ESTIMATION OF JUMPS, RELIABILITY AND HAZARD RATE

By V. K. Murthy

Douglas Aircraft Company

0. Summary. Let $F(x)$ be a probability distribution function. Assuming the singular part to be identically zero, it is well known (see e.g. Cramér [1] pp. 52, 53) that $F(x)$ can be decomposed into $F(x) = F_1(x) + F_2(x)$ where $F_1(x)$ is an everywhere continuous function and $F_2(x)$ is a pure step function with steps of magnitude, say, S_ν at the points $x = x_\nu$, $\nu = 1, 2, \cdots, \infty$ and that finally both $F_1(x)$ and $F_2(x)$ are non-decreasing and are uniquely determined. In this paper the problem of estimating the jump S_i corresponding to the saltus $x = x_i$ is considered. Also considered are the problems of estimation of reliability and hazard rate. Based on a random sample $X_1, X_2, \cdots X_n$ of size n from the distribution $F(x)$, consistent and asymptotically normal classes of estimators are obtained for estimating the jump S_i corresponding to the saltus $x = x_i$. Based on the earlier work of the author [2] on estimation of probability density, consistent and asymptotically normal estimates are obtained for the reliability and hazard rate.

1. Introduction. Let $X_1, X_2, \cdots X_n$ be a random sample of size n from the distribution $F(x)$, i.e. $X_1, X_2, \cdots X_n$ are independently, identically distributed random variables with the same distribution function $F(x)$. In the particular case when the random variable is time to failure of an item, $F(x)$ is the probability of the event that by time x the item has failed and $R(x) = 1 - F(x)$ is the probability of the complementary event that the item survived time instant x and is the so-called reliability of the item. In what follows, for any random variable with distribution function $F(x)$, we call $R(x) = 1 - F(x)$ the reliability function. If x is any point of continuity of the distribution $F(x)$ and if the density at x is denoted by $f(x)$, the function $Z(x) = f(x)/[1 - F(x)]$ will be referred to as the hazard rate.

2. The asymptotic equivalence of an estimate and a class of estimators for the reliability at a point of continuity of $F(x)$. Let

$$F_n(t) = (1/n)[\text{number of observations} \leq t \text{ among } X_1, X_2, \cdots X_n)$$

and

$$R_n(t) = (1/n)[\text{number of observations} > t \text{ among } X_1, X_2, \cdots X_n]. \tag{2.1}$$

Clearly $R_n(t)$ is a binomially distributed random variable with

$$\begin{aligned} E(R_n(t)) &= R(t). \\ \text{Var}\,(R_n(t)) &= (1/n)R(t)(1 - R(t)). \end{aligned} \tag{2.2}$$

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Let $K(x)$ be a function satisfying

(2.3) $K(x) \geqq 0, \quad K(-x) = K(x), \quad \lim_{|x|\to\infty} xK(x) = 0, \quad \int_{-\infty}^{\infty} K(x)\,dx = 1.$

A function $K(x)$ satisfying (2.3) is called a Window. (Murthy [2]). Let B_n be a sequence of non-negative constants depending on the sample size n such that $B_n \to \infty$ as $n \to \infty$.

Let

(2.4) $f_n(t) = \int_{-\infty}^{\infty} B_n K(B_n(x - t))\,dF_n(x) = (B_n/n) \sum_{j=1}^{n} K(B_n(X_j - t)).$

It was shown by the author [2] that the class of estimators $\{f_n(x_0)\}$ given by (2.4) consistently and asymptotically normally estimate the density $f(x_0)$ at every point of continuity x_0 of the distribution $F(x)$ and also of $f(x)$ if $\sum_i S_i/|x_i - x_0| < \infty$. We will now propose the class of estimators $\{R_n^*(t)\}$ for estimating the reliability function $R(t)$ where

(2.5) $R_n^*(t) = \int_t^{\infty} f_n(x)\,dx = (B_n/n) \sum_{j=1}^{n} \int_t^{\infty} K(B_n(X_j - x))\,dx.$

We will now prove that at a point of continuity t of the distribution $F(t)$

$$\lim_{n\to\infty} E(R_n^*(t)) = R(t), \tag{2.6}$$

and

$$\lim_{n\to\infty} [n \operatorname{Var}(R_n^*(t))] = R(t)(1 - R(t)). \tag{2.7}$$

Let

$$G(t) = \int_{-\infty}^{t} K(x)\,dx. \tag{2.8}$$

In terms of $G(t)$, $R_n^*(t)$ can be written as

$$R_n^*(t) = (1/n) \sum_{j=1}^{n} G(B_n(X_j - t)). \tag{2.9}$$

Taking expectation on both sides of (2.9) we obtain

$$\begin{aligned} E(R_n^*(t)) &= \int_{-\infty}^{\infty} G(B_n(x - t))\,dF(x) \\ &= 1 - \int_{-\infty}^{\infty} B_n K(B_n(x - t))F(x)\,dx. \end{aligned} \tag{2.10}$$

Now

(2.11) $\int_{-\infty}^{\infty} B_n K(B_n(x - t))F(x)\,dx = \int_{-\infty}^{\infty} K(\lambda)F(t + \lambda/B_n)\,d\lambda.$

If t is a point of continuity of the distribution $F(x)$, taking limit on both sides of (2.11) as $n \to \infty$ we have

(2.12) $\lim_{n\to\infty} \int_{-\infty}^{\infty} B_n K(B_n(x - t))F(x)\,dx = F(t) \int_{-\infty}^{\infty} K(\lambda)\,d\lambda = F(t).$

Combining (2.10) and (2.12) we have at a point of continuity t of the distribution $F(x)$ that

$$\lim_{n\to\infty} E(R_n^*(t)) = 1 - F(t) = R(t). \tag{2.13}$$

Taking the variance of the estimator $R_n^*(t)$ given by (2.9) we obtain

$$\text{(2.14)} \quad \mathrm{Var}\,(R_n^*(t)) = (1/n)\,\mathrm{Var}\,[G(B_n(x - t))]$$
$$= (1/n)[E(G^2(B_n(x - t))) - E^2(G(B_n(x - t)))].$$

Now

$$\text{(2.15)} \quad E(G^2(B_n(x - t))) = \int_{-\infty}^{\infty} G^2(B_n(x - t))\, dF(x)$$
$$= 1 - 2\int_{-\infty}^{\infty} G(B_n(x - t))B_nK(B_n(x - t))F(x)\, dx,$$

after integration by parts. Substituting $B_n(x - t) = \lambda$, (2.15) can be written as

$$\text{(2.16)} \quad E(G^2(B_n(x - t))) = 1 - 2\int_{-\infty}^{\infty} G(\lambda)K(\lambda)F(t + \lambda/B_n)\, d\lambda.$$

Taking limit as $n \to \infty$ on both sides of (2.16) we have at a point of continuity t of the distribution $F(x)$ that

$$\text{(2.17)} \quad \lim_{n\to\infty} E(G^2(B_n(x - t))) = 1 - 2F(t)\int_{-\infty}^{\infty} G(\lambda)K(\lambda)\, d\lambda$$
$$= 1 - F(t),$$

since $\int_{-\infty}^{\infty} G(\lambda)K(\lambda)\, d\lambda = \frac{1}{2}$. Combining (2.10), (2.13), (2.14) and (2.17) we discover that

$$\text{(2.18)} \quad \lim_{n\to\infty} [n\ \mathrm{Var}\ (R_n^*(t)] = R(t) - R^2(t) = R(t)(1 - R(t)),$$

at every point of continuity of the distribution $F(x)$. Also from (2.9) $R_n^*(t)$ can be written as $R_n^*(t) = (1/n)\sum_{j=1}^{n} V_j$, where $V_j = G(B_n(x_j - t))$ and the V_j's are independently and identically distributed as a random variable

$$\text{(2.19)} \quad y_n = G(B_n(X_j - t)).$$

A sufficient condition for the sequence $\{R_n^*(t)\}$ to be asymptotically normally distributed (see Parzen [3] p. 1069) is that for some $\delta > 0$

$$\text{(2.20)} \quad E|y_n - E(y_n)|^{2+\delta}/\{n^{\delta/2}[\mathrm{Var}\ (y_n)]^{1+\delta/2}\} \to 0\,; \qquad \text{as } n \to \infty.$$

For y_n given by (2.19) Condition (2.20) is easily verified by noting that both $\lim_{n\to\infty} E|y_n|^{2+\delta} < \infty$, and $\lim_{n\to\infty} \mathrm{Var}\ (y_n) < \infty$ at every point of continuity t of the distribution $F(x)$. Summing up we have proved the following

THEOREM 1. *The estimate $R_n(t)$ given by (2.1) and the class of estimators $\{R_n^*(t)\}$ given by (2.5) are both consistent estimates of $R(t)$ at every point of continuity t of the distribution $F(x)$. Further $R_n(t)$ and $\{R_n^*(t)\}$ are both asymptotically equivalent in the sense that they have the same order of consistency and the same asymptotic variance. The sequence $\{R_n^*(t)\}$ is asymptotically normal.*

3. Estimation of the jump S_i at the saltus x_i of the distribution $F(x)$. Assuming the singular part to be identically zero, the distribution $F(x)$ can be decomposed into (see e.g. Cramér [1] pp. 52, 53)

$$\text{(3.1)} \quad F(x) = F_1(x) + F_2(x),$$

where $F_1(x)$ is an everywhere continuous function and $F_2(x)$ is a pure step function with steps of magnitude, say, S_ν at the points $x = x_\nu$, $\nu = 1, 2, \cdots$ and $F_1(x)$ and $F_2(x)$ are non-decreasing and are uniquely determined. Substituting (3.1) in (2.10) we obtain

$$(3.2)\quad E(R_n^*(t)) = \int_{-\infty}^{\infty} G(B_n(x - t))\, dF_1(x) + \int_{-\infty}^{\infty} G(B_n(x - t))\, dF_2(x) = I_1 + I_2, \quad \text{say}.$$

Following the argument of the previous section, we readily obtain $\lim_{n\to\infty} I_1 = F_1(\infty) - F_1(t)$. Since $F_1(x)$ is continuous at $x = x_i$ we have

$$(3.3)\quad \lim_{n\to\infty} I_1 = F_1(\infty) - F_1(x_i),$$

at the saltus $x = x_i$ of the distribution $F(x)$.

Now

$$(3.4)\quad I_2 = \int_{-\infty}^{\infty} G(B_n(x - t))\, dF_2(x) = \sum_{\nu=1}^{\infty} S_\nu G(B_n(x_\nu - t)).$$

Denoting by $\sum_{x_\nu > x_i}$ summation over all ν such that $x_\nu > x_i$ and by $\sum_{x_\nu < x_i}$ summation over all ν such that $x_\nu < x_i$, at the saltus $t = x_i$ of the distribution $F(x)$, I_2 can be written as

$$I_2 = I_{21} + I_{22} + I_{23},$$

where $I_{21} = \sum_{x_\nu < x_i} S_\nu G(B_n(x_\nu - x_i))$, $I_{22} = S_i G(0) = \frac{1}{2} S_i$, and $I_{23} = \sum_{x_\nu > x_i} S_\nu G(B_n(x_\nu - x_i))$. Now

$$I_{21} = \Sigma_1 + \Sigma_2$$

where

$$\Sigma_1 = \sum_{x_\nu < x_i, |\nu| \leqq m} S_\nu G(B_n(x_\nu - x_i)),$$

and

$$\Sigma_2 = \sum_{x_\nu < x_i, |\nu| > m} S_\nu G(B_n(x_\nu - x_i)).$$

It can be argued as in the proof of the lemma (see Murthy [2]) that Σ_2 can be made arbitrarily small, by choosing m sufficiently large, (no matter what n is) and Σ_1, for fixed m, can be made arbitrarily small by choosing n sufficiently large, i.e. $\lim_{n\to\infty} I_{21} = 0$. From the fact that

$$I_{23} = \sum_{x_\nu > x_i} S_\nu - \sum_{x_\nu > x_i} S_\nu [1 - G(B_n(x_\nu - x_i))],$$

we discover $\lim_{n\to\infty} I_{23} = \sum_{x_\nu > x_i} S_\nu$. Of course, it should be noted (see Murthy [2]) that in proving the above statement it is assumed that $\sum_{\nu \neq i} S_\nu / |x_\nu - x_i| < \infty$. We have therefore proved that at the saltus $t = x_i$ of the distribution $F(x)$

$$(3.5)\quad \lim_{n\to\infty} I_2 = \tfrac{1}{2} S_i + \sum_{x_\nu > x_i} S_\nu.$$

Combining (3.2), (3.3) and (3.5) we obtain

(3.6) $\lim_{n\to\infty} E(R_n^*(x_i)) = F_1(\infty) - F_1(x_i) + \frac{1}{2}S_i + \sum_{x_\nu > x_i} S_\nu$,

at the saltus x_i of the distribution $F(x)$.

Now

$$F(x_i) = \int_{-\infty}^{x_i} d(F_1(x) + F_2(x)) = F_1(x_i) + \sum_{x_\nu \leqq x_i} S_\nu,$$

and therefore

(3.7) $$R(x_i) = 1 - F(x_i) = F_1(\infty) + F_2(\infty) - F_1(x_i) - \sum_{x_\nu \leqq x_i} S_\nu$$
$$= F_1(\infty) - F_1(x_i) + \sum_{x_\nu > x_i} S_\nu.$$

Substituting from (3.7) in (3.6) we discover that

(3.8) $$\lim_{n\to\infty} E(R_n^*(x_i)) = R(x_i) + \tfrac{1}{2}S_i.$$

From (2.2) we have that

(3.9) $$E(R_n(x_i)) = R(x_i).$$

Let us write

(3.10) $$H_n(x_i) = 2[R_n^*(x_i) - R_n(x_i)].$$

In view of (3.8) and (3.9) we obtain

(3.11) $$\lim_{n\to\infty} E(H_n(x_i)) = S_i,$$

at the saltus x_i of the distribution $F(x)$.

4. Variance of the estimator $H_n(x_i)$. We have

(4.1) $$\mathrm{Var}\,(H_n(x_i)) = 4[\mathrm{Var}\,(R_n^*(x_i)) + \mathrm{Var}\,(R_n(x_i)) - 2\,\mathrm{cov}\,(R_n^*(x_i), R_n(x_i))].$$

Since we already know Var $(R_n(x_i))$ as given by (2.2), we only have to obtain Var $(R_n^*(x_i))$ and cov $(R_n^*(x_i), R_n(x_i))$ at the saltus $x = x_i$ of the distribution $F(x)$. We have from (2.14) that

(4.2) $$n\,\mathrm{Var}\,(R_n^*(t)) = E(G^2(B_n(x - t))) - E^2(G(B_n(x - t))).$$

Now

$$E(G^2(B_n(x - t))) = \int_{-\infty}^{\infty} G^2(B_n(x - t))\,dF_1(x) + \int_{-\infty}^{\infty} G^2(B_n(x - t))\,dF_2(x)$$
$$= J_1 + J_2, \quad \text{say}.$$

It is easily seen that

(4.3) $$\lim_{n\to\infty} J_1 = F_1(\infty) - F_1(x_i),$$
$$\lim_{n\to\infty} J_2 = \tfrac{1}{4}S_i + \sum_{x_\nu > x_i} S_\nu,$$

at the saltus $t = x_i$ of $F(t)$. Therefore

(4.4) $\lim_{n\to\infty} E[G^2(B_n(x - x_i))] = F_1(\infty) - F_1(x_i) + \frac{1}{4}S_i + \sum_{x_\nu > x_i} S_\nu$
$= R(x_i) + \frac{1}{4}S_i$.

Combining (3.8), (4.2) and (4.4) we obtain

(4.5) $\lim_{n\to\infty} [n \text{ Var } (R_n^*(x_i))] = R(x_i) + \frac{1}{4}S_i - (R(x_i) + \frac{1}{2}S_i)^2$,

at the saltus $t = x_i$ of the distribution $F(t)$.

To find the co-variance between $R_n(t)$ and $R_n^*(t)$ let us recall

(4.6) $$R_n^*(t) = (1/n) \sum_{j=1}^{n} G(B_n(X_j - t)),$$

and

(4.7) $R_n(t) = (1/n)$[number of observations $> t$ among $X_1, X_2, \cdots X_n$]
$= (1/n) \sum_{j=1}^{n} U(X_j - t)$,

where

$$U(x) = 1 \quad \text{for } x > 0$$
$$= 0 \quad \text{for } x \leqq 0.$$

Now

(4.8) $\text{cov } [R_n^*(t), R_n(t)] = (1/n^2) \sum_{j=1}^{n} \text{cov } [G(B_n(x_j - t)), U(X_j - t)]$
$= (1/n) \text{ cov } [G(B_n(x - t)), U(x - t)]$

we have

(4.9) $$\text{cov } [G(B_n(x - t)), U(x - t)] = M_1 - M_2,$$

where

(4.10) $M_1 = \int_{-\infty}^{\infty} U(x - t)G(B_n(x - t))\, d(F_1(x) + F_2(x)) = M_{11} + M_{12}$, say,

and

(4.11) $$M_2 = E(U(x - t))E(G(B_n(x - t))).$$

It can be easily verified that

(4.12) $\lim_{n\to\infty} M_{11} = \lim_{n\to\infty} \int_{-\infty}^{\infty} U(x - t)G(B_n(x - t))\, dF_1(x)$
$= F_1(\infty) - F_1(x_i)$

at the saltus $t = x_i$. Also

$$M_{12} = \int_{-\infty}^{\infty} U(x - t)G(B_n(x - t))\, dF_2(x) = \sum_{x_\nu > x_i} S_\nu G(B_n(x_\nu - x_i)).$$

Hence

(4.13) $$\lim_{n\to\infty} M_{12} = \sum_{x_\nu > x_i} S_\nu,$$

at the saltus $t = x_i$. Summing up

$$\lim_{n\to\infty} M_1 = F_1(\infty) - F_1(x_i) + \sum_{x_\nu > x_i} S_\nu = R(x_i), \tag{4.14}$$

at the saltus $t = x_i$. We have

$$\begin{aligned} E(U(x - x_i)) &= \int_{-\infty}^{\infty} U(x - x_i)\, d(F_1(x) + F_2(x)) \\ &= \int_{x_i}^{\infty} dF_1(x) + \sum_{x_\nu > x_i} S_\nu \\ &= F_1(\infty) - F_1(x_i) + \sum_{x_\nu > x_i} S_\nu = R(x_i). \end{aligned} \tag{4.15}$$

Combining (3.8), (4.11) and (4.15) we have

$$\lim_{n\to\infty} M_2 = R(x_i)(R(x_i) + \tfrac{1}{2}S_i), \tag{4.16}$$

at the saltus $t = x_i$. Combining (4.8), (4.14) and (4.16) we discover that

$$\lim_{n\to\infty} \operatorname{cov}[G(B_n(x - t), U(x - t))] = R(x_i)(1 - R(x_i) - \tfrac{1}{2}S_i), \tag{4.17}$$

at the saltus $t = x_i$. Taking (4.1), (2.2), (4.5) and (4.17) we finally obtain

$$\begin{aligned} \lim_{n\to\infty} [n \operatorname{Var}(H_n(x_i))] &= 4[R(x_i) + \tfrac{1}{4}S_i - R^2(x_i) - \tfrac{1}{4}S_i^2 - S_iR(x_i) \\ &\quad + R(x_i) - R^2(x_i) - 2R(x_i) + 2R^2(x_i) \\ &\quad + S_iR(x_i)] \\ &= S_i(1 - S_i), \end{aligned} \tag{4.18}$$

at the saltus $t = x_i$.

Writing the estimator $H_n(x_i)$ as

$$H_n(x_i) = (1/n)\sum_{j=1}^{n} \xi_j,$$

where $\xi_j = 2[G(B_n(X_j - x_i)) - U(X_j - x_i)]$, one can easily verify that the sufficient condition for asymptotic normality given by (2.20) is satisfied by the sequence $\{\xi_j\}$ of independently and identically distributed random variables. We have therefore proved

THEOREM 2. *The class of estimators $\{H_n(x_i)\}$ are consistent and asymptotically normal for estimating the jump S_i corresponding to the saltus $x = x_i$ of the distribution $F(x)$.*

Consider now the estimator $f_n^*(x_i)$ where

$$f_n^*(x_i) = (1/B_n)f_n(x_i)$$

and $f_n(x_i)$ is given by (2.4) at the saltus $t = x_i$. Since

$$f_n^*(x_i) = (1/n)\sum_{j=1}^{n} K(B_n(X_j - x_i)). \tag{4.19}$$

a straight forward calculation yields that

$$\begin{aligned} \lim_{n\to\infty} E[f_n^*(x_i)] &= K(0)S_i, \\ \lim_{n\to\infty} [n \operatorname{Var}(f_n^*(x_i))] &= K^2(0)S_i(1 - S_i) \end{aligned} \tag{4.20}$$

at the saltus $x = x_i$ of the distribution $F(x)$ where the derivative of the absolutely continuous part $f(x)$ is assumed continuous and finally the estimate $f_n^*(x_i)$ is asymptotically normal. Thus the estimators $H_n(x_i)$ and $[1/K(0)]f_n^*(x_i)$ are asymptotically equivalent for estimating the jump S_i at the saltus $x = x_i$ of the distribution $F(x)$.

5. Estimation of hazard rate. The function $Z(t)$ will be called the hazard rate where

$$Z(t) = f(t)/[1 - F(t)] = f(t)/R(t). \tag{5.1}$$

Let us now propose $Z_n(t)$ as an estimate of the hazard rate $Z(t)$ where

$$Z_n(t) = f_n(t)/R_n(t), \tag{5.2}$$

$f_n(t)$ and $R_n(t)$ being respectively given by (2.4) and (2.1). It was earlier shown by the author (Murthy [2]) that $f_n(t)$ is a consistent estimate of $f(t)$ at every point of continuity t of $F(t)$ and $f(t)$, i.e.

$$\text{Plim}_{n\to\infty} f_n(t) = f(t). \tag{5.3}$$

It follows from (2.2) that

$$\text{Plim}_{n\to\infty} R_n(t) = R(t). \tag{5.4}$$

Combining (5.3) and (5.4) and using a well known convergence theorem (see Cramér [1], p. 254) we at once have

$$\text{Plim}_{n\to\infty} Z_n(t) = f(t)/R(t) = Z(t), \tag{5.5}$$

in other words $Z_n(t)$ is a consistent estimate of the hazard rate $Z(t)$.

It was also shown by the author [2] that at every continuity point $x = t$ of $F(x)$ and $f(x)$

$$\lim_{n\to\infty} P\{(n/B_n)^{\frac{1}{2}}[(f_n(t) - f(t))/(f(t) \textstyle\int_{-\infty}^{\infty} K^2(x)\,dx)^{\frac{1}{2}}] < x\} = (2\pi)^{-\frac{1}{2}} \textstyle\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy. \tag{5.6}$$

Combining (5.4) and (5.6) and using a well known convergence theorem (see Cramér [1], p. 254) we discover

$$\lim_{n\to\infty} P\left\{\left(\frac{n}{B_n}\right)^{\frac{1}{2}} \frac{\dfrac{f_n(t)}{R_n(t)} - \dfrac{f(t)}{R_n(t)}}{\left(\dfrac{f(t)}{R^2(t)} \displaystyle\int_{-\infty}^{\infty} K^2(x)\,dx\right)^{\frac{1}{2}}} < x\right\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy. \tag{5.7}$$

Consider now

$$y_n = (n/B_n)^{\frac{1}{2}}(R_n(t) - R(t)). \tag{5.8}$$

We have in view of (2.2) that

$$\begin{aligned} E(y_n) &= 0 \\ \text{Var}\,(y_n) &= (1/B_n)R(t)(1 - R(t)), \end{aligned} \tag{5.9}$$

and hence

$$\text{plim}_{n\to\infty}\, y_n = 0. \tag{5.10}$$

Combining (5.7) and (5.10) and using the convengence theorem (see Cramér [1], p. 254) again we finally obtain

$$\lim_{n\to\infty} P[(n/B_n)^{\frac{1}{2}}\{(Z_n(t) - Z(t))/[(Z(t)/R(t)) \textstyle\int_{-\infty}^{\infty} K^2(x)\,dx]^{\frac{1}{2}}\} < x] = (2\pi)^{-\frac{1}{2}} \textstyle\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy. \tag{5.11}$$

We have therefore proved the following

THEOREM 3. *The class of estimators $Z_n(t)$ given by (5.2) for estimating the hazard rate $Z(t)$ is asymptotically normally distributed at every point of continuity $x = t$ of the distribution $F(x)$* and the density $f(x)$.

It may be observed that the estimator $Z_n^*(t)$ for estimating $Z(t)$ where

$$Z_n^*(t) = f_n(t)/R_n^*(t),$$

is consistent and asymptotically normal, the proof being exactly similar to the one given for $Z_n(t)$.

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