CONSTRUCTION OF CONFOUNDING PLANS FOR MIXED FACTORIAL DESIGNS

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1. Summary. This paper extends the use of finite fields for the construction of confounding plans, to include "asymmetrical" or "mixed" factorials. The technique employed is to define addition and multiplication of elements from distinct finite fields, by mapping these elements into a finite commutative ring containing sub-rings isomorphic to each of the fields in question. The standard and relatively simple techniques for symmetrical factorials are then applied in a straightforward manner to the asymmetrical case. The paper is concluded with a numerical example for the case of a $3^2 \times 5$ factorial design, and an outline of some possible confounding plans. Fractional factorials are discussed briefly.

2. Introduction. Galois fields, or finite fields, have been used in the past for two major purposes related to confounding of factorial designs. These are:

1. To provide a relatively simple procedure for assigning treatment combinations to blocks so as to make certain treatment effects or interactions identical with the "between block" sum of squares.
2. To provide an equally simple method for obtaining the sum of squares associated with a treatment effect or interaction.

The procedure referred to has been used for factorial designs where all of the factors have the same number of treatment levels, the so-called "symmetrical" designs. The mathematical theory for the construction of confounding plans for symmetrical factorials was first developed in a geometrical framework by Nair and Bose, and an expository article was published by Bose (1947). Later work by Kempthorne (1947) and most recently by Bailey (1959) presented these results in an analytical framework using Galois fields, rather than in the setting originally used by Bose and Nair. A series of articles at the applied level has been published by Kempthorne and Federer (1948a), and (1948b), and Federer (1949) based on the analytic approach, which is the simplest from the point of view of the applied statistician.

Geometrical methods have been successfully applied to the problem of constructing confounding plans for the "asymmetrical" or "mixed" factorial designs. This has sometimes been accomplished by the use of certain types of incomplete block designs; other times by direct geometrical procedures. A general article was published by Nair and Rao (1948); this was followed by Kramer and Bradley (1957), and Zelen (1958), both of whom used group divisible incomplete block designs to construct their confounding plans and to simplify the analysis.

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More recently, Kishen and Srivastava (1959) have developed some more general methods which are based on geometrical considerations. Publications giving concrete examples of confounding plans for asymmetrical or mixed factorials include articles by Inkson (1961), Shah (1960) and books or pamphlets by Cochran and Cox (1957), Kempthorne (1952), Li (1934) and Yates (1937).

While geometrical methods have been successfully used, the analytical method, using Galois fields, breaks down when asymmetrical factorials are considered. One reason for this is that addition and multiplication of elements coming from distinct fields are not defined. The material of this paper extends the use of analytical techniques commonly used for "symmetrical" factorials to include "asymmetrical" or "mixed" factorial designs. The objective, as before, is two-fold:

1. To provide a relatively simple method of constructing confounding plans.
2. To provide a reasonably simple method of analyzing the data.

Section 3 of the paper establishes definitions and theorems for adding and multiplying of elements from two distinct finite fields.

Section 4 includes the statistical notation and definitions to be used throughout, and then uses these definitions to extend the well-known analytical techniques to asymmetrical factorial designs for some three-way cases.

The last section (5) contains a concrete example.

3. Combining of elements from two residue classes modulo distinct prime numbers. In the following, $x(\text{mod } a)$ will sometimes be written $x(a)$. Multiplication is sometimes indicated by a dot—other times simply by juxtaposition.

DEFINITION 3.1. If residue classes of integers (mod p_1) are to be combined with those (mod p_2), then all elements of both sets of residue classes are to be considered as members of the set of residue classes of integers (mod $p_1 \cdot p_2$), where

(i) p_1, p_2 are prime numbers;

(ii) the correspondences between $i(p_1)$ or $j(p_2)$ and the integers (mod $p_1 \cdot p_2$) are determined by the following rules:

We let $\phi[i(p_1)]$ and $\phi[j(p_2)]$ be the elements (mod $p_1 \cdot p_2$) which correspond to $i(p_1)$ and $j(p_2)$, respectively. Then,

$$(3.1) \quad \text{(a)} \quad \phi[1(p_u)] = k_u(p_1p_2) \cdot p_v(p_1p_2), \qquad \text{where } k_up_v(p_1p_2) \equiv 1(p_u)$$

$$\text{(b)} \quad \phi[l(p_u)] \equiv l(p_1p_2) \cdot k_u(p_1p_2) \cdot p_v(p_1p_2),$$

where k_u is defined in (a), and

$$u, v = 1, 2, \qquad u \neq v.$$

DEFINITION 3.2. If integers (mod p_1) and (mod p_2) are to be combined, we define

$$i(p_1) + j(p_2) \equiv \phi[i(p_1)] + \phi[j(p_2)],$$

$$i(p_1) \cdot j(p_2) \equiv \phi[i(p_1)] \cdot \phi[j(p_2)].$$

As an example, we consider the combining of elements from residue classes of integers (mod 2) and (mod 3). (We let $p_1 = 2$, $p_2 = 3$.) By Definition 3.1 we consider both sets as elements in the ring of integers (mod $2 \cdot 3$), or (mod 6). For the integers (mod 3), corresponding to (mod p_2), we look at all multiples of 2 in the integers (mod 6); i.e., at 0, 2, 4. We map $1(3)$ (corresponding to $1(p_2)$) to that multiple of 2 which is congruent to $1(3)$ (i.e., $k_2 \cdot 2 \equiv 1(3)$ corresponds to $k_2 \cdot p_1 \equiv 1(p_2)$). Now, $4 \equiv 1(3)$, so $1(3)$ maps to $4(6)$. Hence $k_2 \equiv 2(6)$, although this fact is not needed for most purposes. We now map $l(3)$ to $l(6) \cdot k_2 \cdot 2(6)$. That is,

$$\begin{aligned} &0(3) \to 0(6) \cdot 4(6) \equiv 0(6) \\ &1(3) \to 1(6) \cdot 4(6) \equiv 4(6) \\ &2(3) \to 2(6) \cdot 4(6) \equiv 2(6). \end{aligned} \tag{3.2}$$

In a similar fashion,

$$\begin{aligned} &0(2) \to 0(6) \cdot 3(6) \equiv 0(6) \\ &1(2) \to 1(6) \cdot 3(6) \equiv 3(6) \end{aligned} \tag{3.3}$$

since $3(6)$ is congruent to $1(\text{mod } 2)$. We note the following results:

$$\begin{aligned} &0(2) + 0(3) \equiv 0(6) \qquad 1(2) + 0(3) \equiv 3(6) \\ &0(2) + 1(3) \equiv 4(6) \qquad 1(2) + 1(3) \equiv 1(6) \\ &0(2) + 2(3) \equiv 2(6) \qquad 1(2) + 2(3) \equiv 5(6). \end{aligned} \tag{3.4}$$

It should be further noted that $i(2) \cdot j(3) \equiv 0(6)$ for all $i(2)$ and $j(3)$.

The following Lemmas follow directly from the definition:

LEMMA 3.1. *If p_1, p_2 are distinct prime numbers greater than one, there always exist k_1, k_2 such that*

$$\begin{aligned} &\text{(i)} \quad 0 < k_1 < p_1, \qquad \text{(ii)} \quad k_1 \cdot p_2 \equiv 1(p_1), \\ &\phantom{\text{(i)} \quad} 0 < k_2 < p_2, \qquad \phantom{\text{(ii)} \quad} k_2 \cdot p_1 \equiv 1(p_2). \end{aligned} \tag{3.5}$$

From this Lemma we know that the correspondences (ii) in Definition 3.1 are always defined.

LEMMA 3.2. *The correspondences* (ii) *of Definition* 3.1 *are isomorphisms.*

DEFINITION 3.3.

$$\begin{aligned} l(p_1) + m(p_1 \cdot p_2) &= \phi[l(p_i)] + m(p_1 \cdot p_2), \\ l(p_1) \cdot m(p_1 \cdot p_2) &= \phi[l(p_i)] \cdot m(p_1 \cdot p_2), \end{aligned}$$

where ϕ is defined in Definition 3.1. $l(p_i) + l^*(p_i)$ and $l(p_i) \cdot l^*(p_i)$ are defined in the usual manner, or alternatively, as $\phi[l(p_i)] + \phi[l^*(p_i)]$ and $\phi[l(p_i)] \cdot \phi[l^*(p_i)]$. The two alternative definitions are equivalent, by Lemma 3.2.

LEMMA 3.3. *If $i \not\equiv 0(p_1)$ or $j \not\equiv 0(p_2)$, then if p_1, p_2 are distinct primes,*

$$i(p_1) + j(p_2) \not\equiv 0(p_1 \cdot p_2). \tag{3.6}$$

LEMMA 3.4. *Each distinct combination* $i(p_1)$, $j(p_2)$ *gives a distinct value for*

$$l(p_1)i(p_1) + m(p_2)j(p_2), \tag{3.7}$$

where $l \not\equiv 0(p_1)$, $m \not\equiv 0(p_2)$, *and* p_1, p_2 *are distinct primes.*

LEMMA 3.5. *If* p_1, p_2 *are distinct primes,*

$$i(p_1)\cdot j(p_2) \equiv 0(p_1\cdot p_2), \qquad \text{for all } i(p_1) \text{ and } j(p_2). \tag{3.8}$$

LEMMA 3.6. $i(p_1) + j(p_2)$ *is a divisor of zero if and only if at least one of* $i(p_1)$, $j(p_2)$ *is equal to zero.*

LEMMA 3.7. *In the residue class ring* (mod $p_1\cdot p_2$) *every element* $x(p_1\cdot p_2)$ *has a unique decomposition,*

$$x(p_1\cdot p_2) \equiv x(p_1) + x(p_2), \tag{3.9}$$

where $x(p_1)$ *and* $x(p_2)$ *are not necessarily the same integers.*

THEOREM 3.1. *Consider the system of h simultaneous linear congruences* (mod m) *in g unknowns, where m is the product of two first power primes:*

$$a_{u1}(p_1p_2)x_1(p_1p_2) + a_{u2}(p_1p_2)x_2(p_1p_2) + \cdots + a_{ug}(p_1p_2)x_g(p_1p_2) \equiv l_u(p_1p_2), \quad u = 1, 2, \cdots, h. \tag{3.10}$$

If $n(p_1p_2)$ *is an element in the residue class ring* (mod $p_1\cdot p_2$), *let*

$$n(p_1p_2) \equiv n(p_1) + n(p_2) \tag{3.11}$$

be the unique decomposition of Lemma 3.7. *Then,*

(1): *a solution to* (3.10) *exists if and only if solutions exist to each of the systems:*

$$a_{u1}(p_1)x(p_1) + a_{u2}(p_1)x_2(p_1) + \cdots + a_{ug}(p_1)x_g(p_1) \equiv l_u(p_1), \quad u = 1, 2, \cdots, h, \tag{3.12}$$

$$a_{u1}(p_2)x_1(p_2) + a_{u2}(p_2)x_2(p_2) + \cdots + a_{ug}(p_2)x_g(p_2) \equiv l_u(p_2), \quad u = 1, 2, \cdots, h. \tag{3.13}$$

(2) *Let* $(x_{1i_1}(p_1), x_{2i_1}(p_1), \cdots, x_{gi_1}(p_1))(x_{1i_2}(p_2), x_{2i_2}(p_2), \cdots, x_{gi_2}(p_2))$ *be the sets of solutions to* (3.12) *and* (3.13), *respectively, where if* $i_t \not\equiv i_t'$ (*hold t constant*), *at least one of the inequalities* $x_{1i_t} \not\equiv x_{1i_t'}(p_t)$, $x_{2i_t} \not\equiv x_{2i_t'}(p_t)$, $\cdots$, $x_{gi_t} \not\equiv x_{gi_t'}(p_t)$ *holds. Then,* $x_{1i_1i_2}(p_1p_2)$, $x_{2i_1i_2}(p_1p_2)$, $\cdots$, $x_{gi_1i_2}(p_1p_2)$ *is the set of solutions to* (3.10), *where*

$$x_{vi_1i_2}(p_1p_2) \equiv x_{vi_1}(p_1) + x_{vi_2}(p_2), \qquad v = 1, 2, \cdots, g. \tag{3.14}$$

The proof of this theorem follows directly from Lemma 3.7.

COROLLARY. *The number of solutions to* (3.10) *is equal to the product of the numbers of solutions to* (3.12) *and* (3.13). *This is known to be, for finite fields,*

$$\left(\prod_{t=1}^{2} \delta_{\rho[A(p_t)],\rho[A(p_t),X(p_t)]}\right)\cdot p_1^{g-\rho[A(p_1)]}p_2^{g-\rho[A(p_2)]}, \tag{3.15}$$

where

$$(1) \qquad \rho[C] = \textit{rank of the matrix, } C,$$

$$(2) \qquad A(p_t) = \begin{bmatrix} a_{11}(p_t) & a_{12}(p_t) & \cdots & a_{1g}(p_t) \\ \vdots & & & \vdots \\ a_{h1}(p_t) & a_{h2}(p_t) & \cdots & a_{hg}(p_t) \end{bmatrix},$$

$$(3) \qquad X(p_t) = \begin{bmatrix} l_1(p_t) \\ \vdots \\ l_h(p_t) \end{bmatrix}.$$

LEMMA 3.8. *In the ring of integers* (mod p_1p_2), *the identity element is equal to* $1(p_1) + 1(p_2)$.

PROOF. In a commutative semi-group there is, at most, one identity (see, e.g., Jacobson, (1951), p. 22). Now every element $n(p_1p_2)$ is expressible in the form $n(p_1) + n(p_2)$, by Lemma 3.7. Further, for arbitrary $n(p_1p_2)$, we have

$$(3.16) \quad [1(p_1) + 1(p_2)]\cdot[n(p_1) + n(p_2)] = n(p_1) + 1(p_2)n(p_1) + 1(p_1)n(p_2) + n(p_2) = n(p_1) + n(p_2),$$

so that $1(p_1) + 1(p_2)$ is an identity. Since, by the above, there is at most one such,

$$(3.17) \qquad 1(p_1) + 1(p_2) = 1(p_1p_2),$$

as required.

LEMMA 3.9. *If* $c(p_1p_2)$ *is not a divisor of zero, then* $c^{-1}(p_1p_2)$ *exists.*

PROOF. $c(p_1p_2) = x(p_1) + y(p_2)$ by Lemma 3.7. By Lemma 3.6, $x \not\equiv 0(p_1)$ and $y \not\equiv 0(p_2)$. Since x and y are both elements of a field, $x^{-1}(p_1)$ and $y^{-1}(p_2)$ exist. Let

$$(3.18) \qquad d(p_1p_2) = x^{-1}(p_1) + y^{-1}(p_2).$$

Then

$$(3.19) \qquad d(p_1p_2)c(p_1p_2) = [x^{-1}(p_1) + y^{-1}(p_2)][x(p_1) + y(p_2)] = 1(p_1) + 1(p_2) = 1(p_1p_2),$$

by Lemma 3.8. This proves the Lemma.

4. Three-way mixed factorials: number of levels for each factor one of two distinct primes. Having defined a method of combining elements from different sets of residue classes, we are now in a position to use the model with subscripts in a finite field ("mod" model, for short) for mixed factorials where two distinct primes are involved. Consider a three-way asymmetrical factorial design, prime number of levels for each factor. The standard model is

$$(4.1) \quad Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijk},$$

where $i = 0, 1, \cdots, p_1$, $j = 0, 1, \cdots, p_2$, $k = 0, 1, \cdots, p_3$.

DEFINITION 4.1. Define the sum of squares for main effects and interactions of model (4.1) in the conventional manner, and denote them by

(i) SS_α , SS_β , SS_γ ,

(ii) $SS_{\alpha\beta}$, $SS_{\alpha\gamma}$, $SS_{\beta\gamma}$,

(iii) $SS_{\alpha\beta\gamma}$

and let SS_{TO} be the corrected total sum of squares.

We let factors A and B have p_1 levels, with p_2 levels for C, and write the model as

$$(4.2) \qquad Y_{ijk} = \mu + \textstyle\sum_{q,r,s} A^{q(p_1)}B^{r(p_1)}C^{s(p_2)}{}_{qi(p_1)+rj(p_1)+sk(p_2)} + \epsilon_{ijk}$$

or, in vector form,

$$(4.3) \qquad Y = \mathbf{1}\mu + \sum X_{A^qB^rC^s}(A^{q(p_1)}B^{r(p_1)}C^{s(p_2)}) + \boldsymbol{\varepsilon}$$

where we wish to omit any $(A^{\bar{q}}B^{\bar{r}}C^{\bar{s}})$ such that

$$(4.4) \qquad c(\bar{q}, \bar{r}, \bar{s}) = (q, r, s)$$

if $(A^qB^rC^s)$ is also in the model, and where c is not a divisor of zero. $\mathbf{1}$ is a vector of units. The standard procedure used to confound the elements of a vector $(A^{q(p_1)}B^{r(p_1)}C^{s(p_2)})$ in model (4.3) with blocks is to require that all treatment combinations $(i(p_1), j(p_1), k(p_2))$ with the property that $qi(p_1) + rj(p_1) + sk(p_2) = h$ be assigned to one and only one block level. That is, there is a one-to-one relationship between the elements of $(A^{q(p_1)}B^{r(p_1)}C^{s(p_2)})$ and those of the vector of block levels. This implies that we would introduce a block vector into model (4.3) when we confound $(A^{\bar{q}(p_1)}B^{\bar{r}(p_1)}C^{\bar{s}(p_2)})$ in the following fashion:

$$(4.5) \quad Y = \mathbf{1}\mu + \textstyle\sum_{(q,r,s)\neq(\bar{q},\bar{r},\bar{s})} X_{A^qB^rC^s}(A^qB^rC^s) + X_{A^{\bar{q}}B^{\bar{r}}C^{\bar{s}}}[(A^{\bar{q}}B^{\bar{r}}C^{\bar{s}}) + \beta] + \boldsymbol{\varepsilon}.$$

It is conceivable that we could have two vectors $(A^{q(p_1)}B^{r(p_1)}C^{s(p_2)})$ and $(A^{\bar{q}(p_1)}B^{\bar{r}(p_1)}C^{\bar{s}(p_2)})$ in model (4.3) confounded in the sense that there is a one-to-one relationship between the elements of the two vectors. While condition (4.4) prevents this from happening, we wish to cover this possibility, in order to make the next definition more useful.

DEFINITION 4.2. In model (4.3), extended to include block vectors, two vectors will be termed "linearly and completely confounded" if there is a one-to-one mapping between their elements, and if the function defining the mapping is linear in i, j, k. Throughout the paper, this term will be shortened to "LC-confounded."

DEFINITION 4.3. With model (4.3), a confounding plan will be defined as a "linearly complete confounding plan" (LC confounding plan) if a component $A^{q(p_1)}B^{r(p_1)}C^{s(p_2)}$ is LC confounded with blocks.

DEFINITION 4.4. The terms "main effect" and "interaction" will refer to the corresponding vectors in the *standard* model (4.1).

DEFINITION 4.5. A "main effect" or "interaction component" is a vector of the form $(A^{q(p_1)}B^{r(p_1)}C^{s(p_2)})$ in the "mod" model (4.3). If all but one of q, r, s is $= 0$, it is a main effect. If more than one of q, r, s is non-zero, it is a component of the interaction obtained by deleting all letters corresponding to the zero terms.

DEFINITION 4.6. In an LC confounding plan, component $(A^{q(p_1)}B^{r(p_1)}C^{s(p_2)})$ is defined to be "expectation-wise confounded with blocks" (ϵ-confounded with blocks) if the expected value of the sum of squares corresponding to $(A^{q(p_1)}B^{r(p_1)}C^{s(p_2)})$ is a function of the elements of the block effect vector.

Definitions for the sum of squares will be given later.

DEFINITION 4.7. A sum of squares is termed "expectation-wise unconfounded with blocks," "ϵ-unconfounded," or "clean," if it is not ϵ-confounded with blocks.

LEMMA 4.1. *If there exists* $c(p_1p_2)$, *not a divisor of zero, such that*

$$c(p_1p_2)[q(p_1)i(p_1) + r(p_1)j(p_1) + s(p_2)k(p_2)] = \bar{q}(p_1)i(p_1) + \bar{r}(p_1)j(p_1) + \bar{s}(p_2)k(p_2) \tag{4.6}$$

for all $i(p_1)$, $j(p_1)$, $k(p_2)$, *then* $(A^{\bar{q}}B^{\bar{r}}C^{\bar{s}})$ *and* $(A^{q}B^{r}C^{s})$ *are* LC-*confounded.*

This lemma follows directly from Lemma 3.9.

THEOREM 4.1. *If it is required that the components* (AC), (BC), (AB^rC) *be in model* (4.1), *where* $r = 1, 2, \cdots, p_1 - 1$, *then every other interaction component involving* C *is* LC-*confounded with one of these.*

PROOF. Consider $(A^{\bar{q}}B^{\bar{r}}C^{\bar{s}})$. Either $q \neq 0$ or $r \neq 0$ (or both). Otherwise, the above is not an interaction component involving C. We have three cases:

$$\begin{aligned} &\text{(a) } \bar{q} = 1, \quad \bar{r} = 0, \\ &\text{(b) } \bar{q} = 0, \quad \bar{r} \neq 0, \\ &\text{(c) } \bar{q} = 1, \quad \bar{r} \neq 0, \qquad \text{in all cases, } \bar{s} \neq 0. \end{aligned} \tag{4.7}$$

(a) $(A^{\bar{q}}B^{\bar{r}}C^{\bar{s}}) = (AC^{\bar{s}})$. Using the inverse exhibited in the proof of Lemma 3.9 we have

$$[1(p_1) + \bar{s}^{-1}(p_2)]\cdot[i(p_1) + \bar{s}(p_2)j(p_2)] = i(p_1) + j(p_2), \tag{4.8}$$

which is the linear function associated with AC. Since $1(p_1) + \bar{s}^{-1}(p_2)$ is not a divisor of zero by Lemma 3.6, (AC^s) and (AC) are LC-confounded, by Lemma 4.1.

(b) $(A^{\bar{q}}B^{\bar{r}}C^{\bar{s}}) = (BC^{\bar{s}})$. By exactly the same reasoning as above, $(BC^{\bar{s}})$ is LC-confounded with (BC).

(c) $)A^{\bar{q}}B^{\bar{r}}C^{\bar{s}}) = (AB^{\bar{r}}C^{\bar{s}})$. Then,

$$[1(p_1) + \bar{s}^{-1}(p_2)][i(p_1) + \bar{r}j(p_1) + \bar{s}k(p_2)] = i(p_1) + \bar{r}j(p_1) + k(p_2). \tag{4.9}$$

Hence, $(AB^{\bar{r}}C\ )$ is LC-confounded with $(AB\ C)$. This completes the proof of Theorem 4.1.

This theorem shows us which of the components of model (4.3) are superfluous, and enables us to reduce our model to:

$$\begin{aligned} Y_{ijk} = \mu &+ A_{i(p_1)} + B_{j(p_1)} + AB_{i(p_1)+j(p_1)} + \cdots + AB^{p_1-1}_{i(p_1)+(p_1-1)j(p_1)} \\ &+ C_{k(p_2)} + AC_{i(p_1)+k(p_2)} + BC_{j(p_1)+k(p_2)} + ABC_{i(p_1)+j(p_1)+k(p_2)} \\ &+ \cdots + AB^{p_1-1}C_{i(p_1)+(p_1-1)j(p_1)+k(p_2)} + \epsilon_{ijk}\,. \end{aligned} \tag{4.10}$$

or in vector form,

$$Y = \mathbf{1}\mu + X_A(A) + X_B(B) + X_{AB}(AB) + \cdots + X_{AB^{p_1-1}}(AB^{p_1-1})$$
$$(4.11) \qquad + X_C(C) + X_{AC}(AC) + X_{BC}(BC) + X_{ABC}(ABC)$$
$$+ \cdots + X_{AB^{p_1-1}C}(AB^{p_1-1}C) + \boldsymbol{\varepsilon}.$$

We note that by Lemma 3.4, $X_{A^qB^rC^s}$ has $p_1^2p_2$ rows, and $p_1^{1-\delta_{0q}\delta_{0r}} \cdot p_2^{1-\delta_{0s}}$ columns.

$$X_{A^qB^rC^s} = \begin{bmatrix} & \vdots & \\ \cdots & \delta_{qi(p_1)+rj(p_1)+sk(p_2),\, h(p_1^{1-\delta_{0q}\delta_{0r}} p_2^{1-\delta_{0s}})} & \cdots \\ & \vdots & \end{bmatrix}.$$

DEFINITION 4.8. In model (4.10), define

$$Y_{(qrs)h} = \sum_R y_{ijk},$$
$$(4.12) \qquad Y_{\cdots} = \sum_{i,j,k} y_{ijk},$$
$$S_{A^qB^rC^s} = \sum_{h=0}^{p_1^{1-\delta_{0q}\delta_{0r}}p_2^{1-\delta_{0s}}} Y^2_{(qrs)h}/p_1^{1+\delta_{0q}\delta_{0r}}p_2^{\delta_{0s}} - Y^2_{\cdots}/p_1^2p_2,$$

where the conditon R under the summation sign of the first expression means that we sum over all y_{ijk} such that

$$(4.13) \qquad qi(p_1) + rj(p_1) + sk(p_2) = h(p_1^{1-\delta_{0q}\delta_{0r}}p_2^{1-\delta_{0s}}).$$

DEFINITION 4.9. Terms of the type $S_{A^qB^rC^s}$ will be called "component sum of squares" or "sum of squares corresponding to the component $(A^qB^rC^s)$."

DEFINITION 4.10. In the model

$$(4.14) \quad y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijk},$$

define SS_α, SS_β, SS_γ, $SS_{\alpha\beta}$, $\cdots$, $S_{\alpha\beta\gamma}$ in the conventional fashion, to be the sum of squares corresponding to the vector in (4.14) indicated by the greek letters. It then follows immediately, that

$$(4.15) \qquad SS_\alpha = S_A, \qquad SS_\beta = S_B, \qquad SS_\gamma = S_C.$$

THEOREM 4.2.

$$SS_{\alpha\beta} = \sum_{r=1}^{p_1-1} S_{AB^r},$$
$$(4.16) \qquad SS_{\alpha\gamma} = S_{AC} - S_A - S_C,$$
$$SS_{\beta\gamma} = S_{BC} - S_B - S_C,$$

and finally,

$$SS_{\alpha\beta\gamma} = \sum_{r=1}^{p_1-1} (S_{AB^rC} - S_{AB^r} - S_C)$$
$$= \sum_{r=1}^{p_1-1} S_{AB^rC} - \sum_{r=1}^{p_1-1} S_{AB^r} - (p_1 - 1)S_C.$$

The proofs for second-order interactions are straightforward. To prove the result for the third-order interaction, we need the following

LEMMA 4.2.

(4.17) $X_{BC}X'_{BC} + \sum_{r=1}^{p_1-1} X_{AB^rC}X'_{AB^rC} + X_{AC}X'_{AC} - X_CX_C' = p_1I.$

PROOF. Using (4.15), some algebraic manipulations yield

$$X_{A^qB^rC^s}X'_{A^qB^rC^s} = \begin{bmatrix} & \vdots & \\ \cdots & \delta_{qi+rj+sk,\, qi^*+rj^*+sk^*} & \cdots \\ & \vdots & \end{bmatrix}. \tag{4.18}$$

Hence, (4.17) equals

$$\begin{bmatrix} & \vdots & \\ \cdots & \sum_{q,r} \delta_{qi+rj+k,\, qi^*+rj^*+k^*} - \alpha_{ijk,\, i^*j^*k^*} & \cdots \\ & \vdots & \end{bmatrix}, \tag{4.19}$$

where $\alpha_{ijk,i^*j^*k^*} = \delta_{k,k^*}$ and subject to the condition that no vector (q, r) is a non-divisor of zero times any other vector, and (0, 0) is excluded. Since

$$\delta_{qi+rj+k,qi^*+rj^*+k^*} = 1 \tag{4.20}$$

if and only if

$$q(i - i^*)(p_1) + r(j - j^*)(p_1) = (k^* - k)(p_2), \tag{4.21}$$

$\sum_{q,r} \delta_{qi+rj+k,qi^*+rj^*+k^*}$ equals the number of non-trivial solutions of (4.21). We put (4.21) in the form of Theorem 3.1:

$$\begin{aligned} &[(i - i^*)(p_1) + 0(p_2)]\cdot q(p_1p_2) \\ &\quad + [(j - j^*)(p_1) + 0(p_2)]\cdot r(p_1p_2) = 0(p_1) + (k^* - k)(p_2), \\ &\qquad [0(p_1) + 1(p_2)]\cdot q(p_1p_2) = 0(p_1) + 0(p_2), \\ &\qquad [0(p_1) + 1(p_2)]\cdot r(p_1p_2) = 0(p_1) + 0(p_2). \end{aligned} \tag{4.22}$$

By the corollary to Theorem 3.1, the number of solutions to (4.22) equals the product of the numbers of solutions of the systems:

$$(i - i^*)q(p_1) + (j - j^*)r(p_1) = 0(p_1), \tag{4.23}$$

$$\begin{aligned} 0\cdot q(p_2) + 0\cdot r(p_2) &= (k^* - k)(p_2), \\ 1\cdot q(p_2) &= 0(p_2), \\ 1\cdot r(p_2) &= 0(p_2). \end{aligned} \tag{4.24}$$

These are $1 + \delta_{ii^*}\delta_{jj^*}p_1$ and δ_{kk^*}, respectively;
Hence,

$$\sum_{q,r} \delta_{qi+rj+k,qi^*+rj^*+k^*} = \delta_{kk^*} + \delta_{ii^*}\delta_{jj^*}\delta_{kk^*}p_1 . \tag{4.25}$$

Thus,

$$X_{BC}X'_{BC} + \sum_{r=1}^{p_1-1} X_{AB^rC}X'_{AB^rC} + X_{AC}X'_{AC} - X_CX_C'$$

$$= \begin{bmatrix} & \vdots & \\ \cdots & (\delta_{kk^*} + \delta_{ii^*}\delta_{jj^*}\delta_{kk^*}p_1) - (\delta_{kk^*}) & \cdots \\ & \vdots & \end{bmatrix} = p_1 I,$$

where I is a $p_1^2p_2$-square matrix. This is the required result, for Lemma 4.2.

We apply this result to establish the last identity of Theorem 4.2. From Lemma 4.2,

$$S_{BC} + \sum_{r=1}^{p_1-1} S_{AB^rC} + S_{AC} - p_1 S_C = SS_{TO}\,. \tag{4.26}$$

Using the identities of Theorem 4.2 for the second-order interactions, we get

$$SS_{\alpha\beta\gamma} = SS_{TO} - \sum_r S_{AB^r} - (S_{AC} - S_A - S_C) - (S_{BC} - S_B - S_C) - S_A - S_B - S_C\,. \tag{4.27}$$

Using (4.26), this becomes

$$SS_{\alpha\beta\gamma} = \sum_{r=1}^{p_1-1} S_{AB^rC} - \sum_{r=1}^{p_1-1} S_{AB^r} - (p_1 - 1)S_C = \sum_{r=1}^{p_1-1} (S_{AB^rC} - S_{AB^r} - S_C), \tag{4.28}$$

which was the required identity.

THEOREM 4.3. *The following statistics are mutually independent*:

(i) $S_A,\ S_B,\ S_C,$
(ii) $S_{AC} - S_A - S_C,\ S_{BC} - S_B - S_C,\ S_{AB^r},\qquad r = 1, \cdots, p_1 - 1,$
(iii) $(S_{AB^rC} - S_{AB^r} - S_C),\qquad r = 1, \cdots, p_1 - 1.$

PROOF. Let $D_{A^qB^rC^s}$ be the matrix associated with the quadratic form

$$S_{A^qB^rC^s} - (1 - \delta_{0q}\delta_{0r})(1 - \delta_{0s})[S_{A^qB^r} + S_{C^s}], \tag{4.29}$$

which is the general form for the expressions of Theorem 4.3. From Theorem 4.2,

$$\sum_{q,r,s} D_{A^qB^rC^s} = I - J/p_1^2 p_2\,, \tag{4.30}$$

where J is a $p_1^2p_2$-square matrix of units and algebraic manipulation verifies that all the $D_{A^qB^rC^s}$ are idempotent. But then, by Theorem 1.68 [Graybill, (1961)]

$$D_{A^qB^rC^s}D_{A^{\bar q}B^{\bar r}C^{\bar s}} = 0, \quad \text{if} \quad (q, r, s) \neq (\bar q, \bar r, \bar s). \tag{4.31}$$

This, in turn, establishes the independence of $Y'D_{A^{\bar q}B^{\bar r}C^{\bar s}}Y$ and $Y'D_{A^qB^rC^s}Y$ by Theorem 4.10 [Graybill, (1961)].

THEOREM 4.4. *If* $(A^qB^rC^s)$ *is* LC-*confounded, then* (A^qB^r) *and* (C^s) *are* ϵ-*confounded.*

PROOF. To confound $(A^qB^rC^s)$ with blocks means that all y_{ijk} such that $qi(p_1) + rj(p_1) + sk(p_2) = h$ are assigned to the hth block. If q, r are both $= 0$, or if $s = 0$, the theorem is trivially true. Consider the case for which at least one of q, r is $\neq 0$, and $s = 1$. Since for each level of $h(p_1p_2)$, there is a unique value of $qi(p_1) + rj(p_1)$ and of $sk(p_2)$ such that $qi(p_1) + rj(p_1) + sk(p_2) = h(p_1p_2)$, every change in the level of $qi(p_1) + rj(p_1)$ or of $sk(p_2)$ results in a different level of h. Hence, no two levels of (A^qB^r), or of (C^s) are assigned to the same block. But if each level of a vector component is assigned to a different block, then the sum of squares corresponding to that component is ϵ-confounded. This completes the argument for Theorem 4.4.

THEOREM 4.5. *Consider a mixed three-way factorial, prime number of levels for each factor and two distinct primes. If* $AB^{r^*}C$ *is completely and linearly confounded* *with blocks, then the following sum of squares are ϵ-unconfounded with block effects:*

(a) $S_{AB^rC} - S_{AB^r} - S_C$, $\quad r \neq r^*$,
(b) S_A,
(c) S_B,
(d) $S_{AC} - S_A - S_C$,
(e) $S_{BC} - S_B - S_C$,
(f) S_{AB^r}, $\quad r \neq r^*$.

PROOF. The model with $AB^{r^*}C$ confounded is

$$(4.32)\quad Y = \textstyle\sum_{q,r,s} X_{A^qB^rC^s}(A^qB^rC^s) + X_{AB^{r^*}C}[(AB^{r^*}C) + \beta] + \boldsymbol{\varepsilon}, \qquad (q, r, s) \neq (1, r^*, 1).$$

We wish the expected value of $S_{A^qB^rC^s} - (1 - \delta_{0q}\delta_{0r})(1 - \delta_{0s})(S_{A^qB^r} - S_C)$, where

$$(4.33)\qquad \begin{array}{ll} \text{(i)} & (q, r, s) \neq (1, r^*, 0), \\ \text{(ii)} & (q, r, s) \neq (0, 0, 1), \\ \text{(iii)} & (q, r, s) \neq (1, r^*, 1). \end{array}$$

We have two cases to consider:

$$\text{(a)}\quad (1 - \delta_{0q}\delta_{0r}) = 1, \qquad (1 - \delta_{0s}) = 0,$$
$$\text{(b)}\quad (1 - \delta_{0q}\delta_{0r}) = 1, \qquad (1 - \delta_{0s}) = 1.$$

CASE (a). We wish the expected value of $S_{A^qB^rC^0}$, where $(q, r, 0) \neq (1, r^*, 0)$. Let $V = \{(q, r, s): (q, r, s) \neq (1, r^*, 1)\}$ and $\bar{V} = \{(\bar{q}, \bar{r}, \bar{s}): (\bar{q}, \bar{r}, \bar{s}) \neq (1, r^*, 1)\}$.

$$\begin{aligned} E(S_{A^qB^r}) &= EY'(X_{A^qB^r}X'_{A^qB^r}/p_1p_2 - J/p_1p_2)Y = E(Y'K_{A^qB^r}Y) \\ &= E\{\epsilon'K_{A^qB^r}\epsilon + \textstyle\sum_V \sum_{\bar{V}} (A^qB^rC^s)'X'_{A^qB^rC^s}K_{A^qB^r}X_{A^{\bar{q}}B^{\bar{r}}C^{\bar{s}}}(A^{\bar{q}}B^{\bar{r}}C^{\bar{s}}) \\ (4.34)\qquad &\quad + \textstyle\sum_V (A^qB^rC^s)'X'_{A^qB^rC^s}K_{A^qB^r}X_{AB^{r^*}C}[(AB^{r^*}C) + \beta] \\ &\quad + \textstyle\sum_{\bar{V}} [(AB^{r^*}C) + \beta]'X'_{AB^{r^*}C}K_{A^qB^r}X_{A^{\bar{q}}B^{\bar{r}}C^{\bar{s}}}(A^{\bar{q}}B^{\bar{r}}C^{\bar{s}}) \\ &\quad + [(AB^{r^*}C) + \beta]'X'_{AB^{r^*}C}K_{A^qB^r}X_{AB^{r^*}C}[(AB^{r^*}C) + \beta]\}. \end{aligned}$$

We propose to show that

$$(4.35)\qquad X'_{AB^{r^*}C}K_{A^qB^r} = 0,$$

when conditions (4.33) hold. If this is true, then all terms in (4.34) involving the vector, β, of block main effects, will vanish, and the theorem will have been established. Now,

$$(4.36)\quad X'_{AB^{r^*}C}K_{A^qB^r} = X'_{AB^{r^*}C}[X_{A^qB^r}X'_{A^qB^r}/p_1p_2 - J/p_1^2p_2]$$
$$= \begin{bmatrix} & \vdots & \\ \cdots & \delta_{l,i+r^*j+k} & \cdots \\ & \vdots & \end{bmatrix} \cdot \begin{bmatrix} & \vdots & \\ \cdots & \delta_{qi+rj,\,q\bar{i}+r\bar{j}}/p_1p_2 - 1/p_1^2p_2 & \cdots \\ & \vdots & \end{bmatrix}$$

$$= \left[\cdots \left(\sum\nolimits_{i,j} \delta_{l(p),\, i+r^*j} \cdot \delta_{qi+rj,\, q\bar{i}+r\bar{j}}\right) J/p_1 p_2 \right.$$

$$\left. - \left(\sum\nolimits_{i,j} \delta_{qi+rj,\, q\bar{i}+r\bar{j}}\right) J/p_1^2 p_2 \cdots \right].$$

Now, $\sum_{i,j} \delta_{l(p_1),i(p_1)+r^*j(p_1)} \cdot \delta_{qi(p_1)+rj(p_1),q\bar{i}(p_1)+r\bar{j}(p_1)}$ is equal to the number of solutions of

$$i + r^*j = l(p_1), \qquad (4.37)$$
$$qi + rj = (q\bar{i} + r\bar{j})(p_1).$$

By (4.33), $(q, r) \neq (1, r^*)$. If $(q, r) = c(1, r^*)$, then since $q \neq 0$, $q = c = 1$, and hence, $(q, r) = (1, r^*)$, a contradiction. Thus, the rank of the matrix corresponding to the left hand side of (4.37) is two. Hence, there is exactly one solution to (4.37), so that

$$\sum\nolimits_{i,j} \delta_{l(p_1),i+r^*j} \cdot \delta_{qi+rj,q\bar{i}+r\bar{j}} = 1. \qquad (4.38)$$

Next, $\sum_{i,j} \delta_{qi+rj,q\bar{i}+r\bar{j}}$ equals the number of solutions of

$$qi + rj = (q\bar{i} + r\bar{j})(p_1), \qquad (4.39)$$

which is p_1. Thus

$$X'_{AB^{r*}C} K_{AB^r} = J/p_1p_2 - p_1J/p_1^2p_2 = 0, \qquad (4.40)$$

which was to be proved, for Case (a). Case (b) is proved similarly.

5. An example. This section will apply the theorems of the preceding sections to a three-way mixed factorial. Table 1 gives some of the possible ways of constructing confounding plans with this method, giving the effects confounded, the appropriate block sizes, etc. The data is taken from a $3 \times 4 \times 5$ experiment reported in Davies (1954), Table 8.1, with the fourth level deleted from the four-level factor, to make a $3^2 \times 5$ experiment. Artificial block effects are introduced, after treatment combinations are allocated for confounding, to indicate which effects remain "clean." The block effects were $-70, \cdots, -10, 0, +10, \cdots, +70$ for the 15 blocks.

The "modular" model for this data is

$$Y_{ijk} = \mu + A_{i(3)} + B_{j(3)} + AB_{i(3)+j(3)} + AB^2_{i(3)+2j(3)} + C_{k(5)} + AC_{i(3)+k(5)}$$
$$+ BC_{j(3)+k(5)} + ABC_{i(3)+j(3)+k(5)} + AB^2C_{i(3)+2j(3)+k(5)} + \epsilon_{ijk}.$$

We need to write out the complete vector of Y_{ijk}'s, with the corresponding "modular" components, in order to assign treatment combinations to blocks. Rather than write out the entire model each time (i.e., $Y_{213} = \mu + A_{2(3)} + B_{1(3)} + AB_{0(3)} + \cdots$ etc.), the subscripts only will be used, and the component given at the head of the column, as in Table 4.1.

TABLE 4.1

The complete model for a $3^2 \times 5$ experiment
(mod numbers omitted to conserve space)

Y_{ijk}	A_i	B_i	AB_{i+j}	AB^2_{i+2j}	C_k	AC_{i+k}	BC_{j+k}	ABC_{i+j+k}	AB^2C_{i+2j+k}
000	0	0	0	0	0	0	0	0	0
001	0	0	0	0	1	6	6	6	6
002	0	0	0	0	2	12	12	12	12
003	0	0	0	0	3	3	3	3	3
004	0	0	0	0	4	9	9	9	9
010	0	1	1	2	0	0	10	10	5
011	0	1	1	2	1	6	1	1	11
012	0	1	1	2	2	12	7	7	2
013	0	1	1	2	3	3	13	13	8
014	0	1	1	2	4	9	4	4	14
020	0	2	2	1	0	0	5	5	10
021	0	2	2	1	1	6	11	11	1
022	0	2	2	1	2	12	2	2	7
023	0	2	2	1	3	3	8	8	13
024	0	2	2	1	4	9	14	14	4
100	1	0	1	1	0	10	0	10	10
101	1	0	1	1	1	1	6	1	1
102	1	0	1	1	2	7	12	7	7
103	1	0	1	1	3	13	3	13	13
104	1	0	1	1	4	4	9	4	4
110	1	1	2	0	0	10	10	5	0
111	1	1	2	0	1	1	1	11	6
112	1	1	2	0	2	7	7	2	12
113	1	1	2	0	3	13	13	8	3
114	1	1	2	0	4	4	4	14	9
120	1	2	0	2	0	10	5	0	5
121	1	2	0	2	1	1	11	6	11
122	1	2	0	2	2	7	2	12	2
123	1	2	0	2	3	13	8	3	8
124	1	2	0	2	4	4	14	9	14
200	2	0	2	2	0	5	0	5	5
201	2	0	2	2	1	11	6	11	11
202	2	0	2	2	2	2	12	2	2
203	2	0	2	2	3	8	3	8	8
204	2	0	2	2	4	14	9	14	14
210	2	1	0	1	0	5	10	0	10
211	2	1	0	1	1	11	1	6	1
212	2	1	0	1	2	2	7	12	7
213	2	1	0	1	3	8	13	3	13
214	2	1	0	1	4	14	4	9	4
220	2	2	1	0	0	5	5	10	0
221	2	2	1	0	1	11	11	1	6
222	2	2	1	0	2	2	2	7	12
223	2	2	1	0	3	8	8	13	3
224	2	2	1	0	4	14	14	4	9

TABLE 4.2

Allocation of treatment combinations to blocks for confounding of AB^2C component of model (4.10)

Block No.				
0	1	2	3	4
000	021	202	003	024
110	101	012	113	104
220	211	122	223	214
5	6	7	8	9
010	001	022	203	004
120	111	102	013	114
200	221	212	123	224
10	11	12	13	14
020	011	002	023	014
100	121	112	103	124
210	201	222	213	204

TABLE 4.3

"Modular" analysis of variance for data from Davies (1954), Table 8.1

Source	df	Formulas	Sum of Squares Without Artificial Block Effects	Sum of Squares With Artificial Block Effects
$A = \alpha$	2	S_A	90,141.91	90,141.91
$B = \beta$	2	S_B	40,513.38	40,513.38
$\alpha\beta$	4	$S_{AB} + S_{AB^2}$	4,102.49	9,262.49
AB	2	S_{AB}	3,312.58	3,312.58
AB^2	2	S_{AB^2}	789.91	5,949.91
$C = \gamma$	4	S_C	360,719.34	271,419.34
$AC = \alpha\gamma$	8	$S_{AC} - S_A - S_C$	40,567.86	40,535.20
$BC = \beta\gamma$	8	$S_{BC} - S_B - S_C$	12,267.06	12,267.06
$\alpha\beta\gamma$	16	$(S_{ABC} - S_{AB} - S_C) + (S_{AB^2C} - S_{AB^2} - S_C)$	6,492.39	73,245.72
ABC	8	$S_{ABC} - S_{AB} - S_C$	2,705.86	2,719.19
AB^2C	8	$S_{AB^2C} - S_{AB^2} - S_C$	3,786.53	70,526.53

We propose to confound AB^2C in blocks. The block size will be $=3$, with 15 blocks. We assign to the lth block those treatment combinations in Table 2 whose element in (AB^2C) is AB^2C_l. The blocks appear in Table 4.2. Table 4.3 provides the numerical analysis of variance for purposes of comparison.

Note that as indicated in Theorem 4.4, AB^2 and C are confounded when AB^2C is and that everything else remains "clean," to within round-off error.

6. Discussion. The material of this paper can be generalized in a straightforward manner to the case of n-way factorials, with the number of levels of each

TABLE 4.4

Possible confounding plans for a $3^2 \times 2^2$

	Effects Confounded With Blocks	Other Effects Also Confounded	No. of Blocks
1	A	—	3
2	B	—	3
3	AB	—	3
4	AB^2	—	3
5	C	—	2
6	AC	A, C	6
7	BC	B, C	6
8	ABC	AB, C	6
9	AB^2C	AB^2, C	6
10	D	—	2
11	AD	A, D	6
12	BD	B, D	6
13	ABD	AB, D	6
14	AB^2D	AB^2, D	6
15	CD	—	2
16	ACD	A, CD	6
17	BCD	B, CD	6
18	$ABCD$	AB, CD	6
19	AB^2CD	AB^2, CD	6

factor one of k distinct prime numbers. As a matter of fact, Theorem 4.4 indicates that less obvious confounding plans which leave main effects clean are only encountered when fourth-order factorials or higher are considered. If we consider a $2^2 \times 3^2$ factorial, the possible plans are as in Table 4.4. Plans 18 and 19 leave main effects clean and may be of some interest. While there are not a lot of plans available in this case, the number of useful plans increases with the number of levels of the factors and also with the number of factors. The experience of other investigators (see, e.g., Kempthorne (1952) p. 348) seems to indicate the paucity of plans which do not confound main effects if the number of levels of the factors are relatively prime. Double confounding is, of course, also possible; it should be noted that in the $3^2 \times 2^2$ example, confounding of AB^2CD, for instance, is equivalent to the double confounding of AB^2 and CD. Generalized interactions are obtained using the definitions of Section 3 of the paper.

Fractional replications are easy to obtain; the alias identities for the $3^2 \times 2^2$ case are given in Table 4.4. The computations are virtually as simple as in the symmetrical case.

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REFERENCES

[1] BAILEY, N. T. J. (1959). The use of linear algebra in deriving prime-power factorial designs with confounding and fractional replications. *Sankhyā* **21** 345–355.

[2] BOSE, R. C. (1947). Mathematical theory of the symmetric factorial design. *Sankhyā* **8** 107–166.

[3] BROWNE, E. T. (1958). *Introduction to the Theory of Determinants and Matrices.* Univ. of North Carolina Press, Chapel Hill.

[4] COCHRAN, W. G. and COX, G. B. (1957). *Experimental Designs* (2nd edition). Wiley, New York.

[5] DAVIES, O. L. (1954). *The Design and Analysis of Industrial Experiments.* Hafner, New York.

[6] FEDERER, W. T. (1949). The general theory of prime-power lattice designs, III. *Biometrics* **5** 144–161.

[7] GRAYBILL, F. A. (1961). *An Introduction to Linear Statistical Models* **1**. McGraw-Hill, New York.

[8] INKSON, R. H. E. (1961). The analysis of a $3^2 \times 2^2$ factorial experiment with confounding. *J. Appl. Statist.* **10** 98–107.

[9] JACOBSON, N. (1951). *Lectures in Abstract Algebra* **1**. Van Nostrand, Princeton.

[10] KEMPTHORNE, O. (1947). A simple approach to confounding and fractional replication in factorial experiments. *Biometrika* **34** 255–272.

[11] KEMPTHORNE, O. and FEDERER, W. T. (1948a). The general theory of prime-power lattice designs, I. *Biometrics* **4** 54–79.

[12] KEMPTHORNE, O. and FEDERER, W. T. (1948b). The general theory of prime-power lattice designs, II. *Biometrics* **4** 109–121.

[13] KEMPTHORNE, O. (1952). *The Design and Analysis of Experiments.* Wiley, New York.

[14] KISHEN, N. and SRIVASTAVA, J. N. (1959). Mathematical theory of confounding in asymmetrical and symmetrical factorial designs. *J. Indian Soc. Agric. Statist.* **11** 73–110.

[15] KRAMER, C. Y. and BRADLEY, R. A. (1957). Intra-block analysis for factorials in two-associate class group divisible designs. *Ann. Math. Statist.* **28** 349–361.

[16] LEVEQUE, W. J. (1962). *Elementary Theory of Numbers.* Addison-Wesley, Reading.

[17] LI, J. (1934). Design and statistical analysis of some confounded factorial experiments. *Iowa Agric. Exp. Sta. Res. Bull.* 333.

[18] NAIR, K. R. and RAO, C. R. (1948). Confounding in asymmetrical factorial experiments. *J. Roy. Statist. Soc. Ser. B* **10** 109–131.

[19] SHAH, B. V. (1960). On a 5×2^2 factorial design. *Biometrics* **16** 115–118.

[20] VAN DER WAERDEN, B. L. (1949). *Modern Algebra* (2nd ed.). Ungar, New York. (English translation.)

[21] YATES, F. (1937). *The Design and Analysis of Factorial Experiments.* Imperial Bureau of Soil Science. Harpenden, England.

[22] ZELEN, M. (1958). The use of group divisible designs for confounded asymmetrical factorial arrangements. *Ann. Math. Statist.* **29** 22–40.