A NOTE ON MIDRANGE

By E. J. Gumbel, P. G. Carlson and C. K. Mustafi

Columbia University; Tulane University; Columbia University

1. Introduction. For certain distributions the sample midrange, as a measure of central tendency, is preferable to the sample mean. This holds, for example, for the rectangular distribution. Rider [2] has shown other distributions where the midrange is more efficient than the range. One of the authors [1] derived the asymptotic distribution of the midrange for initial symmetrical distributions of the exponential type. This asymptotic distribution is unlimited, differentiable, symmetrical and unimodal. In the following, we will prove that this holds for any sample size, provided that the initial distribution is unlimited, differentiable, symmetrical and unimodal.

The usefulness of the midrange is confined to symmetrical distributions. A distribution is called symmetrical about a value $M/2$ if

$$f[(M - x)/2] = f[(M + x)/2], \qquad F[(M - x)/2] = 1 - F[(M + x)/2] \tag{1.1}$$

where $F(x)$ and $f(x)$ are the probability and density functions of the initial distribution.

For a sample of size n $(n \geqq 2)$, ordered in increasing magnitudes, $x_1 \leqq x_2 \leqq \cdots x_{n-1} \leqq x_n$, we define the sample range w and the sample midrange v by

$$w = x_n - x_1; \qquad v = x_n + x_1. \tag{1.2}$$

Some authors use, instead, $(x_1 + x_n)/2$. The constant factor, $\frac{1}{2}$, is here omitted for reasons of analytical simplicity, but does not affect the results to follow. The density function $h_n(v)$ of the midrange is

$$\begin{aligned} h_n(v) = (n(n-1)/2) \int_0^\infty & f[(v-w)/2] \\ & \cdot [F[(v+w)/2] - F[(v-w)/2]]^{n-2} f[(v+w)/2]\, dw. \end{aligned} \tag{1.3}$$

From (1.1) it follows that

$$h_n(M + v) = h_n(M - v). \tag{1.4}$$

If the initial distribution is symmetrical about a value $M/2$, the distribution of the midrange is symmetrical about M, which is evident.

2. Theorem. *If the initial distribution is unlimited, differentiable, symmetrical and unimodal, the distribution of the midrange is also unlimited, differentiable, symmetrical and unimodal.*

For the proof, we consider first the case $n > 2$. Differentiation of the density

Received 16 June 1964; revised 15 December 1964.

function (1.3) of the midrange leads to

$$4h_n'(v)/n(n-1)$$
$$= \int_0^\infty f_v'[(v+w)/2]\{F[(v+w)/2] - F[(v-w)/2]\}^{n-2} f[(v-w)/2]\,dw$$
$$+ (n-2)\int_0^\infty f[(v+w)/2]\{F[(v+w)/2] - F[(v-w)/2]\}^{n-3}$$
$$\cdot\{f[(v+w)/2] - f[(v-w)/2]\} f[(v-w)/2]\,dw$$
$$+ \int_0^\infty f[(v+w)/2]\{F[(v+w)/2] - F[(v-w)/2]\}^{n-2} f_v'[(v-w)/2]\,dw$$

where $f_v' = \partial f/\partial v$ and $f_w' = \partial f/\partial w$. The partial derivatives of the density functions are linked by three relations which result from the symmetry,

$$f_v'[(v+w)/2] = f_w'[(v+w)/2] \tag{2.1}$$
$$f_v'[(v-w)/2] = -f_w'[(v-w)/2] \tag{2.2}$$
$$f_v'[(M+w)/2] = -f_v'[(M-w)/2]. \tag{2.3}$$

Substituting $v = M$ in the expression of $h_n'(v)$, we see that the first and third terms are equal in magnitude but opposite in sign by virtue of (2.3). The second term vanishes due to symmetry. Thus, the distribution of the midrange has a mode at $v = M$. We now show that, for every value of $v > M$, $h_n'(v) < 0$. Suppose v' is a value of $v > M$. Since the initial distribution is unimodal and $w > 0$ we have $f[(v'+w)/2] - f[(v'-w)/2] < 0$, so that the second term of $4h_n'(v)/n(n-1)$ at $v = v'$ is always negative. We now show that the sum of the first and the third terms is also negative at $v = v'$. At $v = v'$, the sum of the first and the third terms is given by

$$\int_0^\infty f_v'[(v'+w)/2]\{F[(v'+w)/2] - F[(v'-w)/2]\}^{n-2} f[(v'-w)/2]\,dw$$
$$+ \int_0^\infty f[(v'+w)/2]\{F[(v'+w)/2] - F[(v'-w)/2]\}^{n-2}$$
$$\cdot f_v'[(v'-w)/2]\,dw$$
$$= \int_0^\infty f_w'[(v'+w)/2]\{F[(v'+w)/2] - F[(v'-w)/2]\}^{n-2} \tag{2.4}$$
$$\cdot f[(v'-w)/2]\,dw$$
$$- \int_0^\infty f[(v'+w)/2]\{F[(v'+w)/2] - F[(v'-w)/2]\}^{n-2}$$
$$\cdot f_w'[(v'-w)/2]\,dw$$
$$= I_1 - I_2 \quad \text{(say)},$$

where we have used (2.1) and (2.2).

Now,

$$I_1 = \int_0^\infty \{F[(v'+w)/2] - F[(v'-w)/2]\}^{n-2} f[(v'-w)/2] f_w'[(v'+w)/2]\,dw$$
$$= -(n-2)\int_0^\infty \{F[(v'+w)/2] - F[(v'-w)/2]\}^{n-3}$$
$$\cdot\{f[(v'+w)/2] + f[(v'-w)/2]\} f[(v'-w)/2] f[(v'+w)/2]\,dw$$

$$+ \int_0^\infty \{F[(v' + w)/2] - F[(v' - w)/2]\}^{n-2} f_w'[(v' - w)/2] f[(v' + w)/2] \, dw$$

$$= -(n-2) \int_0^\infty \{F[(v' + w)/2] - F[(v' - w)/2]\}^{n-3}$$

$$\cdot \{f[(v' + w)/2] + f[(v' - w)/2]\} f[(v' - w)/2] f[(v' + w)/2] \, dw + I_2 .$$

Hence,

$$I_1 - I_2 = -(n-2) \int_0^\infty \{F[(v' + w)/2] - F[(v' - w)/2]\}^{n-3} \cdot \{f[(v' + w)/2] + f[(v' - w)/2]\} f[(v' - w)/2] f[(v' + w)/2] \, dw. \tag{2.5}$$

The right hand side of (2.5) is always negative for $n > 2$. Thus $h_n'(v') < 0$ for $v' > M$. Hence there cannot be any mode to the right of M for $n > 2$. By symmetry there cannot be any mode at the left of M which proves that the distribution of the midrange is unimodal for $n > 2$. We now consider the case $n = 2$. Then the density function of the midrange is

$$h_2(v) = \int_0^\infty f[(v + w)/2] f[(v - w)/2] \, dw \tag{2.6}$$

whence

$$2h_2'(v) = \int_0^\infty f_v'[(v + w)/2] f[(v - w)/2] \, dw + \int_0^\infty f[(v + w)/2] f_v'[(v - w)/2] \, dw.$$

Substituting $v = M$ in (2.6) and applying (2.3), we obtain $h_2'(M) = 0$ which proves that the distribution has a mode at $v = M$. If the distribution possesses a second mode then there exists a value of v, say, v' such that $h_2'(v') > 0$. Hence,

$$2h_2'(v') = \int_0^\infty f_v'[(v' + w)/2] f[(v' - w)/2] \, dw + \int_0^\infty f[(v' + w)/2] f_v'[(v' - w)/2] \, dw$$

$$= \int_0^\infty f_w'[(v' + w)/2] f[(v' - w)/2] \, dw - \int_0^\infty f[(v' + w)/2] f_w'[(v' - w)/2] \, dw > 0.$$

Integrating by parts, we obtain

$$2h_2'(v') = -2f^2(v'/2) > 0$$

which is impossible. Hence there cannot be any other mode of the distribution of midrange for $n = 2$ and the proof is complete for all values of n.

REFERENCES

[1] Gumbel, E. J. (1944). Ranges and midranges. *Ann. Math. Statist.* **15** 414–422.

[2] Rider, P. R. (1957). The midrange of a sample as an estimator of the population midrange. *J. Amer. Statist. Assoc.* **52** 537–542.