BOOK REVIEW

Correspondence concerning reviews should be addressed to the Book Review Editor, Professor William Kruskal, Department of Statistics, University of Chicago, Chicago 37, Illinois.

HARALD BERGSTRÖM, *Limit Theorems for Convolutions*. Almqvist and Wiksell, Stockholm, and John Wiley, New York, 1963. $15.00, £5/4/4, 347 pp.

Review by D. A. DARLING

University of Michigan

The so-called "central limit problem" is quite old, and has been, by and large, the principal impetus in the development of probability theory over the past 200 years. This is the problem of finding necessary and sufficient conditions on a sequence of sums of independent and individually negligible random variables to converge in distribution, to find the class of possible limit distributions and conditions under which a given member obtains. Through the efforts of a great many men over the past 30 or 40 years the problem may be fairly said to be solved today. In 1949 Kolmogorov and Gnedenko published their profound and exhaustive treatise *Limit Distributions for Sums of Independent Random Variables*, which effectively and definitively closes the problem.

In an appendix to the English translation of this book, J. L. Doob states, "It is possible, although undesirable and unnatural, to write a book on limit distributions of sums of independent random variables with essentially the content of this book, but without the use of the name random variable anywhere in the text."

In the book under review Bergström does what Doob said would be possible. This achievement is all the more remarkable in that Bergström does not employ the powerful machinery of Fourier analysis either; thus this book is, if for no other reason, a work of great ingenuity and individuality, but also a marked regression from modern developments. In the preface the author says, "Since I present a new method I need not quote many earlier works on the subject," and he refers the reader to the book by Kolmogorov and Gnedenko for a "historical background"—indeed there is no bibliography and only five works are cited in the footnotes. There is another in one appendix in which "random variables" and "probabilities" are first mentioned; a second appendix, with another reference, consists of a few pages on the Lebesgue-Stieltjes integral.

The author considers convolutions of functions of bounded variation, normalized at points of discontinuity by the average of the right and left-hand limits. The first six chapters give definitions, a restricted development, with a few lapses, of Riemann-Stieltjes integration over the reals, a metrization of the space of functions considered by what the author calls the "Gaussian norm," weak compactness, the selection principle, weak convergence and the algebra of convolutions. All of this is standard material but the development and terminology are the author's own, and not always easly to follow.

The author defines an infinitely divisible function as a function which is the n-fold convolution of another function, for every n. Chapters 7 and 8 consider the limits of convolutions of identical, monotone, uniformly bounded components—i.e., essentially distribution functions. The limits are not specifically exhibited, but given in terms of limits of sequences of other functions, whose convergence is necessary and sufficient for the original limit to exist. From this the author can obtain Lévy's canonical form for the logarithm of the characteristic function of the infinitely divisible distributions. Chapter 8 considers the subclass of stable limits and Chapter 9 the case of functions of bounded variation in the same development, but under the restriction that the convolution of the monotonic sequence of functions, whose members are the variation of the given sequence, also converges, suitably normalized. This severe restriction makes the two developments almost identical, and essentially identical to the case of sums of independent random variables. Chapters 10, 11 and 12 do the same thing for the case of non-identical components. Part II of the book repeats the above material for functions defined over Euclidean spaces of finite dimension.

From Chapter 9 on the limit laws are not exhibited, even implicitly, and the sufficient conditions for convergence are very complicated and uninformative.

One cannot but admire and respect the patience and courage which the author has shown in writing this book, but one is left also with a feeling of dissatisfaction if not disappointment. Most present day mathematicians working in measure—integration, in whichever camp they find themselves, would consider the present book an anachronism, and even those not so disposed would hesitate to recommend it to a student, who, after reading it, would not be in a position to read or appreciate other current research in this area. For better or worse the objects which are nowadays considered and convolved are measures, not functions, and especially in the restricted context of the material in this book, the theory of measure, integration and Fourier analysis is so simple, streamlined and direct that students can be, and need to be, initiated to it early.

Indeed, whereas in the theory of integration over the real line there may be some justification for developing the RS integral, when the author goes to n-dimensions and has to generalize such things as monotonicity, bounded variation, normalized functions, etc. the theory becomes very unwieldy and artificial, and the merit of the general approach, consistent with any co-ordinatization of the space, is strikingly apparent.

The author defends his development by stating in the introduction that convolutions of objects more general than distribution functions occur in applications outside probability theory. It would be very nice to have some of these applications presented. The reviewer is aware that if some of the known limit theorems in probability could be extended to signed, unbounded measures there might be some applications to boundary value problems, quantum mechanics, etc. which cannot be treated in this manner at present. But the author, by restricting his scope, has avoided the necessary delicacies and can not reach (and does not quote) some earlier work in this area; e.g. V. Ju Kry'ov, A Limit Theorem. *Dok. Acad. Nauk. SSSR* **139** (1961) 20–23.

Another unpleasant feature is that the author has made no effort to correlate his theorems with the classical theorems so as to orient the reader as to what results are being "generalized." For example, in Chapter 11 the author treats (translating to probability language) the problem of finding when a series of independent random variables converges in distribution—the fact even that the author is trying to do this is hard to come by. A known theorem states this is true if and only if the series converges with probability one. The author gives sufficient conditions which one realizes, only after considerable strain, to be related to the necessary and sufficient conditions of Kolmogorov's three series theorem. Here and elsewhere an appreciation of the classical material by the author would have been most welcome.

In addition the central role of the individually negligible summands and the Poisson distribution are so disguised as to be essentially invisible, yet they form the cornerstone of the whole theory.

In sum the reviewer feels this work is more remarkable for its existence than its accomplishments, and he doubts that it will take a prominent place in the literature.

In order to present a complete picture, I should add that Professor Bergström has clearly not written this book primarily for people interested or competent in probability, but rather for those with a different mathematical background. He feels that the limit theorems have a content outside of probability theory, and that, for the probabilist, the book is meant as a complement that presents some ideas and results worth consideration. More specifically, the main aim of the book is to demonstrate a very general connection between convergence of products in an algebra and convergence of a sum, in the same way that there is a connection between the convergence of a product of numbers, $\prod a_i$, and of the corresponding sum, $\sum (a_i - 1)$. Professor Bergström feels that he has shown such a general connection, that it leads to the classical probabilistic limit theorems, and that it leads further to fresh generalizations. The book's discussion is in terms of a fairly general algebraic system, in which it is natural to consider the algebra of functions of bounded variation, with convolution as the multiplication rule. Professor Bergström informs me that he has found one new application of his method in the study of stochastic processes of bounded variation, with respect to a suitable norm; he hopes that his book will stimulate interest in his method, and study of other general commutative operators.