ESTIMATION OF THE BISPECTRUM[1]

BY M. ROSENBLATT AND J. W. VAN NESS

University of California, San Diego, and Stanford University

1. Summary. Recently interest has arisen in statistical applications of the bispectrum of stationary random processes. (The bispectrum can be thought of as the Fourier transform of the third-order moment function of the process.) The principal area of statistical harmonic analysis to receive attention previous to this time has been second-order (i.e. spectral) theory on which there is a vast literature. However, the spectrum is most useful in problems of a "linear nature" (see discussion beginning on p. viii of Blackman and Tukey [2]) and provides insufficient information in nonlinear problems.

A desire to study phenomena of a nonlinear character has attracted attention to the higher order theory. Such was the case, for example, in a recent study by Hasselmann, Munk and MacDonald [8] where the bispectrum is used in connection with oceanographic problems, among which, as the authors state, a number of interesting phenomena such as surf beats, wave breaking, and the energy transfer between wave components can be explained only by the nonlinearity of the wave motion.

The bispectrum therefore provides a first glimpse at the nonlinear effects. It is the purpose of the present paper to discuss estimating the weighted and unweighted bispectral density given a set of observations of the process. The relevant properties (consistency and asymptotic unbiasedness) of the estimates are derived for certain general classes of processes.

2. Introduction. The indexed set of random variables, $\{X_t\}$, is called a continuous parameter random process if the index set, T, is the real line and it is called discrete parameter if T is the positive and negative integers. For our purposes we take X_t to have mean zero and finite sixth-order moments and to be real-valued sixth-order weakly stationary all of which implies that for all t,

$$\begin{aligned} Ex_t &= 0, \\ EX_tX_{t+\nu} &= m_2(t, t+\nu) = r(\nu), \\ EX_tX_{t+\nu_1}X_{t+\nu_2} &= m_3(t, t+\nu_1, t+\nu_2) = r_3(\nu_1, \nu_2), \qquad (2.1) \\ &\vdots \\ EX_tX_{t+\nu_1} \cdots X_{t+\nu_5} &= m_6(t, t+\nu_1, \cdots, t+\nu_5) = r_6(\nu_1, \cdots, \nu_5). \end{aligned}$$

Consider $r(\nu)$ and $r_3(\nu_1, \nu_2)$ to be in L_1 (or l_1) and in the continuous parameter case to be continuous. It is well known from second-order harmonic theory that

Received 24 June 1964; revised 17 February 1965.

[1] This research was supported in part by the Office of Naval Research under Contract Nonr 562(29) at Brown University.

$r(\nu)$ has a Fourier-Stieltjes representation in terms of the spectral distribution function, $F(\lambda)$. Take $F(\lambda)$ to be absolutely continuous with a continuous density, $f(\lambda)$; then this representation becomes

$$r(\nu) = \int_{-\infty}^{\infty} e^{i\nu\lambda} f(\lambda)\, d\lambda. \tag{2.2}$$

(Throughout this paper, all formulae shall be written for the continuous parameter case; the discrete case can be obtained by easy modifications.) Further

$$f(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\nu\lambda} r(\nu)\, d\nu. \tag{2.3}$$

Analogously, in third-order harmonic theory we can define the bispectral density function, $g(\lambda_1, \lambda_2)$, as

$$g(\lambda_1, \lambda_2) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\,(-i\nu_1\lambda_1 - i\nu_2\lambda_2) r_3(\nu_1, \nu_2)\, d\nu_1\, d\nu_2 \tag{2.4}$$

and, assuming $g(\lambda_1, \lambda_2)\ \varepsilon\ L_1(R_2)$,

$$r_3(\nu_1, \nu_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\,(i\nu_1\lambda_1 + i\nu_2\lambda_2) g(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2\,. \tag{2.5}$$

(For a discussion of the existence of the higher order spectral representations under some conditions, see Blanc-Lapierre and Fortet [3] and Sinai [20].) This then is the harmonic representation of the third-order moment function under the above assumptions.

Since the process is real, the following symmetries occur in the third-order functions

$$r_3(\nu_1, \nu_2) = r_3(\nu_2, \nu_1) = r_3(-\nu_1, \nu_2 - \nu_1), \tag{2.6}$$

$$g(\lambda_1, \lambda_2) = g(\lambda_2, \lambda_1) = g(\lambda_1, -\lambda_1 - \lambda_2) = \overline{g(-\lambda_1, -\lambda_2)}. \tag{2.7}$$

The symmetries (2.6) imply that $r_3(\nu_1, \nu_2)$ is completely specified over the entire plane by its values in any *one* of the six sectors, ⓐ through ⓕ, shown in Fig. 1. These sectors include their boundaries so that, for example, sector ⓐ is

$$0 \leqq \nu_1 < \infty, \quad 0 \leqq \nu_2 \leqq \nu_1\,. \tag{2.8}$$

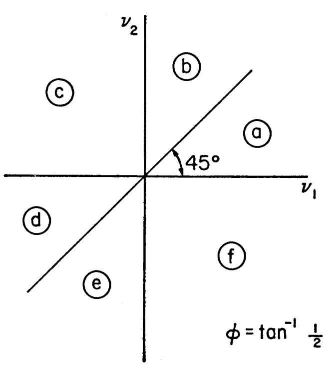

FIG. 1

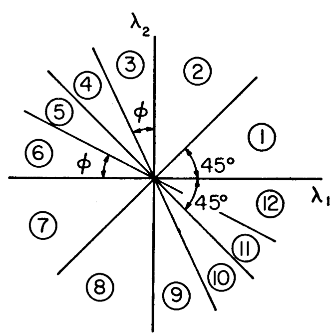

FIG. 2

Similarly $g(\lambda_1, \lambda_2)$ is, by (2.7), completely specified by its values in any one of the twelve sectors (including boundaries) shown in Fig. 2. A later paper will discuss the slightly more complicated situation which arises in the discrete parameter case.

Recall from second-order theory that the process permits the representation

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda}\, dZ(\lambda), \tag{2.9}$$

where $Z(\lambda)$ is a random process of orthogonal increments with (in the case of a real process)

$$\begin{aligned} E\, dZ(\lambda) &= 0, \\ E\, dZ(\lambda_1)\, dZ(\lambda_2) &= \delta(\lambda_1 + \lambda_2)\, dF(\lambda_1), \\ \delta(x) &= 1 \quad \text{if } x = 0 \\ &= 0 \quad \text{if } x \neq 0. \end{aligned} \tag{2.10}$$

Equation (2.10) yields an interpretation of the spectrum. When there exists a Fourier-Stieltjes representation of the third-order moment function in terms of a complex-valued bispectral distribution function of bounded variation, a similar result holds:

$$E\, dZ(\lambda_1)\, dZ(\lambda_2)\, dZ(\lambda_3) = \delta(\lambda_1 + \lambda_2 + \lambda_3)\, dG(\lambda_1, \lambda_2). \tag{2.11}$$

More particularly a real process (implies $dZ(\lambda) = \overline{dZ(-\lambda)}$) has the real representation

$$X_t = \int_0^{\infty} \cos t\lambda\, dZ_1(\lambda) + \int_0^{\infty} \sin t\lambda\, dZ_2(\lambda) \tag{2.12}$$

where $dZ_1(\lambda) = 2 \operatorname{Re} dZ(\lambda)$ and $dZ_2(\lambda) = -2 \operatorname{Im} dZ(\lambda)$. It is interesting that these functions allow us to obtain the following insight into the role of the real and imaginary parts of the bispectrum. In these relations note that when $E\, dZ_{j_1}(\lambda_1)\, dZ_{j_2}(\lambda_2)\, dZ_{j_3}(\lambda_3)$, $j_i = 1, 2$, is nonzero, it is equal to some multiple of the real part of dG if an *odd* number of the j_i's equal 1 and equal to some multiple of the imaginary part of dG if an *even* number of the j_i's equal 1.

$$\begin{aligned} &E\, dZ_1(\lambda_1)\, dZ_1(\lambda_2)\, dZ_1(\lambda_3) \\ &\quad = 2 \operatorname{Re} [dG(\lambda_1, \lambda_2)\delta(\lambda_1 + \lambda_2 - \lambda_3) + dG(\lambda_1, \lambda_3)\delta(\lambda_1 - \lambda_2 + \lambda_3) \\ &\qquad + dG(\lambda_2, \lambda_3)\delta(-\lambda_1 + \lambda_2 + \lambda_3) + dG(0, 0)\delta(\lambda_1)\delta(\lambda_2)\delta(\lambda_3)] \\ &E\, dZ_2(\lambda_1)\, dZ_2(\lambda_2)\, dZ_2(\lambda_3) \\ &\quad = -2 \operatorname{Im} [dG(\lambda_1, \lambda_2)\delta(\lambda_1 + \lambda_2 - \lambda_3) + dG(\lambda_1, \lambda_3)\delta(\lambda_1 - \lambda_2 + \lambda_3) \\ &\qquad + dG(\lambda_2, \lambda_3)\delta(-\lambda_1 + \lambda_2 + \lambda_3)] \\ &E\, dZ_1(\lambda_1)\, dZ_1(\lambda_2)\, dZ_2(\lambda_3) \\ &\quad = 2 \operatorname{Im} [dG(\lambda_1, \lambda_2)\delta(\lambda_1 + \lambda_2 - \lambda_3) - dG(\lambda_1, \lambda_3)\delta(\lambda_1 - \lambda_2 + \lambda_3) \end{aligned} \tag{2.13}$$

$$- dG(\lambda_2, \lambda_3)\delta(-\lambda_1 + \lambda_2 + \lambda_3)]$$

$$E\, dZ_1(\lambda_1)\, dZ_2(\lambda_2)\, dZ_2(\lambda_3)$$

$$= 2 \operatorname{Re} [dG(\lambda_1, \lambda_2)\delta(\lambda_1 + \lambda_2 - \lambda_3) + dG(\lambda_1, \lambda_3)\delta(\lambda_1 - \lambda_2 + \lambda_3) - dG(\lambda_2, \lambda_3)\delta(-\lambda_1 + \lambda_2 + \lambda_3) - dG(0, 0)\delta(\lambda_1)\delta(\lambda_2)\delta(\lambda_3)].$$

It is also worth noting that the symmetries indicate that the integrals in Equations (2.4) and (2.5) can be taken over one (or more) of the sectors instead of over the entire plane.

3. Estimation. Given observations, x_t, for $0 \leqq t \leqq N$, natural estimates for the bispectral density, $g(\lambda_1, \lambda_2)$, are those based on a function, $g_N(\lambda_1, \lambda_2)$, which is analogous to the periodogram of second-order theory (see, for example, Rosenblatt [17] for a discussion of periodogram theory). The usual estimate for $r_3(\nu_1, \nu_2)$ is

$$\rho_N(\nu_1, \nu_2) = N^{-1} \int_{D_N(\nu_1, \nu_2)} x_t x_{t+\nu_1} x_{t+\nu_2}\, dt \tag{3.1}$$

where the interval D_N restricts x_t, $x_{t+\nu_1}$, and $x_{t+\nu_2}$ to the domain in which they are defined (The set Φ is the empty set.):

$$D_N(\nu_1, \nu_2) = \Phi \qquad \text{if } |\nu_1| \text{ or } |\nu_2| \text{ or } |\nu_1 - \nu_2| > N$$
$$= [-\min [0, \nu_1, \nu_2], N - \max [0, \nu_1, \nu_2]] \qquad \text{otherwise.}$$

It is intuitively plausible that if $r_3(\nu_1, \nu_2)$ is replaced by $\rho_N(\nu_1, \nu_2)$ in (2.4), the resulting function is an estimate of $g(\lambda_1, \lambda_2)$, thus

$$g_N(\lambda_1, \lambda_2) = (2\pi)^{-2} \int_{-N}^{N} \int_{-N}^{N} \exp(-i\nu_1\lambda_1 - i\nu_2\lambda_2)\rho_N(\nu_1, \nu_2)\, d\nu_1\, d\nu_2 . \tag{3.2}$$

There are two customary requirements made of such estimates:

(i) the estimate should be asymptotically unbiased,

(ii) the variance of the estimate should go to zero as $N \to \infty$.

Theorem 1 states that under certain conditions on the process, $\{X_t\}$, $\rho_N(\nu_1, \nu_2)$ has both properties. However, $g_N(\lambda_1, \lambda_2)$, like the periodogram, has only Property (i) and not Property (ii). Prompted by the results of second-order theory, one corrects this difficulty by considering weighted estimates of the form

$$g_N^*(W) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(\mu_1, \mu_2) g_N(\mu_1, \mu_2)\, d\mu_1\, d\mu_2 \tag{3.3}$$

where $W(\mu_1, \mu_2)$ is a "bispectral averaging function" to be defined later. By Theorems 2 and 3 below, it is seen that under certain conditions on W and the process, $g_N^*(W)$ is an asymptotically unbiased estimate of

$$g(W) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(\mu_1, \mu_2) g(\mu_1, \mu_2)\, d\mu_1\, d\mu_2 \tag{3.4}$$

and that $\sigma^2(g_N^*(W)) \to 0$ as $N \to \infty$. Therefore $g_N^*(W)$ suffices as an estimate of (3.4) but it obviously is not an asymptotically unbiased estimate of $g(\lambda_1, \lambda_2)$.

To get an estimate of the bispectral density itself with both Properties (i) and (ii), a *sequence* of weight functions, $\{W_N(\mu_1, \mu_2)\}$, can be used so that

$$(3.5) \quad g_N^*(\lambda_1, \lambda_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_N(\mu_1 - \lambda_1, \mu_2 - \lambda_2) g_N(\mu_1, \mu_2)\, d\mu_1\, d\mu_2 .$$

Appropriate choice of the weight functions will give $g_N^*(\lambda_1, \lambda_2)$ Property (ii). As N increases, the weight functions should narrow the region averaged over (that is tend to behave more and more like a two-variate δ-function with the "spike" at the origin); this tends to give the estimate Property (i). A proper rate of concentration by the weight functions must be chosen so that (under certain conditions on the process and on $\{W_N\}$) both properties can be obtained simultaneously (see Theorems 4 and 5).

4. Hypotheses characteristic of this paper. Fourth and sixth-order moments arise during the investigation of the covariance properties of the estimates. It is necessary to make some kind of assumption on these moments. A very advantageous assumption involves the use of cumulant (semi-invariant) functions (see Stratonovich ([21], Chapter 1) and Leonov and Shiryaev [11] for a discussion of cumulant functions). These functions are defined in a manner completely analogous to the way ordinary cumulants are defined for a single random variable.

Let a random vector $\eta = (\eta_1, \cdots, \eta_k)$ be given. Its characteristic function is

$$(4.1) \qquad \varphi_\eta(\alpha_1, \cdots, \alpha_k) = E \exp (i\alpha_1\eta_1 + \cdots + i\alpha_k\eta_k).$$

Assume $E|\eta_j|^n < \infty$; then the mixed moments

$$(4.2) \qquad m^{(\nu_1, \dots, \nu_k)} = E\eta_1^{\nu_1} \cdots \eta_k^{\nu_k}$$

exist for all $\nu_1, \cdots, \nu_k$ such that $\nu_j \geqq 0$ and $\nu_1 + \cdots + \nu_k \leqq n$. Consequently φ_η has the Taylor expansion

$$(4.3) \quad \varphi_\eta(\alpha_1, \cdots, \alpha_k) = \sum_{\nu_1+\dots+\nu_k \leqq n} (i^{\nu_1+\dots+\nu_k}/\nu_1! \cdots \nu_k!) m^{(\nu_1, \dots, \nu_k)} \alpha_1^{\nu_1} \cdots \alpha_k^{\nu_k} + o((|\alpha_1| + \cdots + |\alpha_k|)^n)$$

where the sum is over all non-negative $\nu_1, \cdots, \nu_k$ whose sum does not exceed n. Furthermore, $\log \varphi_\eta$ has a Taylor expansion exactly as in (4.3) except with $m^{(\nu_1, \dots, \nu_k)}$ replaced by the coefficient $s^{(\nu_1, \dots, \nu_k)}$. The quantities $s^{(\nu_1, \dots, \nu_k)}$ are called the cumulants of the vector η. Also $m^{(\nu_1, \dots, \nu_k)}$ can be expressed as a polynomial in the $s^{(\gamma_1, \dots, \gamma_k)}$, $0 \leqq \gamma_j \leqq \nu_j$, $j = 1, \cdots, k$ (and vice versa) (see Leonov and Shiryaev ([11], p. 320)). For a process with zero mean, the two expressions needed in the following are

$$(4.4) \qquad m_4(\nu_1, \cdots, \nu_4) = s_4(\nu_1, \cdots, \nu_4) + \{s_2(\nu_1, \nu_2)s_2(\nu_3, \nu_4)\}_3$$

and

$$(4.5) \quad m_6(\nu_1, \cdots, \nu_6) = s_6(\nu_1, \cdots, \nu_6) + \{s_3(\nu_1, \nu_2, \nu_3)s_3(\nu_4, \nu_5, \nu_6)\}_{10} + \{s_2(\nu_1, \nu_2)s_4(\nu_3, \cdots, \nu_6)\}_{15} + \{s_2(\nu_1, \nu_2)s_2(\nu_3, \nu_4)s_2(\nu_5, \nu_6)\}_{15}$$

where (for $\eta = (X_{\nu_1}, \cdots, X_{\nu_6})$) for example

$$s_2(\nu_1, \nu_2) = s^{(1,1,0,0,0,0)},$$
$$s_2(\nu_1, \nu_3) = s^{(1,0,1,0,0,0)},$$
$$s_4(\nu_1, \nu_2, \nu_5, \nu_6) = s^{(1,1,0,0,1,1)}, \text{ etc.},$$

and where the notation $\{\cdot\}_j$ denotes the sum of all j *different* terms obtained by interchanging the arguments of the terms in brackets (the order of the arguments of the s_j being immaterial) (see Tables I, II and III below.) Thus (4.4) is

$$m_4(\nu_1, \cdots, \nu_4) = s_4(\nu_1, \cdots, \nu_4) + s_2(\nu_1, \nu_2)s_2(\nu_3, \nu_4) + s_2(\nu_1, \nu_3)s_2(\nu_2, \nu_4) + s_2(\nu_1, \nu_4)s_2(\nu_2, \nu_3).$$

Note that in the case of zero mean,

$$(4.6)\qquad \begin{aligned} s_1(\nu) &= 0, \\ s_2(\nu_1, \nu_2) &= m_2(\nu_1, \nu_2), \\ s_3(\nu_1, \nu_2, \nu_3) &= m_3(\nu_1, \nu_2, \nu_3). \end{aligned}$$

Due to stationarity we write

$$(4.7)\qquad \begin{aligned} s_2(t, t + \nu) &= \xi_2(\nu), \\ &\vdots \\ s_6(t, t + \nu_1, \cdots, t + \nu_5) &= \xi_6(\nu_1, \cdots, \nu_5). \end{aligned}$$

Then *the basic assumption on the process used in the following is that* $\xi_4(\nu_1, \nu_2, \nu_3) \,\varepsilon\, L_1(R_3)$ *and* $\xi_6(\nu_1, \cdots, \nu_5) \,\varepsilon\, L_1(R_5)$. This is a large class of processes which includes, for example, normal processes (trivially) and linear processes. More significantly, it includes all k-step dependent processes (see Hoeffding and Robbins [9]) (linear or nonlinear) which have sixth-order moments. The assumption that $\xi_4 \,\varepsilon\, L_1(R_3)$ was used by Magness [12] and Parzen [13] in works on spectral theory.

5. Third-order moment estimation. It was stated in Section 3 that, under certain restrictions, $\rho_N(\nu_1, \nu_2)$ is an asymptotically unbiased estimate of $r_3(\nu_1, \nu_2)$ and that $\sigma^2(\rho_N(\nu_1, \nu_2)) \to 0$ as $N \to \infty$. The specific result is

THEOREM 1. *If* (a) $\{X_t\}$, $EX_t = 0$, *is a real,* 6th-*order weakly stationary process,*
(b) $r(\nu)$, $r_3(\nu_1, \nu_2)$, $\xi_4(\nu_1, \nu_2, \nu_3)$, $\xi_6(\nu_1, \cdots, \nu_5)$, $g(\lambda_1, \lambda_2) \,\varepsilon\, L_1$, *and*
(c) $\gamma(N)$ *is a real function that is* $o(N)$, *then*

$$(5.1)\quad (1)\qquad \lim_{N\to\infty} \gamma(N)|E\rho_N(\nu_1, \nu_2) - r_3(\nu_1, \nu_2)| = 0,$$

$$(2)\quad \lim_{N\to\infty} N \operatorname{cov}[\rho_N(\nu_1, \nu_2), \rho_N(\nu_3, \nu_4)]$$
$$= \int_{-\infty}^{\infty} dy[r_6(\nu_1, \nu_2, y, y + \nu_3, y + \nu_4) - r_3(\nu_1, \nu_2)r_3(\nu_3, \nu_4)]$$
$$(5.2)\qquad = \int_{-\infty}^{\infty} dy[\xi_6(\nu_1, \nu_2, y, y + \nu_3, y + \nu_4)$$
$$+ \{m_3(0, \nu_1, \nu_2)m_3(y, y + \nu_3, y + \nu_4)\}_{10-1}$$

$$+ \{m_2(0, \nu_1)s_4(\nu_2, y, y + \nu_3, y + \nu_4)\}_{15}$$
$$+ \{m_2(0, \nu_1)m_2(\nu_2, y)m_2(y + \nu_3, y + \nu_4)\}_{15}]$$

where

$$\{m_3(0, \nu_1, \nu_2)m_3(y, y + \nu_3, y + \nu_4)\}_{10-1}$$
$$= \{m_3(0, \nu_1, \nu_2)m_3(y, y + \nu_3, y + \nu_4)\}_{10} - r_3(\nu_1, \nu_2)r_3(\nu_3, \nu_4),$$

(3) *Equation* (5.2) *can be written in the "frequency domain" provided* ξ_4 *and* ξ_6 *have Fourier transforms which are suitably regular* (*see* [18]).

PROOF. Assertion (1) is immediate and (2) is almost as immediate but will be proved in order to introduce some notation.

$$NE[\rho_N(\nu_1, \nu_2) - E\rho_N(\nu_1, \nu_2))(\rho_N(\nu_3, \nu_4) - E\rho_N(\nu_3, \nu_4))]$$
$$(5.3) \quad = N^{-1} \int_{D_N(\nu_1,\nu_2)} dt \int_{D_N(\nu_3,\nu_4)} d\tau[m_6(t, t + \nu_1, t + \nu_2, \tau, \tau + \nu_3, \tau + \nu_4) - r_3(\nu_1, \nu_2)r_3(\nu_3, \nu_4)].$$

Let $y = \tau - t$, $t = t$ to get

$$(5.4) \quad \int_{-N}^{N} dy[C_N(\nu_1, \nu_2, \nu_3, \nu_4, y)/N][r_6(\nu_1, \nu_2, y, y + \nu_3, y + \nu_4) - r_3(\nu_1, \nu_2)r_3(\nu_3, \nu_4)]$$

where C_N is defined as follows (see Fig. 3):

1. Construct the set $D_N(\nu_1, \nu_2) \times D_N(\nu_3, \nu_4) = \hat{D}_N$,
2. take the intersection of this set with the line $\tau - t = y$ and call that segment $\hat{C}_N$,
3. then $C_N(\nu_1, \nu_2, \nu_3, \nu_4, y)$ is equal to the length of the projection of $\hat{C}_N$ onto either axis.

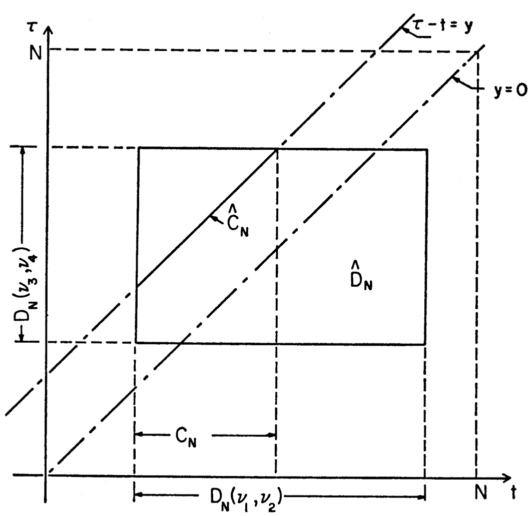

FIG. 3

Note that C_N can be written out analytically in terms of its arguments but the expression is cumbersome and not as enlightening as Fig. 3. Also, $0 \leqq C_N/N \leqq 1$ and $C_N/N \to 1$ as $N \to \infty$ uniformly on any fixed finite set of C_N's arguments. Using (4.5), (5.4) becomes

$$\begin{aligned}\int_{-N}^{N} dy(C_N/N)[\xi_6(\nu_1, \nu_2, y+\nu_3, y+\nu_4) \\ + \{m_3(0, \nu_1, \nu_2)m_3(y, y+\nu_3, y+\nu_4)\}_{10-1} \\ + \{m_2(0, \nu_1)s_4(\nu_2, y, y+\nu_3, y+\nu_4)\}_{15} \\ + \{m_2(0, \nu_1)m_2(\nu_2, y)m_2(y+\nu_3, y+\nu_4)\}_{15}].\end{aligned} \tag{5.5}$$

Tables I, II, and III display the individual terms (later called minor terms) which make up (5.5). Looking at each of these 40 terms separately, it is obvious

TABLE I

The ten terms which sum to give the expression $\{s_3(0, \nu_1, \nu_2)s_3(y, y+\nu_3, y+\nu_4)\}_{10}$.

minor term number	$s_3(\cdot, \cdot, \cdot)s_3(\cdot, \cdot, \cdot)$	
1	$r_3(\nu_1, \nu_2)$	$r_3(\nu_3, \nu_4)$
2	$r_3(\nu_1, y)$	$r_3(y+\nu_3-\nu_2, y+\nu_4-\nu_2)$
3	$r_3(\nu_2, y)$	$r_3(y+\nu_3-\nu_1, y+\nu_4-\nu_1)$
4	$r_3(\nu_1, y+\nu_3)$	$r_3(y-\nu_2, y+\nu_4-\nu_2)$
5	$r_3(\nu_1, y+\nu_4)$	$r_3(y-\nu_2, y+\nu_3-\nu_2)$
6	$r_3(\nu_2, y+\nu_3)$	$r_3(y-\nu_1, y+\nu_4-\nu_1)$
7	$r_3(\nu_2, y+\nu_4)$	$r_3(y-\nu_1, y+\nu_3-\nu_1)$
8	$r_3(y, y+\nu_3)$	$r_3(\nu_2-\nu_1, y+\nu_4-\nu_1)$
9	$r_3(y, y+\nu_4)$	$r_3(\nu_2-\nu_1, y+\nu_3-\nu_1)$
10	$r_3(y+\nu_3, y+\nu_4)$	$r_3(\nu_2-\nu_1, y-\nu_1)$

TABLE II

The fifteen terms which sum to give the expression $\{s_2(0, \nu_1)s_4(\nu_2, y, y+\nu_3, y+\nu_4)\}_{15}$.

minor term number	$m_2(\cdot, \cdot)s_4(\cdot, \cdot, \cdot)$	
1	$r(\nu_1)$	$\xi_4(\nu_2-y, \nu_3, \nu_4)$
2	$r(\nu_2)$	$\xi_4(\nu_1-y, \nu_3, \nu_4)$
3	$r(\nu_3)$	$\xi_4(\nu_1, \nu_2, \nu_4+y)$
4	$r(\nu_4)$	$\xi_4(\nu_1, \nu_2, \nu_3+y)$
5	$r(y)$	$\xi_4(\nu_2-\nu_1, y+\nu_3-\nu_1, y+\nu_4-\nu_1)$
6	$r(\nu_2-\nu_1)$	$\xi_4(-y, \nu_3, \nu_4)$
7	$r(\nu_4-\nu_3)$	$\xi_4(\nu_1, \nu_2, y)$
8	$r(y-\nu_1)$	$\xi_4(\nu_2, y+\nu_3, y+\nu_4)$
9	$r(y-\nu_2)$	$\xi_4(\nu_1, y+\nu_3, y+\nu_4)$
10	$r(y+\nu_3)$	$\xi_4(\nu_1-y, \nu_2-y, \nu_4)$
11	$r(y+\nu_4)$	$\xi_4(\nu_1-y, \nu_2-y, \nu_3)$
12	$r(y+\nu_3-\nu_1)$	$\xi_4(\nu_2, y, y+\nu_4)$
13	$r(y+\nu_3-\nu_2)$	$\xi_4(\nu_1, y, y+\nu_4)$
14	$r(y+\nu_4-\nu_1)$	$\xi_4(\nu_2, y, y+\nu_3)$
15	$r(y+\nu_4-\nu_2)$	$\xi_4(\nu_1, y, y+\nu_3)$

TABLE III

The fifteen terms which sum to give the expression $\{s_2(0, \nu_1)s_2(\nu_2, y)s_2(y + \nu_3, y + \nu_4)\}_{15}$.

minor term number	$s_2(\cdot, \cdot)s_2(\cdot, \cdot)s_2(\cdot, \cdot)$		
1	$r(\nu_1)$	$r(\nu_2 - y)$	$r(\nu_3 - \nu_4)$
2	$r(\nu_2)$	$r(\nu_1 - y)$	$r(\nu_3 - \nu_4)$
3	$r(\nu_1)$	$r(y + \nu_3 - \nu_2)$	$r(\nu_4)$
4	$r(\nu_2)$	$r(y + \nu_3 - \nu_1)$	$r(\nu_4)$
5	$r(\nu_1)$	$r(y + \nu_4 - \nu_2)$	$r(\nu_3)$
6	$r(\nu_2)$	$r(y + \nu_4 - \nu_1)$	$r(\nu_3)$
7	$r(y)$	$r(\nu_2 - \nu_1)$	$r(\nu_4 - \nu_3)$
8	$r(y)$	$r(y + \nu_3 - \nu_1)$	$r(y + \nu_4 - \nu_2)$
9	$r(y)$	$r(y + \nu_3 - \nu_2)$	$r(y + \nu_4 - \nu_1)$
10	$r(y + \nu_3)$	$r(\nu_2 - \nu_1)$	$r(\nu_4)$
11	$r(y + \nu_4)$	$r(\nu_2 - \nu_1)$	$r(\nu_3)$
12	$r(y + \nu_3)$	$r(y - \nu_1)$	$r(y + \nu_4 - \nu_2)$
13	$r(y + \nu_3)$	$r(y - \nu_2)$	$r(y + \nu_4 - \nu_1)$
14	$r(y + \nu_4)$	$r(y - \nu_1)$	$r(y + \nu_3 - \nu_2)$
15	$r(y + \nu_4)$	$r(y - \nu_2)$	$r(y + \nu_3 - \nu_1)$

that the expression in square brackets is absolutely integrable with respect to y on $(-\infty, \infty)$. Thus writing $\int_{-N}^{N} dy$ as $\int_{-\infty}^{\infty} dy\chi_N(y)$ where χ_N is the characteristic function of the interval $[-N, N]$, (5.5) can be looked upon as the infinite integral of a sequence of functions, $\chi_N(y)(C_N/N)[\cdots]$, which (by the properties of C_N) tend to zero pointwise and are dominated by the fixed $L_1(-\infty, \infty)$ function given by the absolute value of the expression in square brackets. Consequently, Lebesgue's dominated convergence theorem gives the result.

Q.E.D.

6. Estimation of the weighted bispectral density. Next consider estimates of the form (3.3),

$$g_N^*(W) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\mu_1, \mu_2)g_N(\mu_1, \mu_2)\, d\mu_1\, d\mu_2 . \tag{3.3}$$

The following conditions will be placed on W.

DEFINITION 1. A bispectral averaging function, $W(\mu_1, \mu_2)$, of order $\alpha > 0$ is

(i) real-valued,

(ii) $\varepsilon\, L_1 \cap L_2$, and

(iii) its Fourier transform (which is bounded and L_2),

$$w(\nu_1, \nu_2) = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-i\nu_1\mu_1 - i\nu_2\mu_2)W(\mu_1, \mu_2)\, d\mu_1\, d\mu_2 \tag{6.1}$$

is such that $|w(\nu_1, \nu_2)| = O(|\nu_1| + |\nu_2|)^{-\alpha}$.

Condition (iii) could be weakened. Note that (iii) implies that there are constants K_1 and $b > 0$ such that for all $|\nu_1|$ and/or $|\nu_2| \geqq b$,

$$|w(\nu_1, \nu_2)| \leqq K_1(|\nu_1| + |\nu_2|)^{-\alpha}. \tag{6.2}$$

THEOREM 2. *If* (a) $\{X_t\}$ *is* 3rd-*order weakly stationary, real,*

(b) $r_3(\nu_1, \nu_2)$, $g(\lambda_1, \lambda_2) \varepsilon L_1(R_2)$, *and*

(c) $W(\mu_1, \mu_2)$ *is a bispectral averaging function of order* $s + \epsilon$, $\epsilon > 0$, $0 < s < 1$, *then* $\lim_{N\to\infty} N^s |Eg_N^*(W) - g(W)| = 0$.

PROOF. By (3.2), (3.3), and (6.1)

$$N^s |Eg_N^*(W) - g(W)|$$

$$= N^s |E \textstyle\int_{-N}^{N} \int_{-N}^{N} w(\nu_1, \nu_2)[\rho_N(\nu_1, \nu_2) - r_3(\nu_1, \nu_2)]\, d\nu_1\, d\nu_2 \tag{6.3}$$

$$- \textstyle\int \int_{|\nu_1| \text{or} |\nu_2| > N} w(\nu_1, \nu_2) r_3(\nu_1, \nu_2)\, d\nu_1\, d\nu_2|.$$

The second integral on the left hand side of (6.3) tends to zero in absolute value by the boundedness of $w(\nu_1, \nu_2)$ and the fact that $r_3(\nu_1, \nu_2) \varepsilon L_1$. The first integral is

$$\leqq N^s |\textstyle\int_{-N}^{N} \int_{-N}^{N} w(\nu_1, \nu_2)[1 - [c_N(\nu_1, \nu_2)/N] - 1] r_3(\nu_1, \nu_2)\, d\nu_1\, d\nu_2| \tag{6.4}$$

where

$$c_N(\nu_1, \nu_2) = \min [\hat{c}_N(\nu_1, \nu_2), N] \tag{6.5}$$

$$\hat{c}_N(\nu_1, \nu_2) = \max [|\nu_1|, |\nu_2|, |\nu_1 - \nu_2|].$$

Since $c_N \leqq |\nu_1| + |\nu_2|$ and by (6.2), the integrand of (6.4), $[c_N(\nu_1, \nu_2)/N^{1-s}] w(\nu_1, \nu_2) r_3(\nu_1, \nu_2)$, is bounded in absolute value on $(-\infty, \infty)$ by $K_2 |r_3(\nu_1, \nu_2)|$ for some K_2. This integrand converges pointwise to zero so that again by Lebesgue's dominated convergence theorem the result is proved.

Q.E.D.

Property (i) of Section 3, the asymptotic unbiasedness, is therefore shown. Now turn to Property (ii) of Section 3.

THEOREM 3. *If* (a) $\{X_t\}$ *is* 6th-*order weakly stationary, real,*

(b) $r(\nu)$, $r_3(\nu_1, \nu_2)$, $\xi_4(\nu_1, \nu_2, \nu_3)$, $\xi_6(\nu_1, \cdots, \nu_5)$, $g(\nu_1, \nu_2) \varepsilon L_1$, *and*

(c) $W_j(\mu_1, \mu_2)$, $j = 1, 2$, *are bispectral averaging functions of order* $1 + \epsilon_j$, $\epsilon_j > 0$, *then*

(1) $\lim_{N\to\infty} N \operatorname{cov} [g_N^*(W_1), g_N^*(W_2)]$

$$= \textstyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu_1 \cdots d\nu_4\, dy w_1(\nu_1, \nu_2) \overline{w_2(\nu_3, \nu_4)}$$

$$\cdot [\xi_6(\nu_1, \nu_2, y, y + \nu_3, y + \nu_4) + \{m_3(0, \nu_1, \nu_2) m_3(y, y + \nu_3, y + \nu_4)\}_{10-1} \tag{6.6}$$

$$+ \{m_2(0, \nu_1) s_4(\nu_2, y, y + \nu_3, y + \nu_4)\}_{15}$$

$$+ \{m_2(0, \nu_1) m_2(\nu_2, y) m_2(y + \nu_3, y + \nu_4)\}_{15}],$$

(2) *Equation* (6.6) *is written in the "time domain"; the result can be written in the "frequency domain"* (*see* [18]).

PROOF. Recalling the proof of Theorem 1,

$$NE[(g_N^*(W_1) - Eg_N^*(W_1))(g_N^*(W_2) - Eg_N^*(W_2))]$$

$$= \textstyle\int_{-N}^{N} \cdots \int_{-N}^{N} \int_{-N}^{N} w_1(\nu_1, \nu_2) \overline{w_2(\nu_3, \nu_4)} [C_N(\nu_1, \cdots, \nu_4, y)/N]$$

$$(6.7)\quad \cdot[\xi_6(\nu_1, \nu_2, y, y + \nu_3, y + \nu_4) + \{m_3(0, \nu_1, \nu_2)m_3(y, y + \nu_3, y + \nu_4)\}_{10-1} + \{m_2(0, \nu_1)s_4(\nu_2, y, y + \nu_3, y + \nu_4)\}_{15} + \{m_2(0, \nu_1)m_2(\nu_2, y)m_2(y + \nu_3, y + \nu_4)\}_{15}]\, d\nu_1 \cdots d\nu_4\, dy.$$

It remains to be shown that the integrand with the (C_N/N) term replaced by 1 is absolutely integrable on $(-\infty, \infty)$. To do this, consider the right hand side of (6.7) as made up of four "major terms" or 40 "minor terms", where the first major term and the first minor term are identical and the remaining minor terms are enumerated in accorance with Tables I, II, and III.

Each minor term is treated separately. Minor term 1 is obviously summable since w_1 and w_2 are bounded and $\xi_6 \varepsilon L_1(R_5)$. The remaining minor terms are grouped according to the major terms within which they are contained.

LEMMA 1. *The hypotheses of Theorem* 3 *imply that*

(1)

$$(6.8)\quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\{m_3(0, \nu_1, \nu_2)m_3(y, y + \nu_3, y + \nu_4)\}_{10-1} w_1(\nu_1, \nu_2)\overline{w_2(\nu_3, \nu_4)}|\, d\nu_1 \cdots d\nu_4\, dy < \infty,$$

(2)

$$(6.9)\quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\{m_2(0, \nu_1)s_4(\nu_2, y, y + \nu_3, y + \nu_4)\}_{15} w_1(\nu_1, \nu_2)\overline{w_2(\nu_3, \nu_4)}|\, d\nu_1 \cdots d\nu_4\, dy < \infty,$$

(3)

$$(6.10)\quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\{m_2(0, \nu_1)m_2(\nu_2, y)m_2(y + \nu_3, y + \nu_4)\}_{15} w_1(\nu_1, \nu_2)\overline{w_2(\nu_3, \nu_4)}|\, d\nu_1 \cdots d\nu_4\, dy < \infty.$$

PROOF. The proof follows using the integrability properties of r_2, r_3 and ξ_4 and (6.1) and (6.2) (see [18] and [19]). Q.E.D.

The previous statements and Lemma 1 show that there is an L_1 upper bound for all N on the integrand. Therefore the properties of (C_N/N) and Lebesgue's dominated convergence theorem give (6.6). Q.E.D.

7. Estimation of the bispectral density. The most interesting properties are exhibited by the third form of estimate introduced in Section 3,

$$(3.5)\quad g_N^*(\lambda_1, \lambda_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_N(\mu_1 - \lambda_1, \mu_2 - \lambda_2)g_N(\mu_1, \mu_2)\, d\mu_1\, d\mu_2.$$

Properties (i) and (ii) of Section 3 will be discussed not for the general form (3.5) but for a subclass of estimates described by first defining a bispectral estimating kernel.

DEFINITION 2. A symmetric bispectral estimating kernel, $w(\nu_1, \nu_2)$, of order $\alpha > 0$ is

(i) real-valued, εL_2,

(ii) $w(\nu_1, \nu_2) \leqq M_1 < \infty$,

(iii) $w(\nu_1, \nu_2) = w(\nu_2, \nu_1) = w(-\nu_1, \nu_2 - \nu_1)$ (same symmetries as those of $r_3(\nu_1, \nu_2)$),

(iv) for every $\epsilon > 0$ there exists an M_2 such that uniformly in ν_2,

$$\int_{|\nu_1|>M_2} |w(\nu_1, \nu_2)|\, d\nu_1 \leqq \epsilon,$$

(v) and

$$w^{(\alpha)} = \sup_P \lim_{\nu_1,\nu_2\to 0} [|1 - w(\nu_1, \nu_2)|/(|\nu_1| + |\nu_2|)^\alpha],$$

where $P \equiv$ all possible paths, is well-defined and finite.

Letting $\{B_N\}$ be a sequence of positive constants tending to zero as $N \to \infty$, the estimates then discussed are of the form

$$(7.1)\quad g_N^*(\lambda_1, \lambda_2) = (2\pi)^{-2} \int_{-N}^{N} \int_{-N}^{N} \exp(-i\lambda_1\nu_1 - i\lambda_2\nu_2) \cdot w(B_N\nu_1, B_N\nu_2)\rho_N(\nu_1, \nu_2)\, d\nu_1\, d\nu_2 .$$

By Condition (i) above, $w(\nu_1, \nu_2)$ is a Fourier transform in the L_2 sense,

$$(7.2)\quad w(\nu_1, \nu_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-i\nu_1\lambda_1 - i\nu_2\lambda_2) W(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2 ,$$

Using this, (7.1) can be rewritten as

$$(7.3)\quad g_N^*(\lambda_1, \lambda_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B_N^{-2} W[(\mu_1 - \lambda_1)/B_N), (\mu_2 - \lambda_2)/B_N] g_N(\mu_1, \mu_2)\, d\mu_1\, d\mu_2$$

which is of the form (3.5). Parzen [13] uses an analogous estimate for the spectral density. The rate at which $B_N \to 0$ governs the rate of concentration of the weight functions.

Condition (iii) of Definition 2 implies that $g_N^*(\lambda_1, \lambda_2)$ has the same symmetries as $g(\lambda_1, \lambda_2)$. It is not necessary in proving the following theorems, but is included merely to permit the statement and proof of Theorem 5 to be written more compactly. A discussion of the nonsymmetric case including a statement of the theorems is in [19].

For convenience of notation, define the generalized qth weighted (with respect to some bounded weight function h) bispectral derivative, $g^{(q)}(\mu_1, \mu_2)$, by

$$(7.4)\quad g^{(q)}(\mu_1, \mu_2, h) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-i\mu_1\nu_1 - i\mu_2\nu_2)(|\nu_1| + |\nu_2|)^q r_3(\nu_1, \nu_2) h(\nu_1, \nu_2)\, d\nu_1\, d\nu_2 .$$

Then the bias of (7.3) satisfies the following.

THEOREM 4. *If* (a) $\{X_t\}$ *is* 3rd-*order weakly stationary, real,*

(b) $r_3(\nu_1, \nu_2)$, $g(\lambda_1, \lambda_2) \in L_1(R_2)$,

(c) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|\nu_1| + |\nu_2|)^q r_3(\nu_1, \nu_2)\, d\nu_1\, d\nu_2 < \infty$, $q > 0$,

(d) $w(\nu_1, \nu_2)$ *is a bispectral estimating kernel of order* $\alpha > q$, *and*

(e) B_N *chosen such that as* $N \to \infty$

(i) $B_N \to 0$,

(ii) $B_N N \to \infty$ *if* $q \leqq 1$

$B_N^q N \to \infty$ *if* $q > 1$

then

$$\text{(7.5)}\quad \lim_{N\to\infty} B_N^{-q}\,|Eg_N^*(\mu_1,\mu_2) - g(\mu_1,\mu_2)| = 0 \qquad \textit{if}\quad \alpha > q$$
$$= |g^{(q)}(\mu_1,\mu_2, w^{(q)})| \quad \textit{if}\quad \alpha = q$$

where in the case $\alpha = q$, $w(\nu_1, \nu_2)$ *is assumed to have the property that for almost all* (ν_1, ν_2),

$$\text{(7.6)}\quad \lim_{N\to\infty} (1 - w(B_N\nu_1, B_N\nu_2))/(|B_N\nu_1| + |B_N\nu_2|)^q = w^{(q)}(\nu_1,\nu_2).$$

PROOF. Break up the bias expression into three terms,

$$B_N^{-q}(Eg_N^*(\mu_1,\mu_2) - g(\mu_1,\mu_2))$$
$$= [B_N^{-q}/(2\pi)^2][\textstyle\int_{-N}^{N}\int_{-N}^{N} \exp(-i\mu_1\nu_1 - i\mu_2\nu_2)(w(B_N\nu_1, B_N\nu_2) - 1)r_3$$
$$\text{(7.7)}\qquad \cdot(\nu_1,\nu_2)\, d\nu_1\, d\nu_2$$
$$- \textstyle\int_{-N}^{N}\int_{-N}^{N} \exp(-i\mu_1\nu_1 - i\mu_2\nu_2)\{\min[\max(|\nu_1|, |\nu_2|, |\nu_1 - \nu_2|), N]/N\}$$
$$\cdot w(B_N\nu_1, B_N\nu_2) r_3(\nu_1,\nu_2)\, d\nu_1\, d\nu_2$$
$$- \textstyle\int\int_{|\nu_1| \text{or} |\nu_2|>N} \exp(-i\mu_1\nu_1 - i\mu_2\nu_2) r_3(\nu_1,\nu_2)\, d\nu_1\, d\nu_2].$$

The third term is

$$\leqq (NB_N)^{-q} \textstyle\int\int_{|\nu_1| \text{or} |\nu_2|>N} (|\nu_1| + |\nu_2|)^q\, |r_3(\nu_1,\nu_2)|\, d\nu_1\, d\nu_2 \to 0$$

by (c) and (e). The second term of (7.7) is

$$\text{(7.8)}\qquad \leqq (M_1/B_N^q N) \textstyle\int_{-N}^{N}\int_{-N}^{N} (|\nu_1| + |\nu_2|)|r_3(\nu_1,\nu_2)|\, d\nu_1\, d\nu_2 .$$

First suppose $q \leqq 1$; then (7.8) is

$$\leqq [M_1/(NB_N)^q] \textstyle\int_{-N}^{N}\int_{-N}^{N} [(|\nu_1| + |\nu_2|)^{1-q}/N^{1-q}](|\nu_1| + |\nu_2|)^q\, |r_3(\nu_1,\nu_2)|\, d\nu_1\, d\nu_2 \to 0$$

by (c) and (e). Next suppose $q > 1$; then (7.8) is

$$\leqq (M_1/NB_N^q) \textstyle\int_{-N}^{N}\int_{-N}^{N} (|\nu_1| + |\nu_2|) r_3(\nu_1,\nu_2)\, d\nu_1\, d\nu_2 \to 0$$

again by (c) and (e). Therefore

$$\lim_{N\to\infty} B_N^{-q}\,|Eg_N^*(\mu_1,\mu_2) - g(\mu_1,\mu_2)|$$
$$\text{(7.9)}\quad = (2\pi)^{-2} \lim_{N\to\infty} |\textstyle\int_{-N}^{N}\int_{-N}^{N} \exp(-i\mu_1\nu_1 - i\mu_2\nu_2)(|\nu_1| + |\nu_2|)^q r_3(\nu_1,\nu_2)$$
$$\cdot[(w(B_N\nu_1, B_N\nu_2) - 1)/(|B_N\nu_1| + |B_N\nu_2|)^\alpha](|B_N\nu_1| + |B_N\nu_2|)^{\alpha-q}\, d\nu_1\, d\nu_2|.$$

Since $w(\nu_1, \nu_2)$ is a bispectral estimating kernel of order α, there exists a constant K_3 such that the integrand of (7.9) is bounded by $K_3(|\nu_1| + |\nu_2|)^q\, |r_3(\nu_1,\nu_2)|$ which is integrable. Q.E.D.

Secondly, the asymptotic behavior of the variance can be described under the assumptions listed in the next theorem. In the proof of this theorem the following lemmas are needed. Lemma 2 is motivated by the Riemann-Lebesgue lemma. The proofs are simple and are omitted (see [18]).

LEMMA 2. *The hypotheses of Theorem* 5 *imply that*

(1) $\int_{-M}^{M}\int_{-M}^{M} d\nu_1\, d\nu_2 \exp[-i\nu_1\mu_1 - i\nu_2(\mu_2/B_N)]w(B_N\nu_1, \nu_2)r(\nu_1) \to 0$ *as* $N \to \infty$ *unless* $\mu_2 = 0$,

(2) $\int_{-M}^{M} \cdots \int_{-M}^{M}\int_{-M}^{M} d\nu_1 \cdots d\nu_4\, dy \exp\{-i\nu_1(\mu_1/B_N) - i\nu_2\mu_2 + i\nu_3\mu_3 + i\nu_4 \cdot(\mu_4/B_N)\}w(\nu_1, B_N\nu_2 + \nu_4)w(B_N\nu_3 + \nu_1, \nu_4)r(y)r(y + \nu_3)r(y - \nu_2) \to 0$ *unless* $\mu_1 = \mu_4 = 0$.

LEMMA 3. *If* (a) $\{\gamma(M, N)\}$ *is a sequence of constants,*

(b) $\gamma(M) = \lim_{N\to\infty}\gamma(M, N)$ *exists for all* M,

(c) $\gamma = \lim_{M\to\infty}\gamma(M)$ *exists, and*

(d) *for any* $\epsilon > 0$ *there exists an* $M_0(\epsilon)$ *such that for all* $N > M > M_0$, $|\gamma(M, N) - \gamma(N, N)| < \epsilon$,

then $\lim_{N\to\infty}\gamma(N, N) = \gamma$.

THEOREM 5. *If* (a) $\{X_t\}$ *is* 6th-*order weakly stationary, real,*

(b) $r(\nu)$, $r_3(\nu_1, \nu_2)$, $\xi_4(\nu_1, \nu_2, \nu_3)$, $\xi_6(\nu_1, \cdots, \nu_5)$, $g(\lambda_1, \lambda_2) \in L_1$,

(c) $w(\nu_1, \nu_2)$ *is a symmetric bispectral estimating kernel,*

(d) B_N *is a sequence of positive constants such that*

(i) $B_N \to 0$ *as* $N \to \infty$

(ii) $B_N^2N \to \infty$ *as* $N \to \infty$,

(e) $w(\nu_1, \nu_2)$ *is continuous* a.e. *and for* $a < \infty$, $w(B_Na, \nu_2) \to w(0, \nu_2)$ *for almost all* ν_2, *and*

(f) *for brevity in writing results,* (μ_1, μ_2) *and* (μ_3, μ_4) *are taken in their first sections of definition as shown in Fig.* 2, i.e. $0 \leqq \mu_1, \mu_3 < \infty$, $0 \leqq \mu_2 \leqq \mu_1$, $0 \leqq \mu_4 \leqq \mu_3$, *then*

$$\begin{aligned}
&\lim_{N\to\infty} NB_N^2 \operatorname{cov}[g_N^*(\mu_1, \mu_2), g_N^*(\mu_3, \mu_4)] \\
(7.10)\qquad &= (2\pi)^{-1}[f(\mu_1)f(\mu_2)f(\mu_1 + \mu_2)f(\mu_3)f(\mu_4)f(\mu_3 + \mu_4)]^{1/2} \\
&\cdot\{w_1\delta(\mu_2)\delta(\mu_4)[1 + 2\delta(\mu_1)][1 + 2\delta(\mu_3)] \\
&+ w_2\delta(\mu_1 - \mu_3)\delta(\mu_2 - \mu_4)[1 + \delta(\mu_1 - \mu_4) + 4\delta(\mu_1)\delta(\mu_2)]\},
\end{aligned}$$

where $w_1 = [\int_{-\infty}^{\infty} w(0, \nu)\, d\nu]^2$, $w_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w^2(\nu_1, \nu_2)\, d\nu_1\, d\nu_2$.

PROOF. The proof will be given in a series of lemmas and individual discussions of major and minor terms. Recalling Theorems 1 and 3, write

$$\begin{aligned}
&NB_N^2 \operatorname{cov}[g_N^*(\mu_1, \mu_2), g_N^*(\mu_3, \mu_4)] \\
&= (B_N^2/(2\pi)^4)\int_{-N}^{N} \cdots \int_{-N}^{N} d\nu_1 \cdots d\nu_4 \exp(-i\mu_1\nu_1 - i\mu_2\nu_2 + i\mu_3\nu_3 + i\mu_4\nu_4) \\
(7.11)\qquad &\cdot w(B_N\nu_1, B_N\nu_2)w(B_N\nu_3, N_N\nu_4)\int_{-N}^{N} dy(C_N(\nu_1, \cdots, \nu_4, y))/N \\
&\cdot[\xi_6(\nu_1, \nu_2, y, y + \nu_3, y + \nu_4) + \{m_3(0, \nu_1, \nu_2)m_3(y, y + \nu_3, y + \nu_4)\}_{10-1} \\
&+ \{m_2(0, \nu_1)s_4(\nu_2, y, y + \nu_3, y + \nu_4)\}_{15} \\
&+ \{m_2(0, \nu_1)m_2(\nu_2, y)m_2(y + \nu_3, y + \nu_4)\}_{15}].
\end{aligned}$$

The boundedness of w and (C_N/N) and the $L_1(R_5)$ property of ξ_6 imply the first major term is $O(B_N^2)$. Further, by also using the integrability properties of r_2, r_3, ξ_4 and w, the second and third major terms are seen to be $O(B_N)$. By these statements

$$\begin{aligned}
&\lim_{N\to\infty} NB_N^2 \operatorname{cov}[g_N^*(\mu_1, \mu_2), g_N^*(\mu_3, \mu_4)] \\
&\quad = \lim_{N\to\infty} (B_N^2/(2\pi)^4)\int_{-N}^{N} \cdots \int_{-N}^{N} \int_{-N}^{N} d\nu_1 \cdots d\nu_4\, dy \\
&\quad\quad \cdot\exp(-i\mu_1\nu_1 - i\mu_2\nu_2 + i\mu_3\nu_3 + i\mu_4\nu_4) w(B_N\nu_1, B_N\nu_2) \\
&\quad\quad \cdot w(B_N\nu_3, B_N\nu_4)(C_N(\nu_1, \cdots, \nu_4, y)/N)\{m_2(0, \nu_1)m_2(\nu_2, y)m_2(y+\nu_3, y+\nu_4)\}_{15}.
\end{aligned}$$

Look at the 15 minor terms, listed by Table III, separately. The first term yields

$$\begin{aligned}
&\lim_{N\to\infty} (B_N^2/(2\pi)^4)\int_{-N}^{N} \cdots \int_{-N}^{N} \int_{-N}^{N} d\nu_1 \cdots d\nu_4\, dy \\
&\cdot\exp(-i\mu_1\nu_1 - i\mu_2\nu_2 + i\mu_3\nu_3 + i\mu_4\nu_4) w(B_N\nu_1, B_N\nu_2) w(B_N\nu_3, B_N\nu_4) \qquad (7.12)\\
&\cdot (C_N(\nu_1, \cdots, \nu_4, y)/N) r(\nu_1) r(\nu_2 - y) r(\nu_3 - \nu_4).
\end{aligned}$$

Let $\hat{y} = y - \nu_2$, $\hat{\nu}_4 = \nu_4 - \nu_3$, $\hat{\nu}_2 = B_N\nu_2$, $\hat{\nu}_3 = B_N\nu_3$; then (7.2) becomes

$$\begin{aligned}
&\lim_{N\to\infty} (2\pi)^{-4}\int_{-N}^{N} d\nu_1 \int_{-NB_N}^{NB_N} d\hat{\nu}_2 \int_{-NB_N}^{NB_N} d\hat{\nu}_3 \int_{-N-(\hat{\nu}_3/B_N)}^{N-(\hat{\nu}_3/B_N)} d\hat{\nu}_4 \int_{-N-(\hat{\nu}_2/B_N)}^{N-(\hat{\nu}_2/B_N)} d\hat{y} \\
&\cdot\exp\{-i\nu_1\mu_1 - i\hat{\nu}_2(\mu_2/B_N) + i\hat{\nu}_3[(\mu_3 + \mu_4)/B_N] + i\hat{\nu}_4\mu_4\} w(B_N\nu_1, \hat{\nu}_2) \qquad (7.13)\\
&\cdot w(-\hat{\nu}_3, B_N\hat{\nu}_4)[C_N(\nu_1, (\hat{\nu}_2/B_N), (\hat{\nu}_3/B_N), \hat{\nu}_4 + (\hat{\nu}_3/B_N), \hat{y} + (\hat{\nu}_2/B_N))/N] \\
&\cdot r(\nu_1) r(\hat{y}) r(\hat{\nu}_4).
\end{aligned}$$

An inspection of Fig. III reveals that (see [18] or [19])

$$[C_N(\nu_1, (\nu_2/B_N), (\nu_3/B_N), \nu_4 + (\nu_3/B_N), y + (\nu_2/B_N))/N] \to 1 \text{ as } N \to \infty. \qquad (7.14)$$

Denote by $\square_M$, the j-dimensional hypercube centered at the origin with sides of length $2M$ parallel to the j axes. The dimension, j, will be obvious from the context. Also let $\square_M'$ denote the complement of $\square_M$ in R_j. We have the following easy lemma (see [18] or [19]).

LEMMA 4. *The hypotheses of Theorem 5 imply that for any $\epsilon > 0$, there is an $M_2(\epsilon)$ independent of N such that for all $M > M_2$ and for all N,*

$$\int\int\int\int\int_{\square_M'} |w(B_N\nu_1, \nu_2) w(-\nu_3, B_N\nu_4) r(\nu_1) r(y) r(\nu_4)|\, d\nu_1 \cdots d\nu_4\, dy < \epsilon.$$

Let

$$\begin{aligned}
\gamma_1(M, N) &= (2\pi)^{-4}\int\int\int\int\int d\nu_1 \cdots d\nu_4\, dy \exp\{-i\nu_1\mu_1 - i\nu_2(\mu_2/B_N) \\
&\quad + i\nu_3[(\mu_3 + \mu_4)/B_N] + i\nu_4\mu_4\} w(B_N\nu_1, \nu_2) w(-\nu_3, B_N\nu_4) r(\nu_1) r(y) r(\nu_4), \\
\gamma_1(M) &= \lim_{N\to\infty} \gamma_1(M, N).
\end{aligned}$$

Lemma 4 and (7.14) indicate that the limit (7.13) is the same as $\lim_{N\to\infty} \gamma_1(N, N)$. Lemma 2 says that $\gamma_1(M) = 0$ unless $\mu_2 = \mu_3 + \mu_4 = 0$. Using the boundedness and continuity properties of w, Lebesgue's convergence theorem, and Lemma 3;

$$\lim_{N\to\infty} \gamma_1(N, N) = (2\pi)^{-1} w_1 f(0) f(\mu_1) f(\mu_3) \delta(\mu_2) \delta(\mu_3 + \mu_4).$$

Minor terms 2 through 7 and 10 and 11 follow by similar arguments. Minor term 8 offers a slight variation. Put $\hat{\nu}_3 = \nu_3 - \nu_1$, $\hat{\nu}_2 = \nu_2 - \nu_4$, $\hat{\nu}_1 = B_N \nu_1$, $\hat{\nu}_4 = B_N \nu_4$ to get (omitting the "hats")

$$\begin{aligned}
&(2\pi)^{-4} \lim_{N\to\infty} \int_{-NB_N}^{NB_N} \int_{-NB_N}^{NB_N} d\nu_1\, d\nu_4 \int_{-N-(\nu_4/B_N)}^{N-(\nu_4/B_N)} d\nu_2 \int_{-N-(\nu_1/B_N)}^{N-(\nu_1/B_N)} d\nu_3 \\
&\quad \cdot \exp\{i\nu_1[(\mu_1 - \mu_3)/B_N] - i\nu_2\mu_2 + i\nu_3\mu_3 + i\nu_4[(\mu_4 - \mu_2)/B_N]\} \\
&\quad \cdot w(\nu_1, B_N\nu_2 + \nu_4) w(B_N\nu_3 + \nu_1, \nu_4) \\
&\quad \cdot \int_{-N}^{N} dy [C_N((\nu_1/B_N), \nu_2 + (\nu_4/B_N), \nu_3 + (\nu_1/B_N), (\nu_4/B_N), y)/N] \\
&\quad \cdot r(y) r(y + \nu_3) r(y - \nu_2).
\end{aligned} \tag{7.15}$$

Again it is seen that $(C_N/N) \to 1$ as $N \to \infty$. Corresponding to Lemma 4 is the following:

LEMMA 5. *The hypotheses of Theorem* 5 *imply that for any* $\epsilon > 0$, *there is an* $M_3(\epsilon)$ *independent of* N *such that for all* $M > M_3(\epsilon)$ *and for all* $N > M$,

$$\iiiint\int_{\square_M} d\nu_1 \cdots d\nu_4\, dy |w(\nu_1, B_N\nu_2 + \nu_4) w(B_N\nu_3 + \nu_1, \nu_4) \cdot r(y) r(y + \nu_3) r(y - \nu_2)| < \epsilon.$$

Proof: The lemma follows using Schwarz's inequality (see [18] or [19]). Q.E.D.

Define

$$\begin{aligned}
\gamma_8(M, N) &= (2\pi)^{-4} \int_{-M}^{M} \cdots \int_{-M}^{M} \int_{-M}^{M} d\nu_1 \cdots d\nu_4\, dy \exp\{i\nu_1[(\mu_1 - \mu_3)/B_N] - i\nu_2\mu_2 \\
&\quad + i\nu_3\mu_3 + i\nu_4[(\mu_4 - \mu_2)/B_N]\} w(\nu_1, B_N\nu_2 + \nu_4) w(B_N\nu_3 + \nu_1, \nu_4) \\
&\quad \cdot r(y) r(y + \nu_3) r(y - \nu_2), \\
\gamma_8(M) &= \lim_{N\to\infty} \gamma_8(M, N).
\end{aligned}$$

Lemma 2, Lebesgue's theorem, and Lemma 3 gives minor term 8 as

$$\lim_{N\to\infty} \gamma_8(M, N) = (2\pi)^{-1} w_2 f(\mu_2) f(\mu_3) f(\mu_2 + \mu_3) \delta(\mu_1 - \mu_3) \delta(\mu_2 - \mu_4).$$

Certain generalizations of Theorem 5 could readily be obtained. For example, two weight functions, w_1 and w_2, could be used; different (but interrelated) sequences, $\{B_N^{(1)}\}$ and $\{B_N^{(2)}\}$, could also be used; and as pointed out earlier certain restrictions on w and the process could be relaxed slightly.

8. Further results. A later paper (part of the results to appear in this paper are contained in Rosenblatt and Van Ness [18].) will discuss the estimates in the discrete parameter case, in particular their asymptotic distribution which (for certain processes) turns out to be complex normal with independent real and imaginary parts. In the second and third-order results the assumptions that the moment functions $r(\nu)$ and $r_3(\nu_1, \nu_2)$ are in L_1 were crucial to the estimation of their transforms. When we go to higher order theory the assumption that $r_n(\nu_1, \cdots, \nu_{n-1}) \,\varepsilon\, L_1$, $n \geqq 4$ is too restrictive. This difficulty can be overcome by estimating the Fourier transforms of the cumulant functions, $\xi_n(\nu_1, \cdots, \nu_{n-1})$, instead of the transforms of the moment functions. (This is actually what we've done in the second and third-order cases since $r(\nu) = \xi(\nu)$ and $r_3(\nu_1, \nu_2) = \xi_3(\nu_1, \nu_2)$.) As stated earlier, the assumption $\xi_n(\nu_1, \cdots, \nu_{n-1}) \varepsilon L_1(R_{n-1})$ is not so restrictive.

REFERENCES

[1] Bartlett, M. S. (1955). *An Introduction to Stochastic Processes*. Cambridge Press.

[2] Blackman, R. B. and Tukey, J. W. (1959). *The Measurement of Power Spectra from the Point of View of Communications Engineering*. Dover, New York.

[3] Blanc-Lapierre, A. and Fortet, R. (1953). *Théorie des Fonctions Aléatoires*. Masson et Cie, Paris.

[4] Doob, J. L. (1953). *Stochastic Processes*. Wiley, New York.

[5] Gnedenko, B. V. and Kolmogorov, A. D. (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Cambridge.

[6] Grenander, U. and Rosenblatt, M. (1957). *Statistical Analysis of Stationary Time Series*. Wiley, New York.

[7] Hannan, E. J. (1960). *Time Series Analysis*. Methuen, London.

[8] Hasselman, K., Munk, W. and MacDonald, G. (1963). Bispectrum of ocean waves. *Proc. Symp. Time Series Analysis* (M. Rosenblatt, ed.). 125–139. Wiley, New York.

[9] Hoeffding, W. and Robbins, H. (1948). The central limit theorem for dependent random variables. *Duke Math. J.* **15** 773–780.

[10] Jenkins, G. M. (1961). General considerations in estimation of spectra. *Technometrics* **3** 319–329.

[11] Leonov, V. P. and Shiryaev, A. N. (1959). On a method of calculation of semi-invariants. (English translation). *Theor. Prob. Appl.* **4** 319–329.

[12] Magness, T. A. (1954). Spectral response of a quadratic device to non-Gaussian noise. *J. Appl. Phys.* **25** 1357–1365.

[13] Parzen, E. (1957). On consistent estimates of the spectrum of a stationary time series. *Ann. Math. Statist.* **28** 329–348.

[14] Parzen, E. (1961). Mathematical considerations in estimation of spectra. *Technometrics* **3** 167–190.

[15] Parzen, E. (1961). An approach to time series analysis. *Ann. Math. Statist.* **32** 951–989.

[16] Rosenblatt, M. (1959). Statistical analysis of stochastic processes with stationary residuals. *Probability and Statistics—The Harald Cramér Volume* (Ulf Grenander, ed.). 125–139. Wiley, New York.

[17] Rosenblatt, M. (1962). *Random Processes*. Oxford, New York.

[18] Rosenblatt, M. and Van Ness, J. W. (1964). Estimates of the bispectrum of stationary random processes. Report Nonr 562(29) 11, Division of Applied Mathematics, Brown Univ.

[19] Rosenblatt, M. and Van Ness, J. W. (1964). Estimation of the bispectrum. Report Nonr 562(29)13, Division of Applied Mathematics, Brown Univ.

[20] Sinai, Ya. G. (1963). On the properties of spectra of ergodic dynamical systems. *Dokl. Akad. Nauk, CCCP* **150** 1235–1237.

[21] Stratonovich, R. L. (1963). *Topics in the Theory of Random Noise*, **1** (English translation). Gordon and Breach, New York.

[22] Tukey, J. W. (1959). An introduction to the measurement of spectra. *Probability and Statistics—The Harald Cramér Volume* (Ulf Grenander, ed.). 300–330. Wiley, New York.

[23] Tukey, J. W. (1959). The estimation of power spectra and related quantities. *On Numerical Approximation* (R. E. Langer, ed.). 389–411. Univ. of Wisconsin Press.