=1}^{k_\alpha} (\lambda_{\alpha k}/2^{k-1}) \\ &\geqq 1 + (1/2^{j-2}) \textstyle\sum_{k=1}^{j-1} \lambda_{\alpha k}, \end{aligned}$$

yielding

$$\liminf_\alpha (\limsup_i x_\alpha(i)) \geqq 1 + (p/2^{j-2}).$$

This shows that

$$\begin{aligned} d(0, x) &= \lim_\alpha d(0, x_\alpha) \\ &\geqq \liminf_\alpha (\limsup_i x_\alpha(i)) \geqq 1 + (p/2^{j-2}), \end{aligned}$$

a contradiction.

Two objections may be raised to this example. First of all, no point in Q_1 is admissible, and secondly, there is no minimax regret point. To answer the first objection, let I now be the entire set of integers, and b_k' the vector

$$(\cdots \tfrac{1}{4} - (1/2^{k+1}), \tfrac{1}{4} - (1/2^{k+1}),$$
$$\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \cdots \tfrac{1}{2}, 0, 0 \cdots 0, 1 + (1/2^{k-1}), 1 + (1/2^{k-1}), \cdots),$$

where there are k $\tfrac{1}{2}$'s and k 0's. That is, b_k' is b_k augmented by $(\tfrac{1}{2}, \tfrac{1}{2}, \cdots) - \tfrac{1}{4}b_k$ in the negative coordinates. Let Q_1' be the closure of $C\{b_k'\}$, $k = 1, 2, \cdots$. By symmetry, every point in Q_1' is admissible. Inspection reveals that $z_h' = 1$ for all h, $v_h'(j) = 0$ for $j \geqq 0$ for all h, and $d(v_h', b_k') = d(v_h, b_k)$ for all k and h. Therefore the iterated minimax regret procedure is identical at every step with the previous case, and yields no rules in the limit.

A similar argument shows that the game formed from the closure of the convex hull of the vectors

$$\begin{aligned} &(-4, x, x, x, x, \cdots), x > 15, \\ &(-1, 3, 3, 3, 3, \cdots), \\ &(-\tfrac{1}{2}, 0, 2, 2, 2, \cdots), \\ &(-\tfrac{1}{2}, 0, 0, 1\tfrac{1}{2}, 1\tfrac{1}{2}, \cdots), \\ &\vdots \\ &(-\tfrac{1}{2}, 0, \cdots, 0, 1 + (1/2^{k-1}), 1 + (1/2^{k-1}), \cdots), \\ &\vdots \end{aligned}$$

has $(-1, 3, 3, 3, \cdots)$ for a minimax regret point, but yields no vector by iterated minimax regret (with $\epsilon_h = \tfrac{1}{3}2^{-h}$).

It has been shown that Criterion 1 does not hold when there are an infinite number of states of nature. Consider now the general decision situation, which, for the purposes here, may be thought of as a convex S-game in the space $L_\infty(\Omega)$, where Ω is the state of nature space. It will be assumed that the set of points in

$L_\infty(\Omega)$ available to the statistician, Q_1, is uniformly bounded below in every co-ordinate (say by zero), in addition to being non-empty and convex.

The definition of the iterated minimax regret procedures extends in an obvious way to the general situation, as do the statements of the eight criteria. (Criterion 7 should now read: If there exists a set C which is a complete class for both of the games A_1 and A_2, then $Q(A_1) = Q(A_2)$. Criterion 8 is defined as before in terms of the symmetric distance function between sets.)

THEOREM 2. *In the general decision situation, the iterated minimax regret procedures satisfy Criteria 2–7. Criterion 8 is also satisfied in the sense that if* $A^N \to A$ ($\lim_N d(Q_1^N, Q_1) = 0$) *and* $\lim_N d(x^N, x) = 0$, *where each* $x^N \varepsilon Q(A^N)$, *then* x *is in the closure of* Q_h *for every* h.

5. Discussion. The verification of Criteria 2–7 differs only slightly in detail from that given in [1]. Lemma 1 remains true as stated in the general situation. A proof can be constructed along very similar lines to the one given, the lack of compactness, and therefore convenient limiting points and values, being paid for in additional ϵ's and δ's. The proof given for Theorem 1 then goes through as before.

It should be noted that none of the more common decision procedures satisfy Criterion 1 in the general situation. The usual resolution of this dilemma works equally well here: the class of "ϵ_h-iterated minimax regret procedures" is defined naturally as $Q_{h+1}(A)$, and by Lemma 1 will satisfy Criterion 8. The other 7 criteria continue to hold, 3 in the usual ϵ_h definition, 7 in the sense that if A_1 and A_2 have the same complete class, then $d(Q_{h+1}(A_1), Q_{h+1}(A_2)) \leqq \epsilon_{h+1}$.

REFERENCES

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