ON MOMENT GENERATING FUNCTIONS AND RENEWAL THEORY[1]

By Charles Stone

University of California, Los Angeles

In this paper, which is complementary to [2], we discuss renewal theory for a distribution function having at least one tail that decreases exponentially fast.

Let F denote a one-dimensional right-continuous probability distribution function and f its characteristic function, defined by

$$f(\theta) = \int_{-\infty}^{\infty} e^{i\theta x}\, dF(x), \qquad -\infty < \theta < \infty.$$

Let $F^{(n)}$ denote the n-fold convolution of F with itself and H the renewal function defined by

$$H(x) = \sum_{n=0}^{\infty} F^{(n)}(x), \qquad -\infty < x < \infty.$$

We call F a lattice distribution function with lattice constant $d > 0$ if the measure corresponding to F is concentrated on the set $\{j\, d \mid -\infty < j < \infty\}$ but not on the set $\{j\, d' \mid -\infty < j < \infty\}$ for any $d' > d$. F is lattice with lattice constant d if and only if f is periodic with period $2\pi\, d^{-1}$ and $f(\theta) \neq 1$ for $-\pi\, d^{-1} \leqq \theta \leqq \pi\, d^{-1}$, $\theta \neq 0$. F is non-lattice if and only if $f(\theta) \neq 1$ for $\theta \neq 0$. As a special case, we call F strongly non-lattice if

$$\liminf_{|\theta|\to\infty} |1 - f(\theta)| > 0.$$

In the non-lattice case set $d = 0$. In general, set $[x]_d = d[x/d]$ for $d > 0$ and $[x]_d = x$ for $d = 0$.

Theorem. *Let F have finite first moment $\mu > 0$. (i) If for some $r_1 > 0$, $F(x) = o(e^{r_1 x})$ as $x \to -\infty$, then for some $r > 0$*

$$(1) \qquad H(x) = o(e^{rx}) \qquad \text{as } x \to -\infty.$$

(ii) *If F has finite second moment μ_2, if for some $r_1 > 0$, $1 - F(x) = o(e^{-r_1 x})$ as $x \to \infty$, and if F is either lattice or strongly non-lattice, then for some $r > 0$*

$$(2) \qquad H(x) = \{[x]_d + (d/2)\}/\mu + \mu_2/2\mu^2 + o(e^{-rx}) \qquad \text{as } x \to \infty.$$

Note that $[x]_d + (d/2) = x$ in the non-lattice case. Actually, in (i) we can allow $\mu = +\infty$ with no essential change in proof. The above theorem was suggested by [2] and a remark of Gelfond [1].

Proof. Let g denote the moment generating function of F defined by

$$g(s) = \int_{-\infty}^{\infty} e^{sx}\, dF(x),$$

the domain being all complex numbers s for which the integral exists absolutely. Then $g(i\theta) = f(\theta)$, $-\infty < \theta < \infty$.

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Suppose that F has finite first and second moments μ and μ_2 respectively. If for some $r_1 > 0$, $1 - F(x) = o(e^{-r_1 x})$ as $x \to \infty$, then $g(s)$ is analytic for $0 < \Re s < r_1$, continuous for $0 \leqq \Re s < r_1$, and

(3) $g(s) = 1 + \mu s + (\mu_2/2)s^2 + o(s^2)$ as $s \to \infty$ and $\Re s \geqq 0$.

If for some $r_1 > 0$, $F(x) = o(e^{r_1 x})$ as $x \to -\infty$, then $g(s)$ is analytic for $-r_1 < \Re s < 0$, continuous for $-r_1 < \Re s \leqq 0$, and

(4) $g(s) = 1 + \mu s + (\mu_2/2)s^2 + o(s^2)$ as $s \to 0$ and $\Re s \leqq 0$.

In the remainder of the paper F is assumed to have finite first moment $\mu > 0$. We first prove (ii) in the lattice case. Suppose that F has finite second moment μ_2, that for some $r_1 > 0$, $1 - F(x) = o(e^{-r_1 x})$ as $x \to \infty$, and that F is lattice with lattice constant $d = 1$ (clearly there is no loss of generality in assuming that $d = 1$). Let $P_n(k)$ denote the jump of $F^{(n)}$ at k and set

$$u_k = \sum_{n=0}^{\infty} P_n(k).$$

Then u_k and $H(k)$ are finite, $u_k = H(k) - H(k-1)$,

$$\lim_{k\to\infty} (u_k - \mu^{-1}) = 0 \quad \text{and} \quad \lim_{k\to\infty} (H(k) - k/\mu - (\mu_2 + \mu)/2\mu^2) = 0.$$

(See [2] for elementary proofs of these known results.) It follows from (20) of [2] that

(5) $u_k - \mu^{-1} = (2\pi)^{-1}\int_{-\pi}^{\pi} e^{-ik\theta}\{1/[1 - g(i\theta)] - 1/[\mu(1 - e^{i\theta})]\}\, d\theta, \quad k \geqq 0.$

Clearly $g(s + 2\pi i) = g(s)$ for $0 \leqq \Re s < r_1$. Recall that $g(i\theta) \neq 1$ unless $\theta/2\pi$ is an integer. Thus by (3) there is an r_2 such that $0 < r_2 \leqq r_1$ and $g(s) \neq 1$ for $0 < \Re s < r_2$. From (3) and (5), the continuity of g on $\Re s = 0$, Cauchy's theorem, and the Riemann-Lebesgue lemma we see that for any r such that $0 < r < r_2$

(6) $$u_k - \mu^{-1} = (2\pi)^{-1}\int_{-\pi}^{\pi} e^{-k(r+i\theta)}\{1/[1 - g(r + i\theta)] - 1/[\mu(1 - e^{r+i\theta})]\}\, d\theta = o(e^{-rk}) \quad \text{as } k \to \infty,$$

from which (2) easily follows. This completes the proof of (ii) in the lattice case.

Suppose next that F is strongly non-lattice and has finite second moment μ_2. Set

$$U(x, h) = \sum_{n=0}^{\infty} (F^{(n)}(x + h) - F^{(n)}(x)), \qquad h > 0.$$

Then $U(x, h)$ and $H(x)$ are finite, $U(x, h) = H(x + h) - H(x)$,

$$\lim_{x\to\infty} (U(x, h) - (h/\mu)) = 0 \quad \text{and} \quad \lim_{x\to\infty} (H(x) - x/\mu - \mu_2/2\mu^2) = 0.$$

(Again, see [2] for elementary proofs of these known results.) Set

$$V(x, h, a) = \int_{-\infty}^{\infty} [1/a(2\pi)^{\frac{1}{2}}]e^{-y^2/2a^2} U(x - y, h)\, dy.$$

Then by a proof similar to that of (36) in [2], we get

(7) $$V(x, h, a) = h/2\mu + (h/2\pi)\int_{-\infty}^{\infty} \Re\{e^{-ix\theta}[(1 - e^{-ih\theta})/ih\theta]e^{-a^2\theta^2/2}[1/(1 - g(i\theta))]\}\, d\theta.$$

It is easily seen that for all $r > 0$

$$(h/2\pi)\int_{-\infty}^{\infty} \Re\{e^{-ix\theta}[(1 - e^{-ih\theta})/ih\theta]e^{-a^2\theta^2/2}(i/\mu\theta)\}\, d\theta$$
$$= \pm(h/2\mu) + o(e^{-rx}) \qquad \text{as } x \to \pm\infty$$

uniformly for a and h is bounded sets.

We now suppose further that for some $r_1 > 0$, $1 - F(x) = o(e^{-r_1 x})$ as $x \to \infty$ and proceed to a proof of (ii) in the strongly non-lattice case. By supposition $g(i\theta) \neq 1$ unless $\theta = 0$ and, in fact, $g(i\theta)$ is uniformly bounded away from 1 if θ is bounded away from 0. Thus by (3) and some elementary continuity properties of g, there is an r_2 such that $0 < r_2 \leqq r_1$ and $g(s) \neq 1$ for $0 < \Re s < r_2$. Thus by (3), the continuity of g on $\Re s = 0$, Cauchy's theorem, and a form of the Riemann-Lebesgue lemma, we obtain that for any r such that $0 < r < r_2$

$$\int_{-\infty}^{\infty} e^{-ix\theta}[(1 - e^{-ih\theta})/i\theta]e^{-a^2\theta^2/2}[1/(1 - g(i\theta)) - 1/-i\mu\theta]\, d\theta$$
$$= \int_{-\infty}^{\infty} e^{-x(r+i\theta)}[(1 - e^{-h(r+i\theta)})/(r + i\theta)]e^{a^2(r+i\theta)^2/2}$$
$$\cdot[1/(1 - g(r + i\theta)) - 1/-\mu(r + i\theta)]\, d\theta$$
$$= o(e^{-rx}(1 + |\log a|)) \qquad \text{as } x \to \infty$$

uniformly for a and h in bounded sets. Combining the above, we have that for $0 < r < r_2$

$$(8) \qquad V(x, h, a) - h/\mu = o(e^{-rx}(1 + |\log a|)) \qquad \text{as } x \to \infty$$

uniformly for a and h in bounded sets.

Let N be a finite number such that $U(x, h) \leqq N$ for $-\infty < x < \infty$ and $h \leqq 2$. There is an x_0 such that, for $x \geqq x_0$, $2e^{-r_2 x} \leqq 1$ and

$$\int_{|y| \geqq x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy \leqq e^{-r_2 x}.$$

Then for $x \geqq x_0$

$$U(x + e^{-r_2 x} - y, 1 - 2e^{-r_2 x}) \leqq U(x, 1)$$
$$\leqq U(x - e^{-r_2 x} - y, 1 + 2e^{-r_2 x}), \quad |y| \leqq e^{-r_2 x},$$

and hence

$$V(x + e^{-r_2 x}, 1 - 2e^{-r_2 x}, e^{-r_2 x}/x) - Ne^{-r_2 x} \leqq U(x, 1)$$
$$\leqq (1 - e^{-r_2 x})^{-1} V(x - e^{-r_2 x}, 1 + 2e^{-r_2 x}, e^{-r_2 x}/x).$$

We now obtain from (8) that for $0 < r < r_2$

$$U(x, 1) - \mu^{-1} = o(xe^{-rx}) \qquad \text{as } x \to \infty,$$

which yields that for $0 < r < r_2$

$$(9) \qquad U(x, 1) - \mu^{-1} = o(e^{-rx}) \qquad \text{as } x \to \infty.$$

Equation (2) is a simple consequence of (9). This completes the proof of (ii).

Finally, we prove (i). Suppose that for some $r_1 > 0$, $F(x) = o(e^{r_1 x})$ as $x \to -\infty$. If we suppose additionally that F has finite second moment μ_2 and is strongly non-lattice, then by an argument similar to the preceding one we get that for some $r > 0$, $H(x) = o(e^{rx})$ as $x \to -\infty$. The general case is easily reduced to this special case. For we can always find a probability distribution function G such that $G(x) = o(e^{r_1 x})$ as $x \to -\infty$, G has positive first moment and finite second moment, G is strongly non-lattice, $G(x) \geqq F(x)$ for $-\infty < x < \infty$, and hence $G^{(n)}(x) \geqq F^{(n)}(x)$ for $-\infty < x < \infty$. Then by the above special case of (i), for some $r > 0$

$$H(x) = \sum_{n=0}^{\infty} F^{(n)}(x) \leqq \sum_{n=0}^{\infty} G^{(n)}(x) = o(e^{rx}) \qquad \text{as } x \to -\infty.$$

This completes the proof of (i).

REFERENCES

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