INTEGRAL KERNELS AND INVARIANT MEASURES FOR MARKOFF TRANSITION FUNCTIONS[1]

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1. Introduction. An important question concerning Markoff transition functions is, when do they possess invariant measures? One aspect of this question is the following: given a measure μ, when will P possess a nontrivial invariant measure $\epsilon > \mu$? If infinite ϵ is permitted, then the question becomes a more difficult one.

Harris [4] showed that if μ is a separable measure such that for each set A with $\mu(A) > 0$ and every x, the probability of ultimately getting from x to A is one, then there is a unique σ-finite invariant measure ϵ, and $\epsilon > \mu$. In [2], the present author attempted to replace this by some sort of almost-everywhere type of assumption (μ-recurrence). The key point seemed to be to require that $\sum_{n=0}^{\infty} 2^{-n}P^n$ consist partly of an integral operator (an assumption which was an automatic consequence of Harris's hypothesis). A theorem was proven there for the more general case of μ-conservative processes, but the assumptions were stronger than necessary. Recently, R. Isaac [7] proved the existence of an invariant measure in the μ-recurrent case, making much weaker assumptions about the integral operator part. He was unable, however, to show the relation between μ and the invariant measure.

In the present paper, we show under Isaac's hypothesis that his invariant measure is equivalent to $\sum_{n=0}^{\infty} 2^{-n}\mu P$ (Theorem 4). Actually, a theorem is proven for the more general μ-conservative case (Corollary to Theorem 4), but this turns out to be easy, for the following rather surprising reason. While in general a μ-conservative transition operator is some sort of integral average of recurrent operators, the presence of a nontrivial integral operator part forces this integral average to be a *discrete* direct sum (Corollary to Theorem 1). In the process of showing Theorem 4, it proves convenient to find out more precisely what the integral operator part of $\sum 2^{-n}P^n$ is like. This is done in Theorem 2.

2. The μ-nonsingular part of P. Let $\mathfrak{X}$ be a σ-algebra on a set X. Let P be a *subtransition* function, i.e. a function on $X \times \mathfrak{X}$ which is, for each $x \in X$, a nonnegative measure on $\mathfrak{X}$ of total mass $\leqq 1$, and for each $A \in \mathfrak{X}$, an $\mathfrak{X}$-measurable function. P induces an operator on $\mathfrak{L}_\infty(\mathfrak{X})$, by the rule $Pf(x) = \int f(y)P(x, dy)$, and also an operator on the nonnegative measures on $\mathfrak{X}$, by the rule

$$\mu P(A) = \int P(x, A)\mu(dy).$$

Let μ be a fixed σ-finite measure on $\mathfrak{X}$. Then $P(x, \cdot)$ has a unique decomposition

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into nonnegative measures $R(x, \cdot)$ and $S(x, \cdot)$ with $R(x, \cdot) \prec \mu$ and $S(x, \cdot) \perp \mu$. R will be called the *μ-nonsingular part of P.*

FACT. In the event that the measure algebra of μ is separable, then a simple Martingale argument shows that there is a nonnegative real-valued function ρ on $X \times X$, measurable with respect to $\mathfrak{X} \times \mathfrak{X}_\mu$ (where $\mathfrak{X}_\mu$ denotes the μ-completion of $\mathfrak{X}$) such that each $\rho(x, \cdot)$ is a Radon-Nikodym derivative of $R(x, \cdot)$ with respect to μ. The fact is originally due to Doob [1]. A function ρ with these properties will be called a *μ-kernel* for P. Observe that the map $x \to R(x, A) = \int_A \rho(x, y)\mu(dy)$ is $\mathfrak{X}$-measurable for each $A \varepsilon \mathfrak{X}$. The measure $R(x, \cdot)$ clearly extends to $\mathfrak{X}_\mu$, and $R(\cdot, A)$ remains $\mathfrak{X}$-measurable for each $A \varepsilon \mathfrak{X}_\mu$.

REMARK. If the measure algebra of μ is *not* separable, then it may happen that there will exist no μ-kernel ρ for P. For the existence of such a ρ implies that the map $x \to \rho(x, \cdot)$ is a pointwise limit in $\mathfrak{L}_1(\mu)$ of a sequence of $\mathfrak{X}$-measurable step-functions, so that the range of the function $x \to R(x, \cdot)$ is separable as a subset of the measures in the total variation norm. But it is easy to construct examples where this is not the case. See [5], [8], for discussion of this point. As to the weaker property, measurability of $R(\cdot, A)$ for each $A \varepsilon \mathfrak{X}$, it is not known whether this holds in the nonseparable case, so far as the present author could ascertain.

DEFINITION. P will be called *μ-trivial* when $\{x \mid P(x, X) > 0\}$ is μ-null.

We recall that subtransition functions P, Q are multiplied by the rule $PQ(x, A) = \int P(x, dy)Q(y, A)$. Also, the notation I_A will represent the subtransition function $I_A(x, B) = 1$ if $x \varepsilon A \cap B$, 0 otherwise; as an operator, it's multiplication by the indicator function of the set A.

DEFINITION. The set $E \varepsilon \mathfrak{X}$ is called *P-invariant modulo μ* (or just *invariant*) provided

(1) $P(\cdot, E)$ vanishes μ-a.e. outside E,

(2) $P(\cdot, E^\perp)$ vanishes μ-a.e. outside $E^\perp$. In other words: $I_E P I_{E^\perp}$ and $I_{E^\perp} P I_E$ are μ-trivial, or: $P - (I_E P I_E + I_{E^\perp} P I_{E^\perp})$ is μ-trivial. We denote by $\mathcal{I}_\mu(P)$ the family of such sets. It is easy to see the following facts.

(a) $\mathcal{I}_\mu(P)$ is a σ-algebra.

(b) $\mathcal{I}_\mu(P^k) \supset \mathcal{I}_\mu(P)$

(c) $Q \leqq P \Rightarrow \mathcal{I}_\mu(Q) \supset \mathcal{I}_\mu(P)$.

For a σ-algebra $\mathcal{S}$, we denote by $\mathcal{S} \mid A$ the σ-algebra $(B \cap A \mid B \varepsilon \mathcal{S})$. If $A \varepsilon \mathcal{S}$, then this is just $(B \varepsilon \mathcal{S} \mid B \subset A)$.

THEOREM 1. *Suppose μ is purely non-atomic on $\mathcal{I}_\mu(P)$. Then the μ-nonsingular part of P is μ-trivial.*

PROOF. Since only the equivalence class of μ up to mutual absolute continuity is relevant here, we may assume $\mu(X) = 1$. From [3], Lemma 1, we see that X may be partitioned, for each n, into E_j^n, $j = 0, \cdots, 2^{n-1}$, with $E_j^n \varepsilon \mathcal{I}_\mu(P)$, $\mu(E_j^n) = 2^{-n}$, and $E_{2j}^{n+1} \cup E_{2j+1}^{n+1} = E_j^n$. For each $x \varepsilon X$, there is a unique integer j, $0 \leqq j < 2^n$, with $x \varepsilon E_j^n$; call this integer $j_n(x)$. If $n > m$, then $E_{j_n(x)}^n \subset E_{j_m(x)}^m$. Let R be the μ-nonsingular part of P. Then $R(x, E_j^n) \leqq P(x, E_j^n)$, so $R(x, E_j^n) = 0$ for μ-a.e. x in $E_j^{n\perp}$. Therefore $R(x, X) = \sum_{j=0}^{2^{n-1}} R(x, E_j^n) = R(x, E_{j_n(x)}^n)$ for μ-a.e. x in X. Since $E_{j_n(x)}^n \downarrow$ and $\mu(E_{j_n(x)}^n) = 2^{-n} \downarrow 0$, we conclude that $R(x, X) = 0$ for μ-a.e. x in X.

COROLLARY. *For any subtransition function P and σ-finite measure μ on $(X, \mathfrak{X})$, there is a partition of X into a finite or countable family $E_0, E_1, \cdots$ of μ-invariant sets such that*

(1) *the μ-nonsingular part of $P^k I_{E_0}$ is μ-trivial for each $k > 0$.*

(2) *If $j > 0$ then E_j is an atom* (modulo μ) *in $\mathcal{J}_\mu(P)$, and $\exists k \geqq 1$ such that $P^k I_{E_j}$ has μ-nontrivial μ-nonsingular part.*

PROOF. Let $\mathcal{E}$ be the family of sets E in $\mathcal{J}_\mu(P)$ for which the μ-nonsingular part of $P^k I_E$ is μ-trivial for all $k > 0$. Let E_0 be a supremum modulo μ for $\mathcal{E}$. Then it is evident that $E_0 \varepsilon \mathcal{E}$.

The maximality of $\mathcal{E}$ implies that for every μ-nonnull set E in $\mathcal{J}_\mu(P)$ with $E \perp E_0$, $P^k I_E$ has μ-nontrivial μ-nonsingular part for some k.

Finally, we show that μ is purely atomic on $\mathcal{J}_\mu(P) \mid E_0^\perp$. For suppose we have a set E in $\mathcal{J}_\mu(P)$, $E \perp E_0$, $\mu(E) > 0$, with μ purely nonatomic on $\mathcal{J}_\mu(P) \mid E$. Then μ would likewise be purely nonatomic on $\mathcal{J}_\mu(P^k) \mid E$, for each $k > 1$, since $\mathcal{J}_\mu(P^k) \supset \mathcal{J}_\mu(P)$ if $k > 1$. So $I_E P^k I_E$ would have μ-trivial μ-nonsingular part, by the previous theorem. Then the same would hold for $P^k I_E$, since $P^k I_E - I_E P^k I_E$ is μ-trivial. Thus $\mu(E) = 0$, by maximality of E_0.

DEFINITION. P will be called *μ-transitive* if $\mu(A) > 0 \Rightarrow \sum_{k=1}^\infty P^k(\cdot, A) > 0$ μ-a.e. Clearly, if P is μ-transitive, then $\mathcal{J}_\mu(P)$ is trivial modulo μ-null sets.

THEOREM 2. *Assume μ separable, and P μ-transitive, let R_k be the μ-nonsingular part of P^k, and suppose the R_k not all μ-trivial. Let $N = \{x \mid \sum_{k=1}^\infty R_k(x, X) = 0\}$. Then*

(1) $\mu(N) = 0$.

(2) *there is a fixed $F \varepsilon \mathfrak{X}$, $\mu(F) > 0$, such that for each $x \varepsilon N^\perp$, the measure $\sum_{k=1}^\infty R_k(x, \cdot)$ is equivalent to μI_F.*

PROOF. Let ρ_k be a μ-kernel for P_k, and let $F = \{y \mid \sum_{k=1}^\infty \int \rho_k(x, y)\mu(dx) > 0\}$; F is $\mathfrak{X}_\mu$-measurable, but we then change it by a set of measure 0 to get a set in $\mathfrak{X}$ (without bothering to change its name).

Since

$$0 < \sum_{k=1}^\infty \int R_k(x, X)\mu(dx) = \int \sum_{k=1}^\infty \int \rho_k(x, y)\mu(dx)\mu(dy),$$

$\mu(F)$ must be >0.

Now, if $A \subset F$ and $\mu(A) > 0$, we show that $\sum_{k=1}^\infty R_k(\cdot, A) > 0$ μ-a.e. It will suffice to show that $\sum_{k=1}^\infty R_k(\cdot, A)$ cannot vanish on any set B of positive μ-measure.

First: $\exists j \geqq 0$ such that $R_j(\cdot, A)$ is a μ-nonnull function, since

$$\sum_{k=1}^\infty \int R_k(x, A)\mu(dx) = \int_A \sum_{k=1}^\infty \int \rho_k(x, y)\mu(dx)\mu(dy) > 0.$$

Since P is μ-transitive, $\exists i \geqq 0$ such that $P^i R_j(\cdot, A)$ is not identically zero on B. Finally: $P^i R_j \leqq P^{i+j}$, and $P^i R_j(x, C) = \int_C \mu(dz)(\int P^i(x, dy)\rho_j(y, z))$, so that $P^i R_j(x, \cdot) \prec \mu$ for each x. So $P^i R_j \leqq R_{i+j}$, and $R_{i+j}(\cdot, A)$ is not identically zero on B. Consequently $\sum_{k=1}^\infty R_k(\cdot, A)$ is not identically zero on B.

Next: let $N = \{x \mid \sum_{k=1}^\infty R_k(x, X) = 0\}$.

Since $R_k(x, X) \geqq R_k(x, F)$ (actually, they are equal, of course), and we have just seen that $\sum_{k=1}^\infty R_k(\cdot, F) > 0$ μ-a.e., it follows that $\mu(N) = 0$. Now choose any fixed x in $N^{\perp}$. If $A \subset F$ and $\mu(A) > 0$, then $\sum_{k=1}^{\infty} R_k(\cdot, A) > 0$ μ-a.e., so

$$0 < \sum_{\ell=1}^{\infty} \int P^l(x, dy) \sum_{k=1}^{\infty} R_k(y, A) = \sum_{k,\ell} P^l R_k(x, A) \leqq \sum_{k,\ell} R_{\ell+k}(x, A) = \sum_{m=2}^{\infty} (m - 1) R_m(x, A).$$

Thus $R_m(x, A) > 0$ for some m. This completes the proof.

3. Conservative and recurrent transition functions.

DEFINITION. P is called μ-conservative if, for each $A \in \mathfrak{X}$,

$$\sum_{n=0}^{\infty} P(I_{A^{\perp}} P)^n(\cdot, A) = 1 \qquad \mu\text{-a.e. on } A.$$

If $P(x, A)$ is interpreted as the probability of a transition from x into the set A, then μ-conservativeness means that from μ-a.e. x in A, return to A is certain.

DEFINITION. P is called μ-recurrent if, whenever

$$\mu(A) > 0, \qquad \sum_{n=0}^{\infty} P(I_{A^{\perp}} P)^n(\cdot, A) = 1$$

μ-a.e. on X. Probabilistically: if $\mu(A) > 0$, then, from μ-a.e. starting point, arrival in A at some later time is certain.

THEOREM 3. *Let μ be finite, and $\tilde{\mu} = \sum_{n=0}^{\infty} 2^{-n} \mu P^n$. Then P is μ-conservative $\Leftrightarrow$ P is $\tilde{\mu}$-conservative, and P is μ-recurrent $\Leftrightarrow$ P is $\tilde{\mu}$-recurrent.*

Proof. $\Leftarrow$ is obvious in both cases. To go in the other direction, observe that $\tilde{\mu} P \prec \tilde{\mu}$, so P induces a Markoff operator on $\mathfrak{L}_\infty(\tilde{\mu})$, in the sense of [2]. There is thus a partition of X into sets C and D such that $\sum_{n=0}^{\infty} P(I_{B^{\perp}} P)^n = 1$ $\tilde{\mu}$-a.e. in B if $B \subset C$, and C is maximal with respect to the above property, up to $\tilde{\mu}$-null sets. This is just a simple exhaustion argument. The splitup (defined differently) is due to Hopf, [6]. Now, Theorem 2.2 of [2] tells us that $P(\cdot, D) = 0$ $\tilde{\mu}$-a.e. in C. Then also $P^n(\cdot, D) = 0$ $\tilde{\mu}$-a.e. in C, and *a fortiori* $P^n(\cdot, D) = 0$ μ-a.e. in C.

To prove P $\tilde{\mu}$-conservative, we show $\tilde{\mu}(D) = 0$. Suppose $\tilde{\mu}(D) > 0$, i.e. $0 < \sum_{n=0}^{\infty} 2^{-n} \mu P^n(D) = \sum_{n=0}^{\infty} 2^{-n} \mu I_D P^n(D)$. Consequently $\mu(D) > 0$. Thus, $\exists B \subset D$, with $\mu(B) > 0$, such that $\sum_{n=0}^{\infty} P(I_{B^{\perp}} P)^n < 1$ $\tilde{\mu}$-a.e. in B, and *a fortiori* μ-a.e. in B, so P cannot be μ-conservative.

Finally, we verify that μ-recurrence implies $\tilde{\mu}$-recurrence. We already know that P is $\tilde{\mu}$-conservative. This implies that the Markoff operator on $\mathfrak{L}_\infty(\tilde{\mu})$ induced by P is $\tilde{\mu}$-conservative as in [2]. So Theorem 2.3 of [2] tells us that all we need show is that $\tilde{\mu}(A) > 0$ and

$$\tilde{\mu}(B) > 0 \Rightarrow \sum_{n=0}^{\infty} \int_B P^n(x, A) \tilde{\mu}(dx) > 0.$$

Now: $\tilde{\mu}(B) > 0 \Rightarrow \mu P^k(B) > 0$ for some k, and $\tilde{\mu}(A) > 0 \Rightarrow \mu P^l(A) > 0$ for some l.

So if we assume μ-recurrence, and we choose $a > 0$, E with $\mu(E) > 0$, $P^l(\cdot, A) \geqq a > 0$ on E, then $\sum_{n=0}^{\infty} P(I_{E^{\perp}} P)^n(\cdot, E) > 0$ μ-a.e., hence $\sum_{n=0}^{\infty} P^n(\cdot, E) > 0$ μ-a.e., and consequently $\sum_{n=0}^{\infty} P^{n+l}(\cdot, A) \geqq a \sum_{n=0}^{\infty} P^n(\cdot, E) > 0$. Thus $\sum_{n=0}^{\infty} P^n(\cdot, A) > 0$ μ-a.e., and $\sum_{n=0}^{\infty} \int P^n(x, A) P^k(x, B) \mu(dx) > 0$, i.e. $\sum_{n=0}^{\infty} \int_B P^n(x, A) P^k \mu(dx) > 0$. So $\sum_{n=0}^{\infty} \int_B P^n(x, A) \tilde{\mu}(dx) > 0$.

4. Existence and uniqueness of invariant measures. The next fact is mainly just a change of terminology in part of Harris's proof in [4].

LEMMA. *Let Q be a subtransition function on $(Y, \mathfrak{Y})$, and γ a finite measure on $\mathfrak{Y}$. Let τ be a $\gamma \times \gamma$-measurable nonnegative function such that for each $x \varepsilon Y$ and $B \varepsilon \mathfrak{Y}$ we have $Q(x, B) \leqq \int_B \tau(x, y)\gamma(dy)$. Assume $\exists b > 0$ and $a > 2^{-\frac{1}{2}}$ such that $\gamma\{y \mid \tau(x, y) > b\} > \gamma(Y)$. Then $Q^n(x, B)$ converges to a number $\delta(B)$ independent of x, exponentially in the sense that $\exists c$, $0 < c < 1$, such that $|Q^n(x, B) - \delta(B)| < c^n$ for all x, A. δ is a finite measure, and $\delta Q = \delta$.*

If η is any nonzero measure on $\mathfrak{Y}$ with $\eta Q \prec \eta$, then $\delta \prec \eta$.

PROOF. This is almost all shown in Lemma 4 and Appendix in [4], the role of our Q being taken by Harris's R. To see that $\delta \prec \eta$: $\eta Q \prec \eta$ implies $\eta Q^n \prec \eta$ for each n. So if $\eta(B) = 0$, $\int Q^n(x, B)\eta(dx) = 0$ for each $n > 0$, and in particular, for each $n \; \exists \; x$ such that $Q^n(x, B) = 0$. Consequently $\delta(B) = 0$.

Next, we introduce a function P_A on $X \times \mathfrak{X}$, which takes on nonnegative values including perhaps $+\infty$. For each $x \varepsilon X$, $P_A(x, \cdot)$ is a measure, and for each $A \varepsilon \mathfrak{X}$, $P_A(\cdot, B)$ is $\mathfrak{X}$-measurable.

DEFINITION.

$$P_A(x, B) = \sum_{n=0}^{\infty} P(I_{A^\perp}P)^n(x, B).$$

P_A again gives rise to an operator on measures and on functions, but we stick to nonnegative functions because of all the infinities. P_A satisfies the following identity:

$$P_A = P + P_A I_{A^\perp} P_A .$$

Probabilistically, $P_A(x, B)$ is the expected number of times that a particle beginning at x will subsequently arrive in B, up to and including the time it first arrives in A. It is then straightforward, both probabilistically and combinatorially, that $P_A I_A$ is a subtransition function, $P_A I_A(x, \cdot)$ being the hitting distribution for the set A starting from x. Furthermore,

$$(P + \cdots + P^k)(\cdot, B) \leqq ((P_A I_A) + \cdots + (P_A I_A)^k)(\cdot, B) \quad \text{for} \quad B \subset A$$

since the left side is the expected number of times in B during k steps, while the right side is the expected number of times during b arrivals in A. Again, we omit the calculations. The probabilistic arguments may be made precise by constructions analogous to those of Lemmas 2.1, 2.2, and in [2].

Finally, notice that if γ is a measure on $\mathfrak{X}$ with $\gamma(A^\perp) = 0$, and $\gamma P_A I_A = \gamma$, then $(\gamma P_A)P = \gamma P_A$. For $(\gamma P_A)P = \gamma P_A(I_A + I_{A^\perp})P = \gamma(P_A I_A)P + \gamma P_A I_{A^\perp} P = \gamma P + \gamma P_A I_{A^\perp} P = \gamma P_A$.

THEOREM 4. *Let μ be a separable measure on $(X, \mathfrak{X})$, and P a μ-recurrent subtransition function. Assume that not all the P^k have μ-trivial μ-nonsingular part. Then there is a σ-finite measure ϵ, invariant under P, and equivalent to $\sum_{n=0}^{\infty} 2^{-n}\mu P^n = \tilde{\mu}$. ϵ is unique up to a multiplicative constant, among the σ-finite measures which are invariant under P and $\prec \tilde{\mu}$.*

PROOF. From Theorem 3, we may as well start with $\tilde{\mu}$, so we can assume that $\mu P \prec \mu$, without loss of generality; and we may as well also assume μ finite.

Next, we prove Harris's Lemma 2 in the present context. That is: for any b, $0 < b < 1$, we show that $\exists a > 0$, an integer m, and a set A in $\mathfrak{X}$ with $0 < \mu(A)$, such that, letting ρ_k be a μ-kernel for P^k, we have

$$\mu\{y \varepsilon A \mid \rho_1(x, y) + \cdots + \rho_n(x, y) > b\} > a\mu(A) \quad \text{for all} \quad x \varepsilon A.$$

From Theorem 2, $\exists F \varepsilon \mathfrak{X}$ with $\mu(F) > 0$ and

$$\sum_{k=1}^{\infty} \rho_k(x, y) > 0 \ \mu\text{-a.e.} \quad \text{on} \quad F \quad \text{for} \quad \mu\text{-a.e. } x \quad \text{in} \quad X.$$

So $\mu\{y \mid \sum_{j=1}^{k} \rho_j(x, y) > k^{-1}\} \uparrow \mu(F)$ for μ-a.e. x in X. Choose n so large that, setting

$$A = \{x \mid \mu\{y \mid \sum_{j=1}^{m} \rho_j(x, y) > m^{-1}\} \leqq [\tfrac{1}{2}(1 + b)]\mu(F)\},$$

we have $\mu(A) \geqq [\frac{1}{2}(1 + b)]\mu(F)$. This A will do the trick.

Now we shall apply the Lemma, with A taking the role of Y, $\mathfrak{X} \mid A$ that of $\mathfrak{Y}$, $\mu \mid A$ that of ν, and $n^{-1}(I_A P_A I_A + \cdots + (I_A P_A I_A)^n)$ (regarded as a subtransition function on A) that of Q. Thus a measure δ is obtained on $\mathfrak{X} \mid A$, as $\lim_{n\to\infty} Q^n(x, \cdot)$, which is invariant under Q. It is likewise invariant under $I_A P_A I_A$, as the argument of Lemma 5 in [4] shows. If we then denote by γ the extension of δ to $\mathfrak{X}$ gotten by setting $\gamma(A^{\perp}) = 0$, we get $\gamma P_A I_A = \delta$. Consequently, $\gamma P_A = \epsilon$ is an invariant measure under P.

Next, we verify that ϵ is nonzero. To see this, it suffices to show that $Q^n(x, A) = 1$ for all n, for μ-a.e. x in A. But this follows from $(P_A I_A)^n(x, A) = 1$ for μ-a.e. x in X. See the remarks after Theorem 2.1 and after Lemma 5.1 in [2].

Since $\delta \prec \nu$, it follows that $\gamma \prec \mu$, and so also $\epsilon \prec \mu$. As for σ-finiteness: it remains only to show γP_A is σ-finite on $A^{\perp}$. Here again we use a probabilistic argument, after Harris in [4]. Let $A_{ij} = \{x \text{ in } A^{\perp} \mid P^i(x, A) > j^{-1}\}$. Then

$$\bigcup_{i,j} A_{ij} \supset A^{\perp} \ \mu\text{-a.e.}$$

Now:

$$\gamma P_A(A_{ij}) = \int \gamma(dx) E_x\{\text{number of visits to } A_i, \text{ before returning to } A\}$$
$$\leqq \int \gamma(dx) \sum (1 - j^{-1})^i < \gamma(A) \cdot j.$$

(A standard "strong Markoff" argument gives this estimate).

Finally: uniqueness is again a consequence of results in [2], specifically Theorem 6.1.

COROLLARY. *Let μ be a separable measure, P μ-conservative, and E_0 the maximal μ-invariant set (see Corollary to Theorem 1) for which $P^k I_{E_0}$ is μ-trivial for all $k > 0$. Then there is a σ-finite measure equivalent to $\tilde{\mu} I_{E_0^{\perp}}$ and invariant under P.*

PROOF. Straightforward combination of Theorem 4 and the Corollary to Theorem 1.

REFERENCES

[1] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.

[2] FELDMAN, J. (1962). Subinvariant measures for Markoff operators. *Duke Math. J.* **29** 71–98.

[3] HALMOS, F. R. (1947). On the set of values of a finite measure. *BAMS* **53** 138–141.
[4] HARRIS, T. (1956). The existence of stationary measures for certain Markoff processes. *Proc. Third Berkeley Symp. Math. Statist. Prob.* **2** 113–124. Univ. of California Press, Berkeley.
[5] HILLE, E. and PHILLIPS, R. (1957). Functional analysis and semigroups. *A.M.S. Colloquium Publications* **31.**
[6] HOPF, E. (1954). The general temporally discrete Markoff process. *J. Rat. Mech. Anal.* **3** 13–45.
[7] ISAAC, R. (1964). Non-singular recurrent Markov processes have stationary measures. *Ann. Math. Statist.* **35** 869–871.
[8] PETTIS, B. J. (1938). On integration in vector spaces. *T.A.M.S.* **44** 277–304.