ON THE EFFICIENCY OF THE NORMAL SCORES TEST RELATIVE TO THE F-TEST[1]

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1. Discussion. Let $X_1, X_2, \cdots, X_m$ be independent and identically distributed with common cdf $F(x)$ and let $Y_1, Y_2, \cdots, Y_n$ be independent and identically distributed with common cdf $F(\theta x)$ where $0 < \theta < \infty$. For testing $H: \theta = 1$, Klotz [2] proposed the normal scores test and showed that the asymptotic efficiency $e_{NS,F}$ of the normal scores test relative to the F-test can take any value between .47 and ∞. It was also conjectured therein that the above efficiency can take any value between 0 and ∞. The purpose of this note is to prove the above conjecture by exhibiting a class of underlying distributions for which the efficiency tends to zero. The expression for $e_{NS,F}$ is given by (see Klotz [1])

$$e_{NS,F} = [(\beta_2 - 1)/2][\textstyle\int_{-\infty}^{\infty} (\Phi^{-1}\{F(x)\}/\varphi[\Phi^{-1}\{F(x)\}])xf^2(x)\, dx]^2$$
$$= [(\beta_2 - 1)/2]I^2 \text{ (say)},$$

where f is the density corresponding to the cdf F and β_2 is the usual kurtosis defined by

$$\beta_2 = E\{X - E(X)\}^4/[E\{X - E(X)\}^2]^2$$

with the random variable X having cdf F. Take

$$f(x) = c(|x|/(1 - |x|))^{-\alpha} \qquad 0 < \alpha < 1, \ -1 \leqq x \leqq 1,$$

where $c = 1/2\Gamma(1 + \alpha)\Gamma(1 - \alpha)$. It has been shown in [1] that $[\Phi^{-1}\{x\}]^2 \leqq [x(1 - x)]^{-\frac{1}{4}}$ so that we have

$$\begin{aligned} [\Phi^{-1}\{F(x)\}]^2 &\leqq [F(x)(1 - F(x))]^{-\frac{1}{4}} \\ &\leqq (\tfrac{1}{2})^{-\frac{1}{4}}[\textstyle\int_x^1 f(y)\, dy]^{-\frac{1}{4}} \quad \text{for} \quad x \geqq 0, \\ &\leqq 2^{\frac{1}{4}}c^{-\frac{1}{4}}(1 - x)^{-(\alpha+1)/4}/(\alpha + 1)^{-\frac{1}{4}}, \quad \text{for} \quad x \geqq 0. \end{aligned} \tag{1.1}$$

Since $f(x)$ is symmetric about zero, we have

$$\begin{aligned} I &= 2\textstyle\int_0^1 (\Phi^{-1}\{F(x)\}/\varphi[\Phi^{-1}\{F(x)\}])f(x)\{xf(x)\}\, dx \\ &= |[\Phi^{-1}\{F(x)\}]^2 xf(x)|_0^1 + \alpha c\textstyle\int_0^1 [\Phi^{-1}\{F(x)\}]^2 x^{1-\alpha}(1 - x)^{\alpha-1}\, dx \\ &\quad - (1 - \alpha)\textstyle\int_0^1 [\Phi^{-1}\{F(x)\}]^2 f(x)\, dx. \end{aligned}$$

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From (1.1) and the fact that $c \to 0$ as $\alpha \to 1$, it can be shown that $I^2 = o((1 - \alpha))$ as $\alpha \to 1$. Further, by easy evaluation, $\beta_2 = 3(4 - \alpha)(3 - \alpha)/10(1 - \alpha)(2 - \alpha)$. It follows now that $e_{NS,F} \to 0$ as $\alpha \to 1$.

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REFERENCES

[1] KLOTZ, JEROME (1960). Unpublished doctoral dissertation. Univ. of California, Berkeley.
[2] KLOTZ, JEROME (1962). Nonparametric tests for scale. *Ann. Math. Statist.* **33** 498–512.