# SOME BASIC PROPERTIES OF THE INCOMPLETE GAMMA FUNCTION RATIO

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**1. Introduction and summary.** We define the incomplete gamma function ratio for positive $N$ by

$$(1) \qquad P(N, b, X) = \int_0^X t^{N-1}e^{-bt}\,dt \Big/ \int_0^\infty t^{N-1}e^{-bt}\,dt$$
$$\equiv \int_0^X D(N, b, t)\,dt, \quad \text{say},$$

where $N$ and $b$ are positive real numbers and $0 < X \leqq \infty$. In what follows, we generally use the notation $P_N$ to denote the same function, unless $b$ or $X$ are given some specific values. It should be noted that the notation $P(N, b, X)$ does not mean that the function is one of three variables, since we have $P(N, b, X) = P(N, 1, bX)$. The positive real number $b$ is only a scale factor. The function $P_N$ is of appreciable importance in probability and statistics and it is known as the gamma distribution and for $b = \frac{1}{2}$ as the chi-square distribution. It is also of importance in many other branches of applied mathematics. Like the incomplete beta distribution, it is also a special case of the more general confluent hypergeometric function. During the last decade, interest has been revived in the development, use and application of these functions and a number of publications, such as Erdelyi et al. (1953), Tricomi (1954) and Slater (1960), appeared. In these publications, the function treated is defined as the numerator in the middle part of (1) with $b = 1$, and the definition is generally extended for all complex values of $N$ such that $R(N)$ is not a non-positive integer. Many of the properties of $P_N$ are derived in these publications. Bancroft (1949) derived some new properties of the incomplete beta function (particularly, recurrence relations) from the more general properties of the parent hypergeometric function. Similar methods may also be used to derive new properties of $P_N$ of use in statistical work. In this paper, however, we derive such properties directly from the well-known difference-differential properties of $P_N$ given by

$$(2) \qquad (\partial^r/\partial X^r)P_N = (-b)^r\Delta^r P_{N-r},$$
$$= (\partial^{r-1}/\partial X^{r-1})D(N, b, X), \qquad N > r,$$

where $r = 0, 1, 2, 3, \cdots$ and where

$$(3) \qquad \Delta^{r+1}P_T = \Delta^r(\Delta P_T) = \Delta^r(P_{T+1} - P_T)$$

are the advancing finite differences of $P_N$ with respect to $N$ and with a unit difference interval. Property (2) for $r = 1$ may be used, in fact, to define the function $P_N$ itself, as in Milne-Thomson (1933). This property was also used in

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1947 by Khamis in an unpublished Ph.D thesis (1950) to derive, *inter alia*, a series expansion in terms of chi-square integrals to approximate to statistical distribution functions and also to formulate a computational method for the tabulation of the chi-square distribution itself. This method was later used by Hartley and Pearson (1950) in computing a five decimal table for this distribution. The chi-square series expansion was later generalized by Khamis (1960a) into an expansion in terms of incomplete gamma function ratios and the computational method was employed by Khamis, (1964a) and (1965) to prepare a six decimal table of the chi-square integral and a master ten decimal table, for very fine $N$ and $X$ intervals, of the incomplete gamma function (1) for $b = \frac{1}{2}$ (a description of this table is given by Khamis (1964b).

In this paper we make further use of the property (2) to derive other new and useful properties of the function (1). In Section 2 we derive a simple recurrence relation for $P(T, b, X)$ for $T = N, N + 1$ and $N + 2$. In Section 3 we derive new $N$-wise sum formulae as consequences of this recurrence relation. We then give in Section 4 an $X$-wise sum formula based on the Taylor expansion of (2) in the neighbourhood of $X$. This is made possible by extending the definition of $P(N, b, X)$ for $N \leqq 0$, and thus enabling the removal of the condition $N > r$ in (2) above. Other examples of the use of property (2) are given in Section 5 where in particular, the Laguerre polynomials are expressed in terms of the differences of the function $P(N, b, X)$. Section 5 contains also examples of some numerical applications of the previous results.

The importance of the properties derived here and the simplicity inherent in such derivations due to the nature of property (2) are further enhanced by work recently carried out by Wise (1950) and developed further by H. O. Hartley and E. J. Hughes (in process of publication) where the incomplete gamma function ratio is shown to provide quite satisfactory approximations to the incomplete beta function ratio and to many other statistical functions. We note finally, that the notation used in (1) is only one of many notations used by different authors. The references given above include most of the different notations used for the function $P_N$ and other related functions.

**2. A recurrence relation for $P(N, b, X)$ and the extension of its definition for $N \leqq 0$.** The relation (2) for $r = 1$ implies a simple and useful recurrence relation. We have, for $r = 1$ and with $N + 1$ instead of $N$,

$$(4) \qquad D(N + 1, b, X) = -b(P_{N+1} - P_N).$$

We also have from (1),

$$(5) \qquad D(N + 1, b, X) = b^{N+1}X^N e^{-bX}/\Gamma(N + 1),$$

where $\Gamma(T)$ is the ordinary (complete) gamma function. Similarly, we have from (5) and from (4), with $N + 1$ instead of $N$ and for $r = 1$,

$$(6) \qquad \begin{aligned} D(N + 2, b, X) &= [bX/(N + 1)]D(N + 1, b, X) \\ &= -b(P_{N+2} - P_{N+1}). \end{aligned}$$

Eliminating $D(N + 1, b, X)$ from (4) and (6), we obtain the recurrence relation

$$(7) \quad (N + 1)P(N + 2, b, X) = (N + 1 + bX)P(N + 1, b, X) - bXP(N, b, X),$$

which holds for all $N > 0$. Equation (7), although very easily derivable from (2), is also derivable from a related recurrence relation for the more general hypergeometric function ${}_1F_1(1, N + 1, X)$ (cf. e.g., Slater (1960) formula 2.2.2). The author has been making use of (7) for over 10 years in overcoming many difficulties in the use of the function $P_N$ for approximation to other statistical distributions. While illustrations of the use of (7) and of its implications are given in the following sections, we give here an immediate example of its use in overcoming the restrictions $N > r$ in equation (2). Such restrictions are quite limiting in numerical applications of the function $P_N$ and the associated expansions referred to in Section 1 above, while in fact they could easily be removed. This may be achieved simply by extending the definition of $P_N$ for all $X > 0$ to cover values of $N \leqq 0$ as follows: *For all $X > 0$, the function $P_N$ is defined by equation* (1) *for all real $N > 0$. For all real $N \leqq 0$ and $X > 0$, the function $P_N$ is defined by the successive $N$-wise backward application of recurrence relation* (7) *starting with values of $N$ in the interval* $0 < N \leqq 2$.

As a simple consequence of this definition we have,

$$(8) \qquad P(N, b, X) \equiv 1,$$

*for all $X > 0$ and all non-positive integral $N$.*

It should be noted that apart from the special and degenerate case of integral $N$, $P_N$, as generalized here, is not a distribution function over the range $X \geqq 0$ for non-positive values of $N$, as the value of $P(N, b, X)$ may be made as large as one pleases by choosing a sufficiently small $X > 0$. In fact $P(N, b, X)$ thus defined has a singularity at $X = 0$. Interpreted as a distribution function over the range $X \geqq 0$ we of course define $P_N$ as identically zero for all $X < 0$. The extended definition makes it possible to retain the finite difference-differential property of $P_N$ for all $X > 0$ by providing a definition for $N < 0$ which preserves the property (2). It is well known that the numerator in the middle part of (1) may be generalized for all $N$ except non-positive integers, but the function $P_N$ may be generalized for all values of $N$. There are other generalizations of $P_N$ which include the values of $X$ less than zero, but we are not concerned here with these generalizations.

**3. $N$-wise sum formulae.** We give in this section some immediate consequences of the difference-differential property (2) and of the recurrence relation (7).

We note that by applying equation (7) successively we obtain for any positive integer $R$,

$$(9) \qquad P_{N+R} \equiv P_N + \Delta P_N \textstyle\sum_{r=1}^{R} [\Gamma(N + 1)/\Gamma(N + r)](bX)^{r-1},$$

where $\Delta P_N = P_{N+1} - P_N$. Equation (9) for $R \geqq 2$ is a sum formula giving $P(N + R, b, X)$ in terms of $P(N, b, X)$ and $P(N + 1, b, X)$. Similarly, successive applications of the recurrence relation (7) *backwards* leads to the sum formula for $R \geqq 1$,

$$(10) \qquad P_{N-R} \equiv P_N - \Delta P_N \textstyle\sum_{r=1}^{R} N(N - 1) \cdots (N - r + 1)(bX)^{-r},$$

where $P_{N-R} = P(N - R, b, X)$ is the generalized incomplete gamma function ratio. When $N$ is a positive integer and $R > N$, the terms in the summation on the right hand side of (10) for $r \geqq N + 1$ vanish. Again (10) is another sum formula expressing $P_{N-R}$ in terms of $P_N$ and $P_{N+1}$.

When $R \rightarrow \infty$, the $N$-wise sum formula (9) leads to the well known infinite series expansion of $P(N, b, X)$. Similarly, when $R \rightarrow \infty$, it may be shown that formula (10) leads to the well known asymptotic expansion of $P(N, b, X)$. Some of the uses of (9) and (10) are indicated in Section 5 below.

Another class of useful $N$-wise summation formulae, not necessarily independent of those given above, may be obtained directly from property (2) for $r = 1$. These are obtained from the identity

$$(11) \qquad P_N = P_{N+R+1} - \textstyle\sum_{r=0}^{R} \Delta P_{N+r}$$

which, by virtue of (2) and (1), reduces to

$$(12) \qquad P_N = P_{N+R+1} + b \textstyle\sum_{r=0}^{R} P'_{N+r-1}$$

where the prime denotes differentiation with respect to $X$. Since for all $T$

$$(13) \quad P'(T + 1, b, X) = D(T + 1, b, X) = [b^{T+1}X^T/\Gamma(T + 1)]e^{-bX},$$

we have by substitution from (13) into (12)

$$\begin{aligned}(14) \quad P_N &= P_{N+R+1} + b^{N-1}X^N \textstyle\sum_{r=0}^{R} b^{r+1}X^r e^{-bX}/\Gamma(N + r + 1) \\ &= P_{N+R+1} + b^{N-1}X^N \textstyle\sum_{r=0}^{R}[\Gamma(r + 1)/\Gamma(N + r + 1)]P'_{r+1}.\end{aligned}$$

Applying (2) again to the last term in (14) under the summation sign we obtain

$$(15) \qquad P_{N+R+1} = P_N + b^N X^N \textstyle\sum_{r=0}^{R} [\Gamma(r + 1)/\Gamma(N + r + 1)]\Delta P_r$$

where $P(0, b, X) \equiv 1$ for all $X > 0$, according to the generalized definition of $P(N, b, X)$. The sum formula (15) gives $P_{N+R+1}$ for any $N > 0$ and non-negative integer $R$ in terms of $P_N$ and the values of $P_r$ for $r = 0, 1, 2, \cdots, R + 1$. Taking $R = 0$ in (15) one obtains the special relation

$$(16) \qquad P_{N+1} = P_N + [b^N X^N/\Gamma(N + 1)](P_1 - 1)$$

or

$$(17) \qquad P_N = P_{N+1} + [b^N X^N/\Gamma(N + 1)](1 - P_1).$$

This last special case is of interest in connection with tables of the function $P$ as will be shown in Section 5.

The sum formula (15) is only one variant of a larger class of formulae that could be easily derived from (12) and (13). Instead of taking outside the summation sign in (14) the power $N$ of $X$ as a factor, one may take out any other convenient factor. For example, one may rewrite (12) in the form

$$P_N = P_{N+R+1} + b^{c-1}X^c \textstyle\sum_{r=0}^{R} b^{N-c+r+1}X^{N-c+r}e^{-bX}/\Gamma(N+r+1)$$

$$= P_{N+R+1} + b^{c-1}X^c \textstyle\sum_{r=0}^{R} [\Gamma(N-c+r+1)/\Gamma(N+r+1)]P'_{N-c+r+1}$$

or

$$(18) \quad P_{N+R+1} = P_N + b^cX^c \textstyle\sum_{r=0}^{R} [\Gamma(N-c+r+1)/\Gamma(N+r+1)]\Delta P_{N-c+r}$$

where $N \geqq c$ and $P(0, b, X) \equiv 1$. Equation (18) reduces to (14) by taking $N = c$. Other variants are obtained by choosing other values for $c \leqq N$. One form that immediately suggests itself is to take $c = N - [N]$, where $[N]$ is the largest integer contained in $N$. The resulting sum formula expresses $P_{N+R+1}$ in terms of $P_N$ and $P_{[N]+r}$, $r = 0, 1, \cdots, R+1$.

Noting that $P_{N+R} \to 0$ as $R \to \infty$ for any finite and fixed $X > 0$, a number of useful relations may be immediately deduced from the $N$-wise sum formulae involving $N + R$. Thus from (9) we obtain

$$\textstyle\sum_{r=1}^{R} [\Gamma(N+1)/\Gamma(N+r)](bX)^{r-1} = (P_{N+R} - P_N)/(P_{N+1} - P_N),$$

or

$$(19) \quad \textstyle\sum_{r=0}^{R} [\Gamma(N+1)/\Gamma(N+r+1)](bX)^r = (P_{N+R+1} - P_N)/(P_{N+1} - P_N),$$

and as $R \to \infty$ and $X > 0$ is held fixed, we have

$$(20) \quad \textstyle\sum_{r=0}^{\infty} [\Gamma(N+1)/\Gamma(N+r+1)](bX)^r = P_N/(P_N - P_{N+1}),$$

and by (2) we obtain the equivalent formula

$$(21) \quad \textstyle\sum_{r=0}^{\infty} (bX)^r/\Gamma(N+r-1) = (bX)^{-N}e^{bX}P_N.$$

Similarly, (19) may be rewritten in the equivalent form:

$$(22) \quad \textstyle\sum_{r=0}^{R} (bX)^r/\Gamma(N+r+1) = (bX)^{-N}e^{bX}(P_N - P_{N+R+1}).$$

Sums of the form (21) and (22) (or of the form (19) and (20)) are thus obtained in terms of the functions $P$ and the related exponential function. In fact formula (21) is nothing but the well known expansion of $P_N$ used to evaluate the corresponding sums on the left hand side when the function $P_N$ is assumed known. Tables for $P(N, \frac{1}{2}, X)$ or for $P(N, 1, X)$ are available. Khamis [(1964b) and (1965)] gives $P(N, \frac{1}{2}, X)$ correct to 10 decimals for fine fractional intervals in $N$ and for fine intervals in $X$ over a wide range of these two variables. Such tables may therefore be used to evaluate sums of the forms (19) and (20) or of the corresponding forms (21) and (22). Special numerical cases of formulae (19) to (22) are obtained by giving $X$ numerical values. In particular, for $X = b^{-1}$ one obtains, from (21) and (22) respectively, the two results:

$$(23) \quad \textstyle\sum_{r=0}^{\infty} 1/\Gamma(N+r+1) = eP(N, b, b^{-1})$$

and

(24) $\sum_{r=0}^{R} 1/\Gamma(N + r + 1) = e[P(N, b, b^{-1}) - P(N + R + 1, b, b^{-1})]$.

Both sides of (23) and (24) are obviously independent of the scale factor $b$. We note incidentally that formula (22) with $N = 0$ leads to the interesting result

(25) $$\sum_{r=0}^{R} (bX)^r/\Gamma(r + 1) = 1 + bX/1! + (bX)^2/2! + \cdots + (bX)^R/R! = e^{bX}(1 - P_{R+1})$$

which is a closed expression for the partial sums of the Taylor expansion of $e^{bX}$, a result already well known in a different context. In fact, the last sum happens to be formally nothing but another form of Molina's (1915) relation between the Poisson and the chi-square distributions obtained here as a special case of the more general relation (21) above. Such sums, and more generally sums like those involved in (19) to (24), are met in applied problems in probability and statistics as well as in other branches of pure and applied mathematics and hence they may be dealt with numerically through tables of the function $P_N$. (For example, cf. Feller (1957) pp. 144 and 417.) We note that the second set of $N$-wise sum formulae and their special cases may also be derived directly from the sum formula (9) but we have preferred to give here a separate proof to illustrate the various methods that may be used.

We conclude this section by noting that, by similar methods, the sum formula (10) leads to the following result:

(26) $$\sum_{r=1}^{R} N(N - 1)(N - 2) \cdots (N - r + 1)(bX)^{-r} = [\Gamma(N + 1)/(bX)^N e^{-bX}](P_{N-R} - P_N),$$

where $P_{N-R}$ is the generalized function for $N \leqq R$. When $N$ is a positive integer less or equal to $R$, the finite series summation formula (26) leads to the finite series results

(27) $$\sum_{r=1}^{N} (bX)^{-r}/(N - r)! = (bX)^N e^{bX}(1 - P_N),$$

and (letting $X = b^{-1}$),

$$\sum_{r=1}^{N} 1/(N - r)! = e(1 - P(N, b, b^{-1})).$$

These last two relations together with (25) and (23) may be used to derive other useful algebraic finite series summation formulae dependent on the function $P_N$.

**4. $X$-wise sum formulae.** An $X$-wise sum formula for the numerator of $P_N$ in the middle part of (1), known as the Nielsen sum formula, has been used in conjunction with tables of the incomplete gamma function (cf., for example, Tricomi, (1954), p. 165). This formula, when rewritten for the function $P_N$ itself, becomes

$$(28)\quad P(N, b, X + Y) = P(N, b, X) - \Delta P(N - 1, b, X) \textstyle\sum_{r=0}^{\infty} C_r(X) P(r, b, Y),$$

where

$$(29)\qquad C_r(X) = (N - 1)(N - 2) \cdots (N - r)(bX)^{-r}.$$

The sum formula (28) is derivable directly by applying the binomial theorem to $P(N, b, X + Y)$ and it holds for all $|Y| < X$.

We derive here another sum formula by writing down the Taylor expansion of $P(N, b, X + Y)$ in the neighbourhood of $X$ and making use of the property (2). Thus, writing $h = bY$, we immediately obtain

$$(30)\quad P(N, b, X + Y) = \textstyle\sum_{r=0}^{n} (-1)^r (h^r/r!) \Delta^r P(N - r, b, X) + R_{n+1},$$

where $R_{n+1}$ is the remainder term in Taylor's theorem. Using the Lagrange form of $R_{n+1}$, we may write

$$(31)\qquad R_{n+1} = [Y^{n+1}/(n + 1)!]\, P^{(n+1)}(N, b, X_0) = (-1)^{n+1}[h^{n+1}/(n + 1)!]\Delta^{n+1} P(N - n - 1, b, X_0),$$

where $X_0$ is strictly between $X$ and $X + Y$.

Formula (30) provides a *finite* $X$-wise sum formula with a remainder term for the function $P_N$. In this paper we assume that $R_{n+1} \to 0$ as $n \to \infty$ for sufficiently small $Y$ or $h$. This and the convergence of the sum formula (30) will be taken up in a further communication where it will be shown that the sum formula

$$(32)\qquad P(n, b, X + Y) = \textstyle\sum_{r=0}^{\infty} (-1)^r (h^r/r!) \Delta^r P(N - r, b, X)$$

holds for all $|Y| < X$.

Formula (30), after simple algebraic manipulation, may be rewritten in the form

$$(33)\qquad P(N, b, X + Y) = \textstyle\sum_{r=0}^{n} (h^r/r!) S_{n-r} P(N - r, b, X) + R_{n+1},$$

where

$$(34)\quad S_{n-r} = \textstyle\sum_{j=0}^{n-r} (-1)^j (h^j/j!) = S_{n-r-1} + (-1)^{n-r} h^{n-r}/(n - r)!$$

and where $h = bY$. Noting that $S_{n-r} \to e^{-h} = e^{-bY}$ as $n \to \infty$ for a fixed $r$, we may rewrite (33) in the form

$$(35)\quad P(N, b, X + Y) = e^{-bY} \textstyle\sum_{r=0}^{n} (h^r/r!) P(N - r, b, X) + K_{n+1},$$

where $K_{n+1}$ is a new remainder term. Using the Lagrange form of the remainder term in the Taylor expansion of $e^{-h}$ in conjunction with (33), one obtains

$$(36)\quad K_{n+1} = R_{n+1} - [h^{n+1}/(n + 1)!] \textstyle\sum_{r=0}^{n} (-1)^{n-r+1} \binom{n+1}{r} P(N - r, b, X) e^{-U_r},$$

where $U_r$ is strictly between 0 and $h$. Applying the same procedure to (32) when $|Y| < X$, one may easily derive the equivalent formula

$$(37)\qquad P(N, b, X + Y) = e^{-bY} \textstyle\sum_{r=0}^{\infty} (h^r/r!) P(N - r, b, X).$$

This is another form of the same $X$-wise sum formula (32) and holds for all $|Y| < X$. Convergence properties of the sum formulae (32) and (37) and upper bounds for $R_{n+1}$ and $K_{n+1}$ will be dealt with in a further communication. We may note here that, since $P_N \equiv 1$ for all non-positive integers $N$, the sum formula (37), and hence (32), will be absolutely convergent for all $Y$ when $N$ is an integer, because the sum of the terms on the right hand of (37) beginning with $r = N$ will be at most equal to $e^{bY}$.

Examples of the use of the $X$-wise and the $N$-wise sum formulae are given in the following section. We conclude this section by noting that Tricomi [(1954), p. 165] derived an $X$-wise sum formula for the numerator of $P_N$ in the middle part of equation (1) which is equivalent to formula (37) above. His result is obtained from a more general $X$-wise sum formula for the corresponding confluent hypergeometric function. Our method, apart from its direct dependence on property (2), has the advantage that it also leads to finite forms of the sum formula, as in the case of equations (30), (33) and (35), with exact remainder terms. Furthermore, our results hold for all real values of $N$. It is possible, however, to generalize $P_N$ by an alternative procedure dependent upon its relation to the known generalization of the numerator in (1) for all $N \neq 0, -1, -2, \cdots$. In fact the gamma function $\Gamma(N)$ assumes infinite values for non-positive integral $N$ and hence $1/\Gamma(N)$ has zeroes at these values of $N$. If we interpret the denominator in the middle part of (1) as the factor $b^N/\Gamma(N)$ multiplied by the numerator in the same part of (1), one may be justified in defining $P(N, b, X) \equiv 1$ for all non-positive integers $N$. This, together with the usual generalization of $P_N$ for the other values of $N$, leads to a generalization of $P_N$ equivalent to that obtained from our recurrence relation (7).

**5. Some numerical examples and other remarks.** We give in this section a few numerical examples illustrating the use of recurrence relation (7), the generalized definition of $P_N$ and the sum formulae derived in the previous sections. We give also further brief indications of other uses of the methods described above.

Tables of $P_N$ usually give the values of the function beginning with a small $N_0 > 0$ and with a given tabular interval of $N$ and a range of tabular values of $0 < X \leqq \infty$. In such tables, Lagrangian and other polynomial interpolation methods usually lead to satisfactory results of $X$-wise interpolation, but unless the interval in $N$ is small (say less than 0.5) polynomial interpolation does not lead to accurate results for $N$-wise interpolation at least near the beginning of the range of values of $N$. This difficulty is completely avoided through the use of recurrence relation (7). This recurrence relation makes it possible to transfer the interpolation from the interval $0 < N < 1$ to values of $N$ in the interval $1 < N < 3$. Thus, for a non-tabular $N = m$ where $0 < m < 1$, particularly when $m$ is nearer to 0 (including the case $m < N_0$), one need only interpolate for $P_{m+1}$ and $P_{m+2}$ for the given $X$ and then by recurrence relation (7) compute the value of $P_m$. An alternative procedure is also possible by using equation (17), where one has to interpolate only for $P_{m+1}$ and then compute $P_m$ making use of a table of the (complete) gamma function. Interpolation for $P_{m+1}$ (and, if the first alternative is used, for $P_{m+2}$) may be carried out by ordinary polynomial interpolation methods. A third interpolation method is also possible through the generalized definition of $P_N$ for non-positive $N$. This consists of the use of recurrence relation (7) backwards to tabulate $P_{N-rk}$, where $r = 1, 2, 3, 4$, say, and $k$ is the tabular $N$-interval. This may require the computation of $P_N$ for non-positive values of $N$. The computed values of $P_N$ together with the tabular values enable the use of ordinary polynomial interpolation to compute $P_m$. This last method involves relatively more computational work than the former two alternatives, but the computational work is generally limited in all cases, and it is easy to undertake on an ordinary desk calculator. It should be noted that these methods apply to interpolation as well as extrapolation for a non-tabular positive $N < N_0$, where $N_0$ is the first tabular value of $N$. As a numerical illustration, we consider the extrapolation for $P(0.05, \frac{1}{2}, 6)$ in a table assumed to begin with $N_0 = 0.1$ and an $N$ tabular interval $k = 0.1$. For this purpose we quote from Khamis (1965) the following tabular values of $P_N$ for $b = \frac{1}{2}$ and $X = 6$:

$$P_{0.1} = 0.9984347283, \quad P_{0.2} = 0.9962968282, \quad P_{0.3} = 0.9935096273;$$
$$P_{0.8} = 0.9673945375, \quad P_{0.9} = 0.9593279040, \quad P_{1.0} = 0.9502129316;$$
$$P_{1.1} = 0.9400246207, \quad P_{1.2} = 0.9287479250, \quad P_{1.3} = 0.9163779717;$$
$$P_{1.8} = 0.8386630191, \quad P_{1.9} = 0.8201865134, \quad P_{2.0} = 0.8008517265;$$
$$P_{2.1} = 0.7807243274, \quad P_{2.2} = 0.7598756671, \quad P_{2.3} = 0.7383818433.$$

Using a six-point Lagrangian polynomial interpolation, we may compute $P_{1.05}$ and $P_{2.05}$ and then use recurrence relation (7) to compute $P_{0.05}$. For this purpose, the two sets of 6 values of $P_N$, one beginning with $N = 0.8$ and the other beginning with $N = 1.8$, given above, are used in the interpolation. The second alternative procedure using equation (17) is based on interpolating directly for $P_{1.05}$, using the six values of $P_N$ beginning with $N = 0.8$ in conjunction with $\Gamma(1.05) = 0.9735042656$. The third method requires the use of recurrence relation (7) to compute from tabular values of $P_N$ for $N = 0.1$ and 0.2 the values of $P_N$ for $N = 0, -0.1$ and $-0.2$. These values, for $X = 6$, are found to be $P_{-0.2} = 1.0017229424$, $P_{-0.1} = 1.0010703212$ and $P_0 = 1$. Using these values, together with the three values of $P_N$ for $N = 0.1$, 0.2 and 0.3, in a six-point Lagrangian interpolation one computes $P_{0.05}$. The application of each of the three methods leads to at least nine decimal accuracy. If a lower accuracy is desired, a lower degree Lagrangian polynomial will be needed. The exact value of $P_{0.05}$ for $X = 6$ and $b = \frac{1}{2}$ to ten decimals is 0.9992840946. In general, the first and the second methods are to be preferred to the third because of the lower amount of computations involved. We may note here that these methods are the best available when $X$ is neither very small nor very large. For small and large values of $X$, a direct computation of $P_N$ from the corresponding expansion and asymptotic expansion, respectively, is also quite easy because only a few terms are needed; and, for sufficiently small or large $X$, the amount of computation is comparable to that involved in either of the first two methods.

Similarly, the $X$-wise sum formulae (30), (33) and (35) may be used for $X$-wise interpolation. When $N$ is an integer or $N > n = 6$, say, these formulae may be used without difficulty. However, when $N$ is small, one may have to compute through recurrence relation (7) a few values of $P_N$ for non-positive $N$ for the given value of $X$. Again in this case, the computation of $P_N$ for non-positive values of $N$ may be avoided by interpolating for $P_{N+M}$ and $P_{N+M+1}$, where $M$ is an integer selected so as to have $N + M$ large enough. One may then compute $P_N$ from $P_{N+M+1}$ and $P_{N+M}$ by successive backward application of recurrence relation (7). When $N$ is small and is not an integer, however, one may also use the Nielsen sum formula (28). These methods are usually resorted to in the absence of a table of polynomial interpolation coefficients. For $N > 6$, the use of polynomial interpolation or of the sum formula (35) usually leads to the same accuracy for the same number of terms used. As a numerical example we interpolate for $P(2, \frac{1}{2}, 0.98)$ using the following values of $P_N$ for $X = 0.97$: $P_2 = 0.0856896628$, $P_1 = 0.3843028032$, $P_0 = 1$ and $P_{-1} = 1$. Here $h = Y/2 = 0.5(0.98 - 0.97) = 0.005$, and the first four terms of formula (35) yield the interpolated value 0.0871866727 which is correct to 10 decimals.

The use of property (2) and of the generalized $P_N$ in connection with the application of the function $P_N$ may take various forms in addition to the results given in this and in the previous sections. We give below a few such examples:

(i) By comparing the Laguerre polynomial series expansion of a distribution function with the equivalent incomplete gamma function expansion, Khamis (1960a) expressed the Laguerre polynomial in terms of the differences of the function $P_N$. This result is in fact more directly derivable from property (2). Thus, defining $L_n(N, b, X)$ by

$$L_n(N, b, X) \equiv (\partial P_N/\partial X)^{-1}(\partial/\partial X)^n(X^n(\partial P_N/\partial X))$$

and noting that $X^n P_N' = [\Gamma(N + n)/\Gamma(N)](\partial P_{N+n}/\partial X)$, we immediately obtain from (2) the relation

$$L_n(N, b, X) \equiv (-b)^n[\Gamma(N + n)/\Gamma(N)][\Delta^{n+1}P(N - 1, b, X)/\Delta P(N - 1, b, X)].$$

The case $N < 1$ is now defined through the generalized definition of $P_N$.

(ii) Property (2) is also useful in the inversion of series expansions in terms of Laguerre polynomials or of the equivalent series expansions in terms of incomplete gamma functions developed by Khamis (1960a). This is best done through the Taylor series expansion of the function inverse to the expanded function as outlines by Khamis (1960b). This method is particularly useful in calculating percentage points of the incomplete gamma function distribution. Again, the generalization of $P_N$ enables the use of this method for all values of $N$.

(iii) We define the ratio $R_N$ for all $N > 0$ by

$$R_N = D(N, b, X)/P(N, b, X). \tag{38}$$

In view of property (2), the ratio $R_N$ may be computed directly from a table of $P_N$ as we have

$$R_N = b(P_{N-1} - P_N)/P_N = b(P_{N-1}/P_N - 1). \tag{39}$$

For $0 < N < 1$, the recurrence relation (7) leads to the more convenient expression

$$R_N = (N/X)(1 - P_{N+1}/P_N), \tag{40}$$

which also holds for all $N > 0$. In statistical applications, however, the ratio $M_N$, known as Mills' ratio for $P_N$ and defined by

$$M_N = (1 - P_N)/D(N, b, X) \equiv (1 - P_N)/P_N R_N \tag{41}$$

is more frequently used than $R_N$ itself. In view of equations (39) to (41), we have

$$M_N = (1 - P_N)/b(P_{N-1} - P_N) \tag{42}$$

and

$$M_N = X(1 - P_N)/N(P_N - P_{N+1}). \tag{43}$$

Some authors define Mills' ratio as the reciprocal of $M_N$ and equations (42) and (43) lead to expressions for this alternative definition given by the reciprocals of the right hand sides of (42) and (43). Equation (43) is more convenient computionally for $0 < N < 1$, while (42) is more convenient for all $N \geqq 1$. In any case, equations (42) and (43) provide simpler methods for the calculation of $M_N$ or its reciprocal from tables of $P_N$ for all values of $X, N > 0$.

By analogous methods, many other similar results may be obtained which render the computational work involved in the use of the function $P_N$ much less involved. Equivalent expressions for other known results may also be obtained.

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