FACTORIAL DISTRIBUTIONS[1]

By W. H. Marlow

The George Washington University

1. Introduction and summary. This note deals with a two-parameter family of discrete distributions which has interesting moment properties. We are indebted to Arthur Schleifer, Jr. for having informed us that the distributions derived below by formal methods are integer parameter cases of what are called *Beta-Pascal* in Raiffa and Schlaifer (1961), p. 238.

We were led to the present family by searching for discrete distributions to be used in numerical work with inventory problems. In particular, we were interested in enlarging our discrete repertoire (binomial, Poisson, geometric, negative binomial, etc.) in two directions. First, we sought convenient closed form expressions for various associated probabilities and expectations. This would of course simplify computation of penalty functions and optimal inventory levels. Second, we sought representation of rather extreme behavior so as to subject our theoretical formulations to stresses such as large dispersions about average future demand. It turns out that the present family meets both criteria. Not only is it highly tractable but corresponding to each value $r = 0, 1, 2, \cdots$ there is a member distribution possessing its rth moment but whose $(r + 1)$st moment fails to exist.

Our approach is entirely analogous to the following formal "development" of *geometric distributions*. We start with the observation that $\sum_{i=0}^{\infty} q^i = 1/(1 - q)$ is a convergent series of positive terms for $0 < q < 1$. Then we normalize to unit sum and consider the series to represent a probability distribution: $\sum_{i=0}^{\infty} pq^i = 1$ where $p = 1 - q$. If this is done, we obtain the one-parameter geometric distribution whose mode is zero, whose mean equals q/p and whose variance-to-mean ratio equals (1 + mean). Our present effort similarly begins with a convergent series (of factorial power functions) and we find a two-parameter family where, interestingly enough, not only is the mode zero but the variance-to-mean ratio, when it exists, approaches the quantity (1 + mean).

2. Factorial power functions. It is a basic fact [Jordan (1950), Section 16] that for factorial power functions $x^{(n)} = \binom{x}{n} n!$, $\Delta x^{(n)} = (x + 1)^{(n)} - x^{(n)} = nx^{(n-1)}$, one of whose consequences is

$$(n + 1) \sum_{x=r}^{s} x^{(n)} = x^{(n+1)} \Big|_{x=r}^{x=s+1} \quad (n \neq -1). \tag{2.1}$$

One extends for $m = 0, 1, 2, \cdots$ through $x^{(-m)} = 1/(x + m)^{(m)}$. This means that in general $x^{(-n)} \neq 1/x^{(n)}$ but instead $1/x^{(n)} = (x - n)^{(-n)}$. It is readily

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verified that (2.1) is valid for negative powers and we obtain in this way [Jordan (1950), Section 42] the following basic expression.

$$\sum_{x=0}^{\infty} [1/(x + m)^{(n)}] = 1/(n - 1)(m - 1)^{(n-1)} \tag{2.2}$$

where $m > n - 1, n = 2, 3, \cdots$. It is of some interest to note that (2.2) qualifies as a "most general" formulation based on factors linear in x: if we start with the analogue of (2.1) with $(px + q)^{(n)}$ in place of $x^{(n)}$ we are again led to an expression equivalent to (2.2).

3. The basic family. Each admissible pair of numbers m, n in (2.2) leads to a probability distribution as follows where for $i = 0, 1, 2, \cdots$,

$$f(i) = (n - 1)(m - 1)^{(n-1)}/(m + i)^{(n)} \tag{3.1}$$

where $m > n - 1, n = 2, 3, \cdots$. This is equivalent to the following convenient form:

$$f(0) = (n - 1)/m,$$

$$f(i + 1) = [(m - n + 1 + i)/(m + 1 + i)]f(i).$$

Direct calculation of the cumulative yields

$$F(k) = 1 - [(m - 1)^{(n-1)}/(m + k)^{(n-1)}] = 1 - [(m - n + 1 + k)/(n - 1)]f(k).$$

The moments can also be computed directly by elementary methods. On account of linearity, for example, $E(m + i) = m + E(i)$ and the left-hand member can

TABLE 1
Distributions with mean 0.10

Distribution	q	$F(0)$	$F(1)$	$F(2)$
Poisson	1	0.90484	0.99532	0.99985
Geometric	1.1	0.90909	0.99174	0.99925
$n = 5$	2.2	0.93023	0.98289	0.99375
$n = 4$	3.3	0.93750	0.98214	0.99245
$n = 3$	∞	0.95238	0.98310	0.99135

TABLE 2
Distributions with $f(0) = 0.5$

Distribution	Mean	Variance
Poisson	0.69	0.69
Geometric	1.00	2.00
$n = 11$	1.11	2.93
$n = 4$	1.50	11.25
$n = 3$	2.00	∞
$n = 2$	∞	∞

be found by summing $(m + i)f(i) = (n - 1)(m - 1)^{(n-1)}/(m + i - 1)^{(n-1)}$. Each distribution has moments through $E(i^{n-2})$; in particular,

$$\text{Mean} = (m - n + 1)/(n - 2),$$

$$(\text{Variance}/\text{Mean}) = [(n - 1)/(n - 3)][1 + \text{Mean}].$$

We record a few more properties as follows.

(a) To each pair, μ and n where $\mu > 0$ and $n = 3, 4, \cdots$, there corresponds a distribution (3.1) with mean equal to μ: $m = (n - 2)\mu + (n - 1)$.

(b) To each pair, μ and q where $\mu > 0$ and $q > 1 + \mu$, there corresponds a distribution (3.1) with *minimum* n, $n = 4, 5, \cdots$, whose mean equals μ and whose variance-to-mean ratio does not exceed q.

(c) The probabilities $f(i)$ are strictly decreasing in i so that the *mode* is always zero.

(d) The median defined as $\text{Min}_i \{F(i) \geqq \frac{1}{2}\}$ equals $\text{Min}_i \{f(i) \leqq (n - 1)/2(m - n + 1 + i)\}$.

(e) If $n > 2$ so that the mean exists then

$$\text{Median} = \text{Min}_i \{f(i) \leqq (n - 1)/2[(n - 2)(\text{Mean}) + i]\},$$

$$\text{Median} = 0 \text{ if and only if Mean} \leqq (n - 1)/(n - 2).$$

4. Illustrations. If $n = 2$ the resulting distribution (3.1) has no expectation, i.e., its mean may be said to be infinite. We find $F(i) = (1 + i)/(m + i)$ which makes it clear that convergence to unity is indeed slow. If $m = 2$ then $f(0) = F(0) = \frac{1}{2}$, $F(1) = \frac{2}{3}, \cdots$; for example, in order to have $F(k) \geqq 0.9999$ we must have $k \geqq 9,998$. In a geometric distribution with $F(0) = \frac{1}{2}$, $k \geqq 13$ would suffice while for the corresponding Poisson distribution merely $k = 5$, i.e., 6 terms, would be enough to exceed 0.9999.

In Table 1 there are illustrated several distributions possessing a common mean of 0.10 with "q" denoting the variance-to-mean ratio. The last three distributions are cases of (3.1) for $m = 4.3$, 3.2 and 2.1, respectively.

Table 2 displays associated moments for several distributions with $f(0) = \frac{1}{2}$. The last four distributions correspond to (3.1) for the respective cases $m = 20, 6, 4, 2$.

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