RECURRENT SETS[1]

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1. Summary and introduction. As is well known ([7], [12], [13] and [14]) Markov processes may be studied via an appropriate potential theory, which for symmetric random walks is discrete Newtonian potential theory (see [8], [19]). Here we will be concerned with the problem of deciding when a set of lattice points in $s \geqq 3$ dimensions is recurrent, that is, when it is visited infinitely often a.s. by the symmetric random walk. In [14] a necessary and sufficient condition was given to decide this problem. Intuitively this test determines whether or not ∞ is a regular point of a lattice set (see [11]). However the test tells one very little about the recurrence properties of an arbitrarily given lattice set.

Here we desire more precise information about lattice sets. Namely can one impose regularity conditions on lattice sets such that there exists a weighting $\mu(a)$, such that the divergence of $\sum_{a \varepsilon A} \mu(a)$ is necessary and sufficient for the recurrence of the lattice set A? As a step in this direction we give a necessary and sufficient condition for recurrence of a set for a general Markov chain in terms of the existence of a non-negative solution to a Wiener-Hopf type equation on the set in question. As a byproduct of this equation for finite lattice sets, we obtain interesting bounds for the probabilistic capacity of finite sets as well as explicit expressions for the probability of leaving a finite set forever.

As an application of our criterion for recurrence in $s = 3$ dimensions, regularity conditions are given on lattice sets so that $\mu(a) = |a|^{-1}$, where $|a|$ is the Euclidean distance from the lattice point a to the origin, is the appropriate weighting so that $\sum_{a \varepsilon A} \mu(a) = \infty$ iff A is recurrent. Further it is shown that the regularity conditions cannot be removed, as we exhibit a set for which the above series is divergent but the set is not recurrent. Further the above regularity conditions are invariant under arbitrary possibly different rotations of *each* lattice point about the origin and hence recurrent lattice sets satisfying these regularity conditions remain recurrent under arbitrary rotation. Finally a necessary condition and a sufficient condition are given for subsets of the axis in 3 dimensions.

2. Definitions and analytic results. We consider stationary transient Markov chains $\{x_n, n = 0, 1, 2, \cdots\}$, with state space $\mathfrak{X}$. Denote by $\bar{K}(a, b)$ for a, b in $\mathfrak{X}$ the expected number of visits of x_n to b given $x_0 = a$. $K(a, b)$ where a and b are

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lattice points will denote $\bar{K}$ for the symmetric random walk. We define recurrence and transience for subsets of the state space $\mathfrak{X}$ as follows:

DEFINITION 2.1. $A \subseteq \mathfrak{X}$ is *recurrent* iff there exists an $a \varepsilon A$ such that $P[x_n \varepsilon A$ infinitely often $| x_0 = a]$ is positive.

DEFINITION 2.2. $A \subseteq \mathfrak{X}$ is *transient* iff for all $a \varepsilon A$ $P[x_n \varepsilon A$ finitely often $| x_0 = a] = 1$.

DEFINITION 2.3. $A \subseteq \mathfrak{X}$ is *almost closed* iff $P[x_n \varepsilon A$ infinitely often$] = P[x_n \varepsilon A$ eventually$] > 0$.

In the particular case of the random walk, S_n, the state space will be the $s \geqq 3$ dimensional lattice. A set is recurrent iff it is visited infinitely often almost surely by the random walk (see [1]). Information on the behavior of $K(a, b)$ is given in [9]; that is, there exist constants depending only on s, for $s \geqq 3$, g_s and $f_s > 0$ such that

$$f_s|a - b|^{2-s} \leqq K(a, b) \leqq g_s|a - b|^{2-s} \tag{2.1}$$

for $|a - b| > 0$. Following Duffin [7] we define the boundary ∂A of a lattice set A as

$$\partial A = \{a \varepsilon A \mid a \pm e_i \varepsilon A^c \quad \text{for some} \quad i, \quad 1 \leqq i \leqq s\}, \tag{2.2}$$

with the e_i unit vectors in each axis direction.

3. Recurrence for sets. We now characterize transience as a set property for stationary transient Markov chains with the following theorem, and by its corollaries, transience of lattice sets for symmetric random walk in $s \geqq 3$ dimensions.

THEOREM 3.1. *Let A be a subset of the state space of a stationary transient Markov chain $\{x_n\}$. Then A is transient* iff *the system of equations* (3.1) *possesses a nonnegative solution $\mu(\cdot)$.*

$$\sum_{b \varepsilon A} \bar{K}(a, b)\mu(b) = 1 \qquad \textit{for all} \quad a \varepsilon A. \tag{3.1}$$

PROOF. Suppose A is transient. Consider x_n with $x_0 = a$ for any $a \varepsilon A$. Now define

$$\begin{aligned} W_{bj} &= 1 \quad \text{if} \quad x_j \quad \text{in} \quad A \quad \text{for the last time at} \quad b \\ &= 0 \quad \text{otherwise} \end{aligned}$$

but since A is transient

$$\sum_{b \varepsilon A} \sum_{j=0}^{\infty} W_{bj} = 1 \quad \text{a.s.} \tag{3.2}$$

However

$$\begin{aligned} E(W_{bj} \mid x_0 = a) &= P[x_j = b, x_{j+1} \not\varepsilon A \quad \text{for all} \quad r > 0 \mid x_0 = a] \\ &= P^{[j]}(a, b)P[x_r \not\varepsilon A \quad \text{for all} \quad r > 0 \mid x_0 = b] = P^{[j]}(a, b)e_A(b) \end{aligned} \tag{3.4}$$

by stationarity and the Markov property. The expected value of (3.2) yields the equality, $1 = \sum_{b \varepsilon A} \bar{K}(a, b)e_A(b)$, when the order of summation and expectation interchanged, a valid operation by Fubini's theorem, and (3.4) is used. However as a was arbitrary it follows that $e_A(\cdot)$ is a non-negative solution of (3.1). e_A will be called the last exit probability of A.

Now suppose A is recurrent and (3.1) possesses a non-negative solution $\mu(\cdot)$ then there exist an $a_0 \varepsilon A$ such that $P[x_n \varepsilon A$ infinitely often $| x_0 = a_0] > 0$. Now set $x_0 = a_0$ and let $K = \{\omega: x_n \varepsilon A$ infinitely often$\}$. Since (3.1) has a solution $\mu(\cdot)$, which is extended to the entire state space by defining it to be zero off A, it follows that this extension satisfies,

$$1 = \sum_{b\varepsilon A} K(a_0, b)\mu(b) = E_{a_0} \sum_{n=0}^{\infty} \mu(x_n). \tag{3.5}$$

Define

$$T_m = \min \{j > m \mid x_j \varepsilon A\}$$
$$= \infty \quad \text{otherwise}$$

and $K_m = [T_m < \infty]$. Now $T_m < \infty$ on K.

Let I_A denote the indicator function of the set A, then

$$\begin{aligned} 1 &\geqq E_{a_0} \sum_{n=0}^{m} \mu(x_n) + E_{a_0} I_{K_m} \sum_{n=m+1}^{\infty} \mu(x_n) \\ 1 &\geqq E_{a_0} \sum_{n=0}^{m} \mu(x_n) + E_{a_0} I_{K_m} \sum_{n=T_m}^{\infty} \mu(x_n) \qquad (3.6) \\ &\geqq E_{a_0} \sum_{n=0}^{m} \mu(x_n) + P(K_m \mid x_0 = a_0) \end{aligned}$$

by the strong Markov property and stationarity. The steps from line 2 to line 3 of (3.6) is easy, just expand it. Letting m tend to infinity, (3.6) implies

$$P(K \mid x_0 = a_0) \leqq 0, \tag{3.7}$$

which is a contradiction.

REMARKS. The author is indebted to F. Spitzer for asking why our previous theorem, Corollary 3.2 here, did not hold in a more general setting. Theorem 3.1 is the result. The essence of the sufficiency proof of Theorem 3.1 is also due to Spitzer and seems more direct and simple than our original proof using potential theory or a later proof using a martingale convergence theorem of [2].

Now we specialize to the random walk in $s \geqq 3$ dimensions and to lattice sets. We obtain the following

COROLLARY 3.1. *Given a lattice set A then A is transient* iff *the system*

$$\sum_{b\varepsilon\partial A} K(a, b)\mu(b) = 1 \qquad \textit{all} \quad a \varepsilon A \tag{3.8}$$

has a non-negative solution $\mu(\cdot)$.

PROOF. The necessity follows immediately from the proof of Theorem 3.1 by noting that $e_A(a)$ for $a \varepsilon A$ is null unless $a \varepsilon \partial A$. Sufficiency is also the same noting that K and K_m are equal a.s. to Ω, the probability space.

COROLLARY 3.2. *Suppose a lattice set A is not almost closed then A is transient* iff *the system*

$$\sum_{b\varepsilon\partial A} K(a, b)\mu(b) = 1, \qquad \textit{all} \quad a \varepsilon \partial A \tag{3.9}$$

has a non-negative solution $\mu(\cdot)$.

PROOF. Necessity is immediate from Corollary 3.1.

Sufficiency; suppose (3.9) has a solution and A is recurrent, then Theorem 3.1 implies ∂A is transient. Consequently $S_n \varepsilon A$ eventually a.s., or A is almost closed, which is a contradiction.

We introduce the capacity of a lattice set as follows (see [14]).

DEFINITION 3.1. The *capacity of a finite lattice set* A is $C(A) = \sum_{a\varepsilon\partial A} e_A(a)$ (with $e_A(\cdot)$ defined as in the proof of Theorem 3.1).

COROLLARY 3.3. *For a finite lattice set* A, $C(A)$ *satisfies*

$$\frac{|A|}{\max_{b\varepsilon\partial A} \sum_{a\varepsilon A} K(a, b)} \leqq C(A) \leqq \frac{|A|}{\min_{b\varepsilon\partial A} \sum_{a\varepsilon A} K(a,b)} \tag{3.10}$$

where $|A|$ *is the number of elements of* A.

PROOF. Clearly every finite lattice set is transient by the Borel Cantelli lemma. Hence $e_A(\cdot)$ is a solution of Equation (3.8). Bounds of the sum of both sides of (3.8), over $a \varepsilon A$, give (3.10).

COROLLARY 3.4. *Suppose* A *is a transient lattice set then the system* (3.9) *has the unique non-negative solution* $e_A(\cdot) = P(S_n \not\varepsilon A \text{ for all } n > 0 \mid S_0 = \cdot)$.

PROOF. From the proof of Theorem 3.1, $e_A(\cdot)$ is a solution. Now suppose there exists another non-negative solution μ. Form

$$f(x) = \sum_{b\varepsilon\partial A} K(x, b)\mu(b)$$
$$g(x) = \sum_{b\varepsilon\partial A} K(x, b)e_A(b).$$

These functions are finite valued and hence potentials in the sense of [14]. For any $x \not\varepsilon \partial A$ there exists an $a_0 \varepsilon \partial A$ of minimum Euclidean distance from x hence

$$|a - a_0| \leqq |x - a_0| + |x - a| \leqq 2|x - a|$$

or

$$|x - a|^{2-s} \leqq 2^{s-2}|a - a_0|^{2-s}.$$

In view of (2.1), for example f satisfies

$$f(x) \leqq g_s f_s^{-1} 2^{s-2} \quad \text{for all} \quad x.$$

Now application of the maximum principle (see [14]) yields $f(x) \leqq g(x)$ and $g(x) \leqq f(x)$. But a potential in the sense of [14] uniquely determines its charge. See [7] for an example of a potential in the sense of Doob, (i.e. non-finite valued) which does not uniquely determine its charge.

COROLLARY 3.5. *Suppose* A *is a transient lattice set. Then there exists a probability measure* $\nu(\cdot)$ *with support on* A *such that* $e_A(\cdot) = \nu(\cdot)/K(\cdot, a)$ *where* $a \varepsilon A$.

PROOF. This follows immediately from (3.8). In fact $\nu(\cdot)$ is the measure in the Riesz-Hunt representation of the excessive function, the entrance probability of A (see [12]).

REMARKS. For finite sets we have shown that solving a matrix equation determines the capacity of a finite set and the last exit probabilities. Since for $s = 3$,

K is tabulated by Duffin in [9], numerical values for these quantities are possible. For finite sets, Spitzer has obtained in [17] the analogous result for $s = 2$ and *recurrent* random walks. The above corollaries can be given suitably modified for transient stationary Markov chains. For example, the generalization of Corollary 3.4 can be proved with a maximum principle for potentials in the sense of Doob, finite only on the supports of their charges. (Private communication McKean.)

EXAMPLE 1. Consider $s = 3$, define $R(\cdot, \cdot, \cdot)$ as $R(a_1 - b_1, a_2 - b_2, a_3 - b_3) = K(a, b)$. Using Corollary 3.1 we compute the last exit probabilities for the sets, $A = \{(0, 0, 0), (0, 0, 1)\}$, $B = \{(a, 0, 0), (b, 0, 0)\}$ and $C = \{(0, 0, 0), (1, 0, 0), (2, 0, 0)\}$ as

$$e_A(0, 0, 0) = e_A(1, 0, 0) = \frac{1}{R(0,0,0) + R(1,0,0)}$$

$$e_B(a, 0, 0) = e_B(b, 0, 0) = \frac{1}{R(0,0,0) + R(a-b,0,0)}$$

$$e_C(0, 0, 0) = e_C(2, 0, 0,) = \frac{R(0,0,0) - R(1,0,0)}{R^2(0,0,0) + R(0,0,0)R(2,0,0) - 2R^2(1,0,0)}$$

$$e_C(1, 0, 0) = \frac{R(0,0,0) + R(2,0,0) - 2R(1,0,0)}{R^2(0,0,0) + R(0,0,0)R(2,0,0) - 2R(1,0,0)^2}.$$

Now we give a more analytic criterion for transience of a lattice set.

THEOREM 3.2. *A set A is transient* iff *there exists a $d \varepsilon A^c$ and a real number γ with $0 < \gamma < 1$ such that for every finite subset $A_N = \{a_1 \cdots a_N\}$ and every collection of positive real numbers $\{c_i\}_{i=1 \cdots N}$*

$$\gamma \sup_{b \varepsilon \partial A} \textstyle\sum_{k=1}^{N} c_k K(b, a_k) \geqq \sum_{i=1}^{N} c_i K(d, a_i). \tag{3.11}$$

PROOF. For sufficiency, suppose the condition holds and A is recurrent. Then by the Hewitt Savage 0-1 law, $\varphi(d) = P(S_n \varepsilon A \text{ for some } n \geqq 0 \mid d) = 1$ for every $d \varepsilon A^c$. Choose A_m finite subsets of A with m elements such that $A_m \uparrow A$. Since A_m are finite, Theorem 3.1 implies there exists a positive lattice function on ∂A_m, $e_{A_m}(\cdot)$ such that

$$\begin{aligned} \varphi_m(d) &= P(S_n \varepsilon A_m \quad \text{for some} \quad n \geqq 0 \mid S_0 = d) \\ &= \textstyle\sum_{b \varepsilon \partial A_m} \sum_{j=0}^{\infty} P(S_\sigma = b, \sigma = j \mid s_0 = d) \\ &= \textstyle\sum_{b \varepsilon \partial A_m} \sum_{j=0}^{\infty} P^{(j)}(d, b) e_{A_m}(b) = \sum_{b \varepsilon \partial A_m} K(d, b) e_{A_m}(b), \end{aligned} \tag{3.12}$$

where σ is the last exit time of A_m. Now $\varphi_m(d) \uparrow \varphi(d)$ as $m \to \infty$ by the countable additivity of the probability measure. But (3.11) implies

$$\gamma \sup_{b \varepsilon \partial A} \varphi_m(b) \geqq \varphi_m(d) \tag{3.13}$$

by choosing

$$\begin{aligned} c_i &= e_{A_m}(b_i) \qquad \text{for} \quad b_i \varepsilon \partial A_m, \\ &= 0, \qquad\qquad \text{otherwise.} \end{aligned}$$

But (3.13) implies

$$\gamma \geqq \varphi_m(d) \qquad \text{for all} \quad m \tag{3.14}$$

a contradiction, since $\varphi_m(d) \to 1$.

For necessity, suppose A is transient, then there exists a $d \,\varepsilon\, A^c$ and a γ, $0 \leqq \gamma < 1$, such that $\varphi(d) = \gamma$. Letting ρ_A be the first entrance time of the walk starting at d to A then it follows that

$$\textstyle\sum_{a \varepsilon \partial A} K(b, a)\mu_d(a) = K(b, d) \qquad \text{for all} \quad b \,\varepsilon\, A \tag{3.15}$$

where

$$\mu_d(a) = P(S_{\rho_A} = a, \quad \rho_a < \infty \mid S_0 = d).$$

Equation (3.15) follows from an argument of Doob [6], namely that $K(b, S_{\rho_A})$ and $K(b, d)$ form a martingale sequence, since $S_n \to \infty$ a.s. (see [6]). Also see Hunt [12] and [13]. But $\varphi(d) = \sum_{a \varepsilon \partial A} \mu_d(a)$, so that Equation (3.15) has a solution of l_1 norm γ. The theorem follows since (3.11) is obviously a necessary condition for (3.15) to have a non-negative solution of l^1 norm γ.

Equation (3.15) gives the harmonic measure of ∂A relative to d. Corollaries 3.1 and 3.2 deal with the existence of a positive solution to a discrete Wiener-Hopf equation and one might hope that it would be possible to practically decide the recurrence question with the Wiener-Hopf theory. But unfortunately in probabilistically interesting situations $K(a, b)$, restricted to ∂A, is not l^1 and the existence problem for positive solutions seems quite delicate.

4. Recurrence for subsets of the three dimensional lattice. Now we seek more specific information about three-dimensional lattice sets. Our results will be stated in three dimensions with their obvious analogue in higher dimensions left to the reader. The following definitions are appropriate:

Definition 4.1. An infinite sequence of real numbers $\{a_n\}$ is *superlinear* iff for all but a finite number of n's $a_n \geqq a_{n-i} + a_i$ for all i, $0 < i < n$,

Definition 4.2. A lattice set A is *radially finite* iff for every positive real number c there exist at most a uniformly in c bounded number of elements of A with Euclidean norm c.

Definition 4.3. A radial skeleton A_s of a radial finite set A is *any collection of elements of A with distinct Euclidean norms* such that for any $a \,\varepsilon\, A \;\exists\; a_s \,\varepsilon\, A_s$ such that $|a| = |a_s|$.

Given two lattice sets A and B with $B \supseteq A$ define $g_{B,A}(b)$ for any $b \,\varepsilon\, B$ as

$$g_{B,A}(b) = \min_{a \varepsilon A} |b - a|. \tag{4.1}$$

We first prove the following corollary, as its proof is computationally simple, and sketch the proof of its generalization Theorem 4.1.

Corollary 4.1. *Let A be a subset of the three dimensional lattice satisfying*:

(i) *A is radially finite*;

(ii) *The norms $r(n)$ of the elements a_n of a radial skeleton of A form a superlinear sequence.*

Then A is transient iff

$$\sum_{n=1}^{\infty} [r(n)]^{-1} < \infty. \tag{4.2}$$

PROOF. If the series converges then $\sum_{n=1}^{\infty} K(0, a_n)$ converges by (2.1) and the Borel Cantelli lemma implies A is transient.

Suppose $\sum_{n=1}^{\infty} 1/r(n)$ is divergent and A is transient. Without loss of generality we may assume $A = \{a_n\}$, where the a_n have distinct increasing norms. By Theorem 3.1 there exists $e(\cdot) \geqq 0$ such that

$$\sum_{n=1}^{\infty} K(a_m, a_n)e(a_n) = 1 \qquad \text{for all} \quad m. \tag{4.3}$$

Now define

$$\lambda^N(m) = \sum_{n=1}^{N} [r(n)]^{-1} K(a_m, a_n)[\sum_{n=1}^{N} [r(n)]^{-1}]^{-1}. \tag{4.4}$$

But (4.3) and (4.4) imply

$$\sum_{m=1}^{\infty} \lambda^N(m)e(a_m) = 1 \tag{4.5}$$

and for fixed m

$$\lambda^N(m) \to 0 \quad \text{as} \quad N \to \infty. \tag{4.6}$$

However, assume $m < N$, then using (2.1) it follows that

$$0 \leqq \lambda^N(m) \leqq \frac{c_1 \sum_{n=1, n \neq m}^{N} \frac{1}{r(n)} \frac{1}{|a_m - a_n|} + c_2 \frac{1}{r(m)}}{\sum_{n=1}^{N} \frac{1}{r(n)}}. \tag{4.7}$$

Now elementary inequalities can be used to yield

$$\begin{aligned} |a_m - a_n|^{-1} &\leqq [r(m) - r(n)]^{-1}, & m > n, \\ |a_n - a_m|^{-1} &\leqq [r(n) - r(m)]^{-1}, & n > m. \end{aligned} \tag{4.8}$$

But it is clear that for $m > n$

$$\{r(n)[r(m) - r(n)]\}^{-1} \leqq [r(m)]^{-1}\{[r(n)]^{-1} + [r(m-n)]^{-1}\} \tag{4.9}$$

while for $n > m$

$$\{r(n)[r(n) - r(m)]\}^{-1} \leqq [r(m)]^{-1}\{[r(n)]^{-1} + [r(n-m)]^{-1}\}$$

using Assumption (ii). Now it follows that

$$\sum_{n=1}^{m-1} [r(m-n)]^{-1} \leqq \sum_{n=1}^{N} [r(n)]^{-1}$$

$$\sum_{n=m+1}^{N} [r(m-n)]^{-1} \leqq \sum_{n=1}^{N} [r(n)]^{-1}.$$

Hence for $m < N$ it follows that

$$\lambda^N(m) \leqq [r(m)]^{-1}[4c + K(0, 0)r(1)]. \tag{4.10}$$

Similarly (4.10) holds for $m \geqq N$. The dominated convergence theorem in view of (4.6) and (4.10) with (4.3) implies

$$\lim_{N\to\infty} \sum_{m=1}^{\infty} \lambda^{N}(m)e(m) = 0. \tag{4.11}$$

But (4.11) contradicts (4.5)

THEOREM 4.1. *Let A be a radially finite subset of the three-dimensional lattice and suppose there exists a lattice set B containing a radial skeleton A_s of A such that g_{B,A_s} is bounded on B and the norms of elements of B are a superlinear sequence then A is transient* iff

$$\sum_{a\varepsilon A} |a|^{-1} < \infty. \tag{4.12}$$

PROOF. If (4.12) converges the Borel Cantelli lemma implies A is transient.

Suppose A_s is transient but (4.12) diverges. Then Theorem 3.1 asserts the existence of $\mu(a)$ satisfying (3.8) relative to A_s and it follows that $F(\cdot)$, the potential induced by the charge $\mu(\cdot)$ satisfies,

$$F(b) = \sum_{a\varepsilon\partial A_s} K(b, a)\mu(a) \geqq K/[g_{B,A_s}(b) + 1], \qquad b \varepsilon B^{\cdot}$$

Define

$$A_N = \{a \varepsilon B \mid |a| \leqq N\}$$

$$\lambda_N(a) = \sum_{b\varepsilon A_N} |b|^{-1}K(b, a)/\sum_{b\varepsilon A_N} |b|^{-1}(g_{B,A_s}(b) + 1)^{-1}.$$

Then since $\lambda_N(a) \to 0$ as $N \to \infty$ and $0 \leqq \lambda_N(a) \leqq c/|a|$ (see the proof of the above corollary) but $\sum_{a\varepsilon A}\lambda_N(a)\mu(a) \geqq K$, a contradiction.

SUB-COROLLARY 4.1. *Let $\alpha(n)$ be an increasing sequence of non-negative integers. Denoting $\Delta(n) = \alpha(n) - \alpha(n - 1)$ suppose $\Delta(n)$ decreases at most finitely often. Then $A = \{(0, 0, \alpha(n)), n = 1, 2, \cdots\}$ is transient* iff $\sum_{n=1}^{\infty} 1/\alpha(n) < \infty$.

PROOF. Let $\{\alpha'(n)\}_{n=1}^{\infty}$ be the set $\{\alpha(n)\}_{n=1}^{\infty}$ finitely augumented so that $\Delta'(n)$ is non-decreasing, where

$$\alpha'(n) - \alpha'(n - 1) = \Delta'(n) \quad \text{with} \quad \Delta'(1) = \alpha'(1).$$

Hence

$$\alpha'(n) = \sum_{j=1}^{n} \Delta'(j) \geqq \sum_{j=1}^{n-i} \Delta'(j) + \sum_{j=1}^{i} \Delta'(j) = \alpha'(n - i) + \alpha'(i)$$

or $\alpha'(n)$ is a superlinear sequence and the result follows.

EXAMPLES.

(1) Let $\alpha(n) = [n \log n]$ where $[x]$ denotes greatest integer in x then A is recurrent.

(2) Let $\alpha(n) = [n^{1+\epsilon}]$ $\epsilon > 0$ then A is transient.

(3) Let $\alpha(n) = [\log n!]$ then A is recurrent.

REMARKS. Note that the result of Ito and McKean ([13], (6.7), (6.8)) is considerably weaker than Sub-corollary 4.1. Given a set A satisfying the hypothesis of Sub-corollary 4.1, perform an arbitrary possibly different rotation of *each* of its lattice points about the origin to a new lattice position. Then Corollary

4.1 shows that the new set obtained has the same recurrence properties as A. It is easily seen that a lattice point of distance r from the origin has asymptotically r possible new positions under the rotation described above. Let A denote a subset of the axis and RA the resultant set obtained from A by arbitrary rotation of each of the points of A. Now *with no conditions* on A it follows easily that if RA is transient then A is transient and if A is recurrent then RA is recurrent. The referee suggested a proof of the foregoing using Theorem 3.2. An open question is whether RA can be recurrent but A transient. Likewise clearly conditions of Corollary 4.1 are invariant with respect to arbitrary lattice rotation of each point of a lattice set about the origin. The motivation for these theorems is a theorem of Breiman [3] for the returns to zero in the fair coin tossing game, which in turn was stated in [5], p. 1009, for the coin tossing game.

The question remains; is the gap condition of Sub-corollary 4.1 best possible in the sense that it cannot be entirely removed? We show that it is by an example of a transient subset of the line, $((0, 0, \lambda_n)\ n = 1, 2, \cdots)$ for which $\sum \lambda_n^{-1} = \infty$; (see [4]).

Counterexample. Let $\Lambda_r = \{i \in I \mid 2^r \leqq i < [2^r(1 + r^{-1})]\}$ and $\Lambda = \cup \Lambda r = (\lambda_n, n = 1, 2, \cdots)$. Then $\sum_{r=0}^{\infty} \lambda_n^{-1} = \sum_{r=1}^{\infty} \sum_{j \in \Lambda_r} j^{-1} = \sum_{r=1} \ln(1 + r^{-1}) = \infty$.

Recalling Corollary 3.3, it follows that

$$C(\Lambda_r) \leqq |\Lambda_r'|[\min_{i \in \Lambda_r'} \textstyle\sum_{j \in \Lambda_r'} K(i, j)]^{-1} \sim 2^r[r \sum_{j=1}^{2^r/r} j^{-1}]^{-1} \sim 2^r[r(r - \ln r)]^{-1}$$

where $A' = \{(0, 0, a) \mid a \in A\}$. Hence $\sum C(\Lambda_r)/2^r < \infty$, but Wiener's test (see Appendix 1 and [14]) implies $((0, 0, \lambda_n), n = 1, 2, \cdots)$ is transient.

In view of the specific nature of the counter-example we give necessary and sufficient conditions for transience of sets of this type satisfying a regularity condition as follows:

Theorem 4.2. *Let* $A = \{(i, 0, 0) \mid i \in B\}$ *where* $B = \bigcup_{r=1}^{\infty} \{j \mid m_r \leqq j < M_r\}$ *and* $M_r > m_r > M_{r-1}$ *with* $\{M_r\}$, $\{m_r\}$ *increasing sequences. Suppose*

(1) *For all but a finite number of* n's *and some finite positive* s *and* k *independent of* n

$$(M_n - M_j)(m_n - M_j)^{-1} \leqq M_{n-j}^k M_{n-j}^{-k} \qquad \text{for all } j < n$$

$$M_n - m_n \leqq \textstyle\prod_{i=1}^{n} M_i^s m_i^{-s}.$$

Then A *is transient* iff

$$\sum_{n=1}^{\infty} (M_n - m_n)/m_n < \infty. \tag{4.13}$$

Proof. Necessity is trivial. Sufficiency is shown by following the argument of Corollary 4.1 where here

$$\lambda_N(a) = [\textstyle\sum_{j=1}^{N} \sum_{i=m_j}^{M_j} i^{-1} K(a, (0, 0, i))][\sum_{j=1}^{N} \sum_{i=m_j}^{M_j} i^{-1}]^{-1}.$$

Using the conditions of this theorem, it easily follows that $\lambda_N(a) \leqq K|a|^{-1}$, and the theorem results.

Example. Let $M_n = n + n^2$, $m_n = n^2$ then A is recurrent.

For subsets of an axis in the lattice, define

$$m_A(n) = |A \cap \{a \mid 2^n \leqq |a| < 2^{n+1}\}|/2^n$$

where $|A|$ is the number of points in A. We have the following,

THEOREM 4.3. *Let A be a subset of the line in the three dimensional lattice. If $\sum_{n=1}^{\infty} m_A(n)/n$ diverges, then A is recurrent. Moreover, if $\sum_{n=1}^{\infty} m_A(n)$ converges, then A is transient.*

PROOF. Let $A_n = A \cap \{a \mid 2^n \leqq |a| < 2^{n-1}\}$. Consider the capacity of A_n, then by the remarks following Corollary 3.1

$$C(A_n) = \sum_{a \varepsilon \partial A_n} \mu(a) \tag{4.14}$$

where $\mu(a)$ is the unique solution of

$$\sum_{b \varepsilon \partial A_n} K(a, b)\mu(b) = 1, \qquad a \varepsilon \partial A_n . \tag{4.15}$$

Using Corollary 3.3, it is evident that

$$\frac{|\partial A_n|}{\max_{b \varepsilon \partial A_n} \sum_{a \varepsilon \partial A_n} K(a, b)} \leqq C(A_n) \leqq \frac{|\partial A_n|}{\min_{b \varepsilon \partial A_n} \sum_{a \varepsilon \partial A_n} K(a, b)} . \tag{4.16}$$

Now

$$\max_{b \varepsilon \partial A_n} \sum_{a \varepsilon \partial A_n} K(a, b) \leqq K(0, 0) + 2\int_1^{2^n} g_3/x \, dx \tag{4.17}$$

(see 2.1) and the result follows by Wiener's test (see [14] and Appendix). Note that one can replace $\sum m_A(n)/n$ by $\sum m_A(n)/\ln m_A(n)$ in Theorem (4.3).

Appendix. [Wiener's Test] (see [14]). Let A be a lattice set in $s \geqq 3$ dimensions then A is recurrent iff

$$\sum_{n=1}^{\infty} C(A \cap \{a \mid 2^n \leqq |a| < 2^{n+1}\})/2^{n(s-2)} = \infty .$$

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