RENEWAL THEORY IN THE PLANE

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1. Summary. This paper presents some generalizations of the elementary renewal theorem (Feller [9]) and the deeper renewal theorem of Blackwell [1], [2] to planar walks.

Let $U[A]$ denote the expected number of visits of a transient 2-dimensional nonarithmetic random walk to a Borel set A in R^2. Let $S(\mathbf{y}, a)$ denote the sphere of radius a about the point $\mathbf{y}$ for a given norm $\|\cdot\|$ of the Euclidean topology. Then, the elementary renewal theoem for the plane, given in Section 2, states that $\lim_{a\to\infty} U[S(\mathbf{0}, a)]/a = 1/\|E[\mathbf{X}_1]\|$, where $\mathbf{X}_1 = (X_{11}, X_{21})$ is the first step of the walk, if $E[\mathbf{X}_1]$ exists. Farrell has obtained similar results for nonnegative walks in [7].

Section 4 contains the main result of the paper, the generalization of the Blackwell renewal theorem in the case of polygonal norms for random walks which have both $E[X_{11}^2]$ and $E[X_{21}^2]$ finite and one of $E[X_{11}]$, $E[X_{21}]$ different from 0.

The theorem states that

$$\lim_{a\to\infty} \{U[S(\mathbf{0}, a + \Delta,)] - U[S(\mathbf{0}, a)]\} = \Delta/\|E[\mathbf{X}_1]\|$$

for every $\Delta \geqq 0$ and $\|\cdot\|$ specified above.

This result is also established with no restrictions on $E[X_{11}^2]$, $E[X_{21}^2]$ under different regularity conditions, in particular, for the L_∞ norm if both $E[X_{11}]$ and $E[X_{21}]$ are different from 0, and correspondingly for the L_1 norm if $|E[X_{11}]| \neq |E[X_{21}]|$.

Farrell in [8] has obtained more general results for nonnegative walks under somewhat more restrictive regularity conditions and by a different method.

The next section gives the Blackwell theorem for totally symmetric transient walks with finite step expectations, both of whose marginal walks are recurrent.

We conclude with a discussion of extensions of these results to higher dimensions and some open questions.

2. Introduction and elementary renewal theorem. On a triple $(\Omega, \mathfrak{A}, P)$ let $\mathbf{X}_i = (X_{1i}, X_{2i})$, $i \geqq 1$, be a sequence of independent identically distributed random vectors with $E[\mathbf{X}_1]$ finite. Then if $\mathbf{S}_n = \sum_{i=1}^n \mathbf{X}_i = (S_n^{(1)}, S_n^{(2)})$, is the process of sums, suppose that

$$\sum_{k=1}^\infty P[\mathbf{S}_n \varepsilon K] < \infty \tag{1}$$

for any compact K. In other words the plane walk is transient.

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We define $Z(A)$ = number of times $\mathbf{S}_n \varepsilon A$, where A is a Borel set in R^2. Then Z is clearly measurable and following Section 1

$$U[A] = E[Z(A)] = \sum_n P[\mathbf{S}_n \varepsilon A]. \tag{2}$$

Then continuing the notation of Section 1 we can now prove the elementary renewal theorem. The method of proof is essentially that of Doob [4].

THEOREM 1. *If* $\|\cdot\|$ *is any norm generating the Euclidean topology, then* $\lim_{a\to\infty} U[S(\mathbf{0}, a)]/a = 1/\|E[\mathbf{X}_1]\|$, *where the right side equals* ∞ *if* $E[\mathbf{X}_1] = \mathbf{0}$.

PROOF. By the strong law of large numbers

$$P[\mathbf{S}_n/n \,\varepsilon\, S[E(\mathbf{X}_1), \epsilon] \text{ eventually}] = 1. \tag{3}$$

Suppose first that $E[\mathbf{X}_1] \neq \mathbf{0}$. Now, since $|\|\mathbf{S}_n\| - n\|E[\mathbf{X}_1]\|| \leqq \|\mathbf{S}_n - nE[\mathbf{X}_1]\|$ it follows that

$$P[n(\|E[\mathbf{X}_1]\| - \epsilon) \leqq \|\mathbf{S}_n\| \leqq n(\|E[\mathbf{X}_1]\| + \epsilon), \text{ eventually}] = 1. \tag{4}$$

From this one may readily derive the inequality

$$\limsup_m n^{-1} Z\{S[\mathbf{0}, m(\|E[\mathbf{X}_1]\| + \epsilon)]\} \leqq 1 \text{ a.s.}$$

where $n = [(\|E[\mathbf{X}_1]\| + \epsilon)(\|E[\mathbf{X}_1]\| - \epsilon)^{-1}m] + 1$. As usual, $[x]$ denotes the greatest integer in x. Similarly we obtain, $\liminf_m q^{-1}Z\{S[\mathbf{0}, m(\|E[\mathbf{X}_1]\| + \epsilon)]\} \geqq 1$ a.s., where $q = [(\|E[\mathbf{X}_1]\| - \epsilon)(\|E[\mathbf{X}_1]\| + \epsilon)^{-1}m]$. Noting that

$$Z\{S[\mathbf{0}, m(\|E[\mathbf{X}_1]\| - \epsilon)]\} \leqq Z[S(\mathbf{0}, m\|E[\mathbf{X}_1]\|)] \leqq Z\{S[\mathbf{0}, m(\|E[\mathbf{X}_1]\| + \epsilon)]\},$$

we obtain

$$\text{a.s.} \quad \lim_m Z[S(\mathbf{0}, m\|E[\mathbf{X}_1)\|)]/m\|E[\mathbf{X}_1]\| = 1/\|E[\mathbf{X}_1]\| \tag{5}$$

from which it readily follows that a.s. $\lim_{a\to\infty} Z[S(\mathbf{0}, a)]/a = 1/\|E[\mathbf{X}_1]\|$.

For any norm as above there exists a square centered at the origin containing the unit sphere for that norm. Then, if $\hat{Z}_a$ denotes the number of visits to a square of side a, there exists a B such that $Z[S(\mathbf{0}, a)] \leqq \hat{Z}_{aB}$ for every a. But Chung [3] has shown that if $E[\mathbf{X}_1] \neq \mathbf{0}$ then $\sup_{|a|\geqq\epsilon} E[\hat{Z}_a^{\,4}]/a^4 < \infty$ for $\epsilon > 0$, and hence so is $\sup_{|a|>\epsilon} E[Z(S(\mathbf{0}, a))]^4/a^4$. The theorem follows if $E[\mathbf{X}_1] \neq \mathbf{0}$.

If $E[\mathbf{X}_1] = \mathbf{0}$, then by arguments similar to the above we obtain

$$\text{a.s.} \quad \lim_{a\to\infty} Z[S(\mathbf{0}, a)]/a = \infty.$$

But then by Fatou's lemma $\lim_a U[S(\mathbf{0}, a)]/a = \infty$. The theorem follows.

In particular, if $\|\mathbf{x}\| = [\sum_{i=1}^{2} x_i^2]^{\frac{1}{2}}$, where $\mathbf{x} = (x_1, x_2)$, the theorem gives the order of the expected number of visits to circles centered at the origin. If $\|\mathbf{x}\| = \max(|x_1|, |x_2|)$, the L_∞ norm, we find the order of the expected number of visits to squares centered at the origin with sides parallel to the axes to be $\frac{1}{2}$ the reciprocal of the maximum projection of $E[\mathbf{X}_1] \times$ the side length. A similar remark holds for the rhombuses defined by the inequality $|x_1| + |x_2| < C$ upon considering the L_1 norm, $\|\mathbf{x}\| = |x_1| + |x_2|$. The above theorem holds trivially for recurrent walks, i.e., those for which the expected number of visits to some closed sphere is infinite.

The class of sets treated of course, coincides with the class of symmetric (about $\mathbf{0}$) convex sets.

3. A useful theorem. In this section we will prove a theorem which plays a substantial role in the proof of the Blackwell type theorem. Let the line segment which starts at the origin, passes through $E[\mathbf{X}_1]$ and continues indefinitely, be called the line of expectation. Let C be any cone containing the line of expectation in its interior. Let $\{V_a\}$ be a sequence of sets such that $d(\mathbf{0}, V_a) \to \infty$ as $a \to \infty$, where $d(\mathbf{x}, B)$ is the infimum Euclidean distance between $\mathbf{x}$ and the elements of B. Then we have

THEOREM 2. (a) *If* $E[X_{11}^2] + E[X_{21}^2] < \infty$ *and* $E[\mathbf{X}_1] \neq \mathbf{0}$ *then* $U(V_a \cap C^c) \to 0$ *as* $a \to \infty$.

(b) *Let* $L(\Delta, r, \alpha)$ *denote an infinite strip of width* Δ *at distance* r *from the origin making an angle* α *with the* x *axis. Suppose* $E[\mathbf{X}_1] \neq \mathbf{0}$ *and suppose* $L(\Delta, r, \alpha)$ *intersects the line of expectation. Then* $U[L(\Delta, r, \alpha) \cap C^c] \to 0$ *as* $r \to \infty$.

PROOF. (a) The condition given implies that

$$(6) \qquad P[\mathbf{S}_k \,\varepsilon\, V_a \cap C^c] \to 0 \quad \text{as} \quad a \to \infty \quad \text{for every } k.$$

Without loss of generality assume $E[X_{21}] \neq 0$ and take $C = \{(x, y): E[X_{11}]/E[X_{21}] - \epsilon < x/y < E[X_{11}]/E[X_{21}] + \epsilon\}$. Now choose δ so that $|x - E[X_{11}]| < \delta$ and $|y - E[X_{21}]| < \delta \Rightarrow |x/y - E[X_{11}]/E[X_{21}]| < \epsilon$. Then

$$(7) \quad P[\mathbf{S}_k \,\varepsilon\, C^c] = P[|(S_k^{(1)}/S_k^{(2)}) - (E[X_{11}]/E[X_{21}])| \geqq \epsilon]$$
$$\leqq P[|(S_k^{(1)}/k) - E[X_{11}]| \geqq \delta] + P[|(S_k^{(2)}/k) - E[X_{21}]| \geqq \delta].$$

Since $\sum_{k=1}^{\infty} P[|S_k^{(i)} - kE[X_{i1}]| \geqq k\delta] < \infty$ for $i = 1, 2$, by a fundamental theorem of Erdös [5], (6) and (7) establish Part (a) of the theorem.

PROOF. (b) Let $Z_r^* =$ number of visits of $\mathbf{S}_n$ to $L(\Delta, r, \alpha) \cap C^c$. We require

LEMMA 2.1. $\lim_{r\to\infty} P[Z_r^* \geqq 1] = 0$.

PROOF. $P[Z_r^* \geqq 1] \leqq P[\mathbf{S}_n \,\varepsilon\, L(\Delta, r, \alpha), \mathbf{S}_n \,\varepsilon\, C^c$, for some $n]$. Now choose C and δ as before. Then

$$P[\mathbf{S}_n \,\varepsilon\, L(\Delta, r, \alpha), \mathbf{S}_n \,\varepsilon\, C^c, \text{ for some } n]$$
$$(8) \qquad \leqq P[\mathbf{S}_n \,\varepsilon\, L(\Delta, r, \alpha), |S_n^{(1)}/n - E[X_{11}]| \geqq \delta \text{ for some } n]$$
$$+ P[\mathbf{S}_n \,\varepsilon\, L(\Delta, r, \alpha), |S_n^{(2)}/n - E[X_{21}]| \geqq \delta \text{ for some } n].$$

Now each of the probabilities on the right of (8) converges to 0 as $r \to \infty$ by the strong law of large numbers and the lemma follows.

We proceed to prove the theorem. Let $M_r = L(\Delta, r, \alpha) \cap C^c$. Since $U[M_r] = E[Z_r^*] = \sum_{n=1}^{\infty} P[Z_r^* \geqq n]$ it suffices to show upon applying Lemma 2.1 and the dominated convergence theorem, that there exists a sequence $\{a_n\}$, $\sum_n a_n < \infty$ such that $P[Z_r^* \geqq n] \leqq a_n$ for all r. Define $\hat{r}(r)$ to be the first n such that

$\mathbf{S}_n \varepsilon M_r$ and $\mathbf{S}_{\hat{t}(r)}$ to be the composite random variable. Then,

$$P[Z_r^* \geqq n] = \int_{\mathbf{x}\varepsilon M_r} P[Z\{M_r - \mathbf{x}\} \geqq n - 1]\, dP[\mathbf{S}_{\hat{t}(r)} \leqq \mathbf{x}]$$

by the strong Markov property.

Let $L(\Delta, \alpha)$ denote the infinite strip of width Δ making angle α with the x axis and centered at the origin. By the formula given above we have $P[Z_r^* \geqq n] \leqq P[Z\{L(2\Delta, \alpha)\} \geqq n - 1]$ for $n \geqq 2$ and $P[Z_r^* \geqq 1] \leqq 1$. Let T denote the rotation that maps $L(w, \alpha)$ onto $L(w, \pi/2)$. Let $T_1(\mathbf{x})$ be the first coordinate of $T(\mathbf{x})$. Then the hypothesis that the line of expectation intersects $L(\Delta, r, \alpha)$ implies that $T_1(E[\mathbf{X}_1]) \neq 0$ which in turn implies that $T_1(\mathbf{S}_n)$ is a transient walk. Therefore $\sum_n P[\{L(2\Delta, \alpha)\} \geqq n] < \infty$ which yields the theorem.

4. The Blackwell theorem. In [2] Blackwell proved that, for one-dimensional transient nonarithmetic walks with $E[X_1] > 0$, $\lim_{a\to\infty} U[(a, a+ \Delta)] = \Delta/E[X_1]$. We employ this result heavily in the following generalization. Call a norm whose unit sphere is a polygon a polygonal norm.

THEOREM 3. *Suppose* $E[\mathbf{X}_1] \neq \mathbf{0}$. *Let* $\|\cdot\|$ *be any polygonal norm and suppose the union of the compact supports of the* $\|\mathbf{S}_n\|$ *is dense in* (b, ∞) *for some* b.

I. *If* $E[X_{11}^2] + E[X_{21}^2] < \infty$, *then*

$$\lim_{a\to\infty} U[S(\mathbf{0}, a + \Delta) \sim S(\mathbf{0}, a)] = \Delta/\|E[\mathbf{X}_1]\|.$$

II. *If* $S(\mathbf{0}, a)$ *does not have any face parallel to the line of expectation, then*

$$\lim_{a\to\infty} U[S(\mathbf{0}, a + \Delta) \sim S(\mathbf{0}, a)] = \Delta/\|E[X_1]\|.$$

PROOF. I and II will be proved simultaneously, but the proof will be separated into two special cases.

Case (i). The line of expectation crosses $S(\mathbf{0}, a)$ through one of its faces. For this case, by Theorem 2 of Section 3 it is enough to find the limit as $a \to \infty$ of the expected number of visits to the infinite strip of width Δ which is the extension of $\{S(\mathbf{0}, a + \Delta) \sim S(\mathbf{0}, a)\}$ at the face which intersects the line of expectation. Let $L(\Delta, r, \alpha)$ denote that strip and let T be the rotation which maps $L(w, r, \alpha)$ onto $L(w, r, \pi/2)$. Then we have $T_1(E[\mathbf{X}_1]) = \|E[\mathbf{X}_1]\|$ where $T_1(\mathbf{x})$ is the first coordinate of T. By the Blackwell theorem for the univariate case we have

$$\lim_{r\to\infty} U[L(\Delta, r, \alpha)] = \Delta/T_1(E[\mathbf{X}_1]). \tag{9}$$

This completes the proof of Case (i).

Case (ii). The line of expectation crosses $S(\mathbf{0}, a)$ at one of its vertices.

By Theorem 2 of Section 3 we know that the only faces of $S(\mathbf{0}, a)$ that are of interest to us are the two faces which are adjacent to the vertex on the line of expectation. If $\mathbf{b}$ is a unit vector parallel to one face and $\mathbf{c}$ is a unit vector parallel to the other face then one can find a nonsingular linear transformation which maps $\mathbf{b}$ into $(0, 1)$ and $\mathbf{c}$ into $(1, 0)$. In the new coordinate system the vertex of the transformed $S(\mathbf{0}, a)$ at the transformed line of expectation will have a 90°

angle. So by the above nonsingular transformation, then by rotation and change of scale and by using Theorem 2 one can reduce the general Case (ii) to the following problem.

Let $E[X_{11}] = E[X_{21}] = 1$, $K(a) = \{\mathbf{x}: \mathbf{x} < \mathbf{a}\}$. Find $\lim_{a\to\infty} U[K(a + \Delta) \sim K(a)]$. Let $A_1(a, b, \Delta)$ denote the horizontal semi-infinite strip of width Δ whose initial points are (a, b) and $(a, b + \Delta)$ which is infinite in the positive direction. Let $A_2(a, b, \Delta)$ denote the vertical semi-infinite strip whose initial points are (a, b) and $(a + \Delta, b)$ which is infinite in the positive direction. By the Blackwell theorem for the univariate case and by Chung [3] we know that $U[K(a + \Delta) \sim K(a)] + U[A_1(a, a, \Delta)] + U[A_2(a, a, \Delta)] \to 2\Delta$ as $a \to \infty$. Therefore it suffices to show that $U[A_1(a, a, \Delta)] + U[A_2(a, a, \Delta)] \to \Delta$ as $a \to \infty$.

Let $\tau(a)$ denote the first n for which $\mathbf{S}_n \varepsilon [K(a)]^c$ and let $\mathbf{S}_{\tau(a)}$ denote the location of the walk at that time. We employ $U[B \mid \cdot]$ to denote the expected number of visits of $\mathbf{S}_n$ to a set B given the conditions $(\,\cdot\,)$, i.e.,

$$U(B \mid \cdot\,) = \textstyle\sum_n P[\mathbf{S}_n \varepsilon B \mid \cdot\,].$$

Finally let C denote a cone of fixed acute angle less than 90° about the 45° line through the origin. We can now state

LEMMA 3.1. *For every $\epsilon > 0$, there exists a $B_1(\epsilon)$ such that* $|U[A_1(a, a, \Delta)|S_{\tau(a)}^{(2)} \leqq a - B_1, S_{\tau(a)}^{(1)} > a] - \Delta\,| < \epsilon$ *for all a sufficiently large.*

PROOF. For simplicity denote the condition $[S_{\tau(a)}^{(2)} \leqq a - B, S_{\tau(a)}^{(1)} > a]$ by $*$. We note that for B sufficiently large

$$|U[L(a, \Delta, \pi)| *] - \Delta| < \epsilon/2 \text{ for all } a, \tag{10}$$

where $L(a, \Delta, \pi)$ is the full strip obtained by extending $A_1(a, a, \Delta)$ indefinitely. This is possible by applying the strong Markov property and Blackwell's theorem to the walk $S_n^{(2)}$.

Now, $U[L(a, \Delta, \pi) \sim A_1(a, a, \Delta) \mid *] = \int U[L(a - y, \Delta, \pi) \sim A_1(a - x, a - y, \Delta)]\, dP[\mathbf{S}_{\tau(a)} \leqq x \mid *]$ and $U[L(a - y, \Delta, \pi) \sim A_1(a - x, a - y, \Delta)] = U\{[L(a - y, \Delta, \pi) \sim A_1(a - x, a - y, \Delta) \cap C^c]\}$ for $x > a$, $y \leqq a - B$. The last term converges to 0 as $B \to \infty$ uniformly in a, by an argument similar to that used to establish Theorem 2.

Now, choose $B = B_1$ sufficiently large to satisfy (10) and $U[L_1(a, \Delta) \sim A_1(a, a, \Delta) \mid *] < \epsilon/2$. The lemma now follows.

We may similarly establish that for suitable $B_2(\epsilon)$, $|U[A_2(a, a, \Delta) \mid S_{\tau(a)}^{(1)} \leqq a - B_2, S_{\tau(a)}^{(2)} > a] - \Delta| < \epsilon$ for a sufficiently large.

Together these statements are equivalent to

$$\lim_{B\to\infty} \lim\,(\sup, \inf)_{a\to\infty} U[A_1(a, a, \Delta) \mid S_{\tau(a)}^{(2)} \leqq a - B, \cdot\, S_{\tau(a)}^{(1)} > a] = \Delta, \tag{11}$$

$$\lim_{B\to\infty} \lim\,(\sup, \inf)_{a\to\infty} U[A_2(a, a, \Delta) \mid S_{\tau(a)}^{(1)} \leqq a - B, \quad S_{\tau(a)}^{(2)} > a] = \Delta, \tag{12}$$

where the inner limit may be either lim inf or lim sup. Now,

$$\begin{aligned} &U[A_1(a, a, \Delta)] + U[A_2(a, a, \Delta)] \\ &\quad = U[A_1(a, a, \Delta) \mid S_{\tau(a)}^{(2)} \leqq a - B, S_{\tau(a)}^{(1)} > a]\cdot P[S_{\tau(a)}^{(2)} \leqq a - B, S_{\tau(a)}^{(1)} > a] \end{aligned}$$

$$+ U[A_1(a, a, \Delta) \mid S^{(1)}_{\tau(a)} \leqq a - B, S^{(2)}_{\tau(a)} > a] \cdot P[S^{(1)}_{\tau(a)} \leqq a - B, S^{(2)}_{\tau(a)} > a]$$
$$+ U[A_2(a, a, \Delta) \mid S^{(2)}_{\tau(a)} \leqq a - B, S^{(1)}_{\tau(a)} > a] P[S^{(2)}_{\tau(a)} \leqq a - B, S^{(1)}_{\tau(a)} > a]$$
$$+ U[A_2(a, a, \Delta) \mid S^{(1)}_{\tau(a)} \leqq a - B, S^{(2)}_{\tau(a)} > a] P[S^{(1)}_{\tau(a)} \leqq a - B, S^{(2)}_{\tau(a)} > a]$$
$$+ U[A_1(a, a, \Delta) \mid \textstyle\bigcap_{1 \leqq i,j \leqq 2} [S^{(i)}_{\tau(a)} \leqq a - B, S^{(j)}_{\tau(a)} > a]^c]$$
$$\cdot P[\textstyle\bigcap_{1 \leqq i,j \leqq 2} [S^{(i)}_{\tau(a)} \leqq a - B, S^{(j)}_{\tau(a)} > a]].$$

Upon consideration of this sum it follows that to establish the theorem, it suffices to show that

$$(13) \quad \liminf_{B\to\infty} \liminf_{a\to\infty} \{P[S^{(1)}_{\tau(a)} \leqq a - B, S^{(2)}_{\tau(a)} > a] + P[S^{(2)}_{\tau(a)} \leqq a - B, S^{(1)}_{\tau(a)} > a]\} = 1$$

and, for $i = 1, 2$,

$$(14) \quad \lim_{B\to\infty} \lim (\sup, \inf)_{a\to\infty} U[A_i(a, a, \Delta) \mid S^{(i)}_{\tau(a)} \leqq a - B, S^{(j)}_{\tau(a)} > a, i \neq j] = 0.$$

Equation (14) follows by an argument analogous to that used to prove (10) and Theorem 3. Proving (13) reduces to showing

$$(15) \quad \limsup_{a\to\infty} \{P[S^{(2)}_{\tau(a)} \leqq -a] + P[S^{(1)}_{\tau(a)} \leqq -a]) = 0,$$

$$(16) \quad \limsup_{B\to\infty} \limsup_{a\to\infty} P[a - B < S^{(1)}_{\tau(a)} < a + B, a - B < S^{(2)}_{\tau(a)} < a + B] = 0,$$

$$(17) \quad \limsup_{B\to\infty} \limsup_{a\to\infty} \{P[a - B < S^{(1)}_{\tau(a)} < a + B, S^{(2)}_{\tau(a)} \geqq a + B] + P[a - B < S^{(2)}_{\tau(a)} < a + B, S^{(1)}_{\tau(a)} \geqq a + B]\} = 0.$$

Equation (16) follows as a consequence of the main theorem of Chung [3]. Equation (15) follows as a consequence of Blackwell's theorem for the line. To prove (17) we let, following Blackwell [2], $t_1, t_2, \cdots$ be those values of n at which the successive maxima of $S_n^{(1)}$ are reached. Let $Z_i = S^{(1)}_{t_{i+1}} - S^{(1)}_{t_i}$ be the "ladder point" random variables. The Z_i are nonnegative, independent, identically distributed, and have finite expectation (Blackwell [2]). Let $S_n^* = S^{(1)}_{t_n} = \sum_{i=1}^{n} Z_i$. Then $P[S^{(1)}_{\tau(a)} - a \geqq B] \leqq P[S^*_{\tau(a)} - S^*_{\tau(a)-1} \geqq B] \to [E[Z_1]]^{-1} \cdot E[Z_1 I^+(Z_1 - B)]$ as $a \to \infty$, by the key renewal theorem (Smith [10]), where $I^+(x) = 1$ if $x \geqq 0$ and 0 otherwise. Hence $\lim_{B\to\infty} \limsup_{a\to\infty} P[a - B < S^{(2)}_{\tau(a)} < a + B, S^{(1)}_{\tau(a)} \geqq a + B] = 0$.

A similar argument for the second probability involved in (13) now establishes the theorem.

5. Blackwell's theorem for totally symmetric transient walks with recurrent marginal walks. It was pointed out to the authors by D. Freedman that transient walks in the plane with recurrent marginal walks could be constructed. Examples of such walks are those with independent coordinates whose steps follow a stable law of index $p < 2$. Since such walks have $E[\mathbf{X}_1] = \mathbf{0}$, by the elementary theorem, if the Blackwell theorem holds in this instance, convergence should be to ∞, as in the recurrent case. We have been able to prove this for totally symmetric walks, that is, walks for which (X_{11}, X_{21}), $(X_{11}, -X_{21})$, and $(-X_{11}, -X_{21})$ are identically distributed. The methods are essentially those of Chung [3].

THEOREM 5. *Suppose* $\mathbf{S}_n$, $n \geqq 1$, *is a totally symmetric random walk in the plane. Then, if* $E[\mathbf{X}_1]$ *exists and* $S_n^{(1)}$, $S_n^{(2)}$ *are both nonarithmetic,*

$$\lim_{a\to\infty} U[S(\mathbf{0}, a + \Delta) \sim S(\mathbf{0}, a)] = \infty.$$

We will prove the theorem first for the L_∞ norm and then indicate the modifications required for the general case. Let $K(\mathbf{x}, a)$ denote the sphere of radius a about $\mathbf{x}$ for the L_∞ norm. We require first,

LEMMA 5.1.

$$\lim_{a\to\infty} \inf_{\mathbf{x}\varepsilon R^2} U[K(\mathbf{x}, a)]/a = \infty.$$

PROOF.

$$U[K(\mathbf{x}, a)] = \int_{\mathbf{x}'\varepsilon K(\mathbf{x},a)} (1 + U[K(\mathbf{x} - \mathbf{x}', a)])\, dG(\mathbf{x}')$$

where $G(\mathbf{x}')$ is the distribution function of the first sum to enter $K(\mathbf{x}, a)$. Hence,

$$\begin{aligned} \inf_{\mathbf{x}\varepsilon R^2} U[K(\mathbf{x}, a)]/a &\geqq \inf_{\mathbf{x}\varepsilon R^2} \inf_{\mathbf{x}'\varepsilon K(\mathbf{x}',a)} (1 + [K(\mathbf{x} - \mathbf{x}', a)])/a \\ &= \inf_{\mathbf{x}\varepsilon K(\mathbf{0},a)} U[K(x, a)]/a + a^{-1} \end{aligned}$$

and it suffices to show that the last quantity goes to ∞. Let $O_i(a)$, $1 \leqq i \leqq 4$, denote the intersections of the respective quadrants of the plane with $K(\mathbf{0}, a)$. Let $Z_i(a)$, $1 \leqq i \leqq 4$, denote the number of visits of the walk to $O_i(a)$. By Theorem 1 of Section 2, $\lim_{a\to\infty} (1/a)E[\sum_{i=1}^4 Z_i(a)] = \infty$ a.s. By the total symmetry of the walk, $U[O_i(a)] = E[Z_i(a)] = E[Z_1(a)] = U[O_1(a)] = \frac{1}{4}E[\sum_{i=1}^4 Z_i(a)]$. We can conclude from the preceding remarks that

$$\lim_a \min_i U[O_i(a)]/a = \infty.$$

The lemma follows.

Our next lemma again exploits the symmetry of the walk.

LEMMA 5.2. *Let* $\{\mathbf{S}_n\}$ *be totally symmetric and* $S_n^{(1)}$, $S_n^{(2)}$ *both be nonarithmetic. Then, if* K_n *denotes the compact support of* $\mathbf{S}_n$, *we have that* $C = \bigcup_{n=1}^\infty K_n$ *is dense in* R^2.

PROOF. Since $S_n^{(1)}$, $S_n^{(2)}$ are nonarithmetic, the projections of C on both axes are dense in R. But if $(x, y) \varepsilon C$ so is $(-x, y)$ by the total symmetry of the walk, and hence since C is a semigroup $(0, 2y) \varepsilon C$ for every y in the projection of C on the y axis. Similarly $(2x, 0) \varepsilon C$ for every x in the projection of C on the axis. Hence, $(s, t) \varepsilon C$ for all $s \varepsilon C_1$, $t \varepsilon C_2$, where C_1, C_2 are dense in R. The lemma follows.

We require further notation before starting the next lemma. Let $U(\mathbf{x}, a) = U[K(\mathbf{x}, a + \Delta) \sim K(\mathbf{x}, a)]$. Denote $\inf_{\mathbf{x}} \lim \inf_{a\to\infty} U(\mathbf{x}, a)$ by U^*. It follows that,

LEMMA 5.3. *If* $a(n, \mathbf{x}_m) \to \infty$ *as* $n \to \infty$ *for each fixed* m *and*

$$\lim_m \lim_n U[\mathbf{x}_m, a(n, \mathbf{x}_m)] = U^*,$$

then $\lim_j \lim_k U[\mathbf{x}_{m_j} - \mathbf{v}, a(n_k, \mathbf{x}_{m_j})] = U^*$ *for a dense set* V *of* $\mathbf{v} = (v_1, v_2)$ *in* R^2 *(possibly dependent on* $\mathbf{x}_m$*), for some subsequences* $\mathbf{x}_{m_j}$, $a(n_k, \mathbf{x}_{m_j})$ *both possibly depending on* $\mathbf{v}$.

PROOF. $U[\mathbf{x}_m, a(n, \mathbf{x}_m)] \geqq \int_{A_{mn}} U(\mathbf{x}_m - \mathbf{v}_k, a(n, \mathbf{x}_m))\, dG(\mathbf{v}_1, \cdots, \mathbf{v}_k)$ where $A_{mn} = \{(\mathbf{v}_1, \cdots, \mathbf{v}_k): \mathbf{v}_j \not\in K(\mathbf{x}_m, a(n, \mathbf{x}_m) + \Delta) \sim K(\mathbf{x}_m, a(n, \mathbf{x}_m))\}$ and G is the joint distribution of $\mathbf{S}_1, \cdots, \mathbf{S}_k$, by the Markov property.

By Fatou's lemma, $\liminf_n U(\mathbf{x}_m, a(n, \mathbf{x}_m)) \geqq \int \liminf_n I(A_{mn}) U(\mathbf{x}_m - \mathbf{v}_k, a(n, \mathbf{x}_m))\, dG(\mathbf{v}_1, \cdots, \mathbf{v}_k) = \int \liminf_n U(\mathbf{x}_m - \mathbf{v}, a(n, \mathbf{x}_m))\, dF^k(\mathbf{v})$ where F^k is the distribution of $\mathbf{S}_k$. The last assertion follows since $I(A_{mn}) \to 1$ for m fixed as $n \to \infty$. But $U^* = \lim_m \liminf_n U(\mathbf{x}_m, a(n, \mathbf{x}_m)) \geqq \inf_m \liminf_n U(\{\mathbf{x}_m, a(n, \mathbf{x}_m)) \geqq \inf_m \int \liminf_n U(\mathbf{x}_m - \mathbf{v}, a(n, \mathbf{x}_m))\, dF^k(\mathbf{v}) \geqq \int \inf_m \liminf_n \cdot U(\mathbf{x}_m - \mathbf{v}, a(n, \mathbf{x}_m))\, dF^k(\mathbf{v}) \geqq U^*$.

From Lemma 5.2 and Fubini's theorem we can conclude that

$$\inf_m \liminf_n U(\mathbf{x}_m - \mathbf{v}, a(n, \mathbf{x}_m)) = U^*$$

for a dense set of $\mathbf{v}$ in R^2. But now we can first choose $\mathbf{x}_{m_j}$ depending on $\mathbf{v}$ such that $\lim_j \liminf_n U(\mathbf{x}_{m_j} - \mathbf{v}, a(n, \mathbf{x}_{mj})) = U^*$ and hence $a(n_k, \mathbf{x}_{m_j})$ which depends on $\mathbf{x}_{m_j}$ and $\mathbf{v}$ such that $\lim_j \lim_k U(\mathbf{x}_{m_j} - \mathbf{v}, a(n_k, \mathbf{x}_{m_j})) = U^*$, and the lemma is proved.

Suppose now that $\liminf_a U(\mathbf{0}, a) < \infty$. Hence, $U^* < \infty$ and we can choose $\{\mathbf{x}_m\}$ and $\{a(n, \mathbf{x}_m)\}$, $m, n \geqq 1$ such that $\lim_m \lim_n U(\mathbf{x}_m, a(n, \mathbf{x}_m)) = U^*$.

Let $B > 0$ and $J = [4B/\Delta] + 1$. By successive applications of Lemma 5.3 we can define a finite sequence $\mathbf{V}^{(0)}, \cdots, \mathbf{V}^{(J)}$ such that,

(1) $\mathbf{V}^{(0)} = \mathbf{0}$.

(2) $\Delta/2 \leqq V_i^{(k)} - V_i^{(k-1)} \leqq \Delta$, $1 \leqq k \leqq J$.

(3) $\lim_j \lim_k U[\mathbf{x}_{m_j} - \mathbf{v}^{(s)}, a(n_k, \mathbf{x}_{m_j})] = U^*$

for a subsequence $\mathbf{x}_{m_j}$ and subsequences $a(n_k, \mathbf{x}_{m_j})$ of $a(n, \mathbf{x}_{m_j})$ which depend on $\mathbf{x}_{m_j}$, and $0 \leqq s \leqq J$.

Let $\mathbf{u}(k, j) = (a(n_k, \mathbf{x}_{m_j}) - B, a(n_k, \mathbf{x}_{m_j}) - B)$. Then by elementary geometry for k so large that $a(n_k, \mathbf{x}_{m_j}) \geqq B$,

$$U[K(\mathbf{x}_{m_j} + \mathbf{u}(k, j), B)] \leqq \sum_{r=0}^{j} U(\mathbf{x}_{m_j} - \mathbf{v}^{(r)}, a(n_k, \mathbf{x}_{m_j})). \tag{17}$$

Hence, $\limsup_j \limsup_k U[K(\mathbf{x}_{m_j} + \mathbf{u}(k, j), B)] \leqq \lim_j \sum_{r=0}^{j} U(\mathbf{x}_{m_j} - \mathbf{v}^{(r)}, a(n_k, \mathbf{x}_{m_j})) = (J + 1)U^* \leqq 4(B + \Delta)U^*\Delta^{-1}$. Finally, we have

$$\limsup_{B\to\infty} \limsup_j \limsup_k B^{-1} U[K(\mathbf{x}_{m_j} + \mathbf{u}(k, j), B)] \leqq 4U^*/\Delta < \infty.$$

But now Lemma 5.1 provides a contradiction and the theorem is proved for the L_∞ norm.

To prove the theorem in full generality consider first a polygonal norm whose unit sphere has a vertex (v, v) on the diagonal in the positive quadrant and is such that the sides meeting at that vertex are perpendicular to each other. Let l_1, l_2 denote the lengths of these sides. Remark that $S(\mathbf{0}, a)$ in this case has a vertex on the diagonal at (av, av) and the sides meeting at that vertex are perpendicular and have lengths al_1, al_2, respectively. We can now repeat for this norm the program followed for the L_∞ norm by first defining

$$U^* = \inf_{\mathbf{x}} \liminf_a U[S(\mathbf{x}, a + \Delta) \sim S(\mathbf{x}, a)]$$

and then stating and proving the appropriate version of Lemma 5.3 by substituting $S(\mathbf{x}, a)$ for $K(\mathbf{x}, a)$ throughout.

We define $\mathbf{u}(k, j) = (a(n_k, \mathbf{x}_{m_j})v - B, a(n_k, \mathbf{x}_{m_j})v - B)$ and consider $K(\mathbf{x}_{mj} + \mathbf{u}(k, j), B)$. The proof now goes through verbatim by substituting S for K in the quantities on the right side of (17), and changing the requirement $a(n_k, \mathbf{x}_{m_j}) \geqq B$ to $\min(l_1, l_2)a(n_k, \mathbf{x}_{m_j}) \geqq B$.

We remark that the preceding proofs continue to hold for any walk with expectation $\mathbf{0}$ satisfying the statements of Lemmas 5.1 and 5.2.

Suppose now that we are given any polygonal norm. As we have seen in the preceding section there exists a nonsingular linear transformation T which maps the boundary of $S(\mathbf{0}, 1)$ onto a polygon with a vertex (v, v) on the positive diagonal such that the sides meeting in that vertex are perpendicular. Denote this convex polygon and its interior by $S^*(\mathbf{0}, 1)$ and in general let $S(\mathbf{x}, a)$ be the sphere of radius a about $\mathbf{x}$ of the norm whose unit sphere is $S^*(\mathbf{0}, 1)$. Let $\mathbf{S}_n^* = T\mathbf{S}_n$, K_n^* be the compact support of $\mathbf{S}_n^*$, $U^*[A]$ the expected number of visits of S_n^* to A, etc. Then, clearly $U[S(\mathbf{0}, a + \Delta) \sim S(\mathbf{0}, a)] \to \infty$ for every $\Delta > 0 \Leftrightarrow U^*[S^*(\mathbf{0}, a + \Delta) \sim S^*(\mathbf{0}, a)] \to \infty$.

From our preceding remarks it suffices to show that $\mathbf{S}_n^*$ satisfies Lemmas 5.1 and 5.2. Since T is 1-1 and continuous, $\bigcup_{n=1}^{\infty} K_n^* = \bigcup_{n=1}^{\infty} T(K_n) = T(\bigcup_{n=1}^{\infty} K_n)$ which is dense. Hence, $\mathbf{S}_n^*$ satisfies Lemma 5.2. On the other hand, $U^*[K(\mathbf{0}, 1)] = U\{T^{-1}[K(\mathbf{0}, 1)]\}$. But $T^{-1}[K(\mathbf{0}, 1)]$ is a bounded open set and hence, $U\{T^{-1}[K(\mathbf{0}, 1)]\} \geqq U[K(\mathbf{0}, c)]$ for some $c > 0$. Hence, $U^*[K(\mathbf{x}, a)] = U\{T^{-1}[K(\mathbf{x}, a)]\} = U\{T^{-1}(\mathbf{x}) + aT^{-1}[K(\mathbf{0}, 1)]\} \geqq U[K(T^{-1}(\mathbf{x}), ac)]$. We conclude that $\inf_{\mathbf{x}} U^*[K(\mathbf{x}, a)]/a \geqq c \inf_{\mathbf{x}} U[K(\mathbf{x}, ac)]/ac$ and hence

$$\lim_{a\to\infty} \inf_{\mathbf{x}} U^*[K(\mathbf{x}, a)]/a = \infty$$

and Lemma 5.1 is satisfied.

The theorem is proved.

6. Conclusions. We remark first that the restriction to 2 dimensions may be immediately dispensed with, with the possible exception of Theorem 3. There Case (ii) which now corresponds to the line of expectation exiting through the boundary of a face presents some difficulty.

There are clearly many directions of generalization, such as more precise estimates of the renewal function $U[S(\mathbf{0}, a)]$ in the presence of higher moments, consideration of the renewal density in the presence of absolute continuity of F and similar extensions of the type discussed in Smith [10]. The most attractive generalization (to nonpolygonal norms) has been dealt with to some extent in Farrell [8].

It also seems that generalizations (or parallel theorems) for arithmetic walks can be made without difficulty.

Unfortunately, the Erdös-Feller-Pollard [6] approach in the above case, and the Feller-Orey [10] approach to the nonarithmetic case in one dimension does not seem to generalize at all.

The remark made by Smith in [11] about univariate renewal theory, specifically Blackwell's theorem, we feel has even more force in the multivariate case, i.e., there should be a general relatively simple argument which would establish these results. Unfortunately, so far none has presented itself.

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