A LEMMA FOR MULTIPLE INFERENCE

By William Knight

University of New Brunswick

1. Introduction and summary. Multiple comparison methods have been constructed for an analysis of variance to test simultaneously a set of hypotheses so that with a predetermined joint confidence all hypotheses rejected are false. A general description of these is given by Scheffé ([6], Chapter 3). A lemma is presented to facilitate the construction of similar methods in other areas.

2. A lemma. A family of distributions, $F(X, \theta)$, with parameter, $\theta \varepsilon \Omega$, and a set of hypotheses, $\mathbf{H}$, about θ are given. A hypothesis is taken to be a subset of Ω, heuristically the set of θ for which the hypothesis is true, Hypotheses are ordered by set inclusion, heuristically $H_1 \subset H_2$ means that H_1 implies H_2 or that H_1 is the stronger hypothesis.

Assume each H in $\mathbf{H}$ is tested by a statistic, $T(H) = T(X, H)$. For convenience arrange that H is rejected when $T(H)$ is large.

The following simultaneous test for all H in $\mathbf{H}$ is proposed. Assume that $\bar{H}$, the intersection of all members of $\mathbf{H}$, is a non-empty member of $\mathbf{H}$ and that a critical point, c, for $\bar{H}$ is defined by

$$\text{Prob}\,(T(\bar{H}) \geqq c) \equiv \alpha, \tag{1}$$

the identity holding for all $\theta \varepsilon \bar{H}$. All hypotheses in $\mathbf{H}$ for which $T(H) \geqq c$ are rejected.

Lemma. *If*

(i) $\mathbf{H}$ *is closed under intersection in the sense that the intersection of all members of any subset of* $\mathbf{H}$ *is a member of* $\mathbf{H}$, *and moreover* $\bar{H}$, *the intersection of all members of* $\mathbf{H}$ *is non-empty,*

(ii) T *is non-increasing in* H *in the sense that* $H_1 \subset H_2$ *implies* $T(H_1) \geqq T(H_2)$, *and*

(iii) *the distribution of* $T(H)$ *is identical for all* θ *in* H *provided* H *is in* $\mathbf{H}$,
then the probability that no true hypothesis in $\mathbf{H}$ *is rejected is at least* $1 - \alpha$.

Proof. Let $\mathbf{K}$ be the set of true hypotheses in $\mathbf{H}$ and $\bar{K}$ be the intersection of all members of $\mathbf{K}$. By (i) $\bar{K}$ is a non-empty member of $\mathbf{H}$ and being true it is in $\mathbf{K}$. Since $T(\bar{K})$ is an upper bound of $T(K)$, $K \varepsilon \mathbf{K}$, it is sufficient to show that $\text{Prob}\,(T(\bar{K}) \geqq c \mid \theta \varepsilon \bar{K}) \leqq \alpha$.

$$\begin{aligned} \text{Prob}\,(T(\bar{K}) \geqq c \mid \theta \varepsilon \bar{K}) &= \text{Prob}\,(T(\bar{K}) \geqq c \mid \theta \varepsilon \bar{H}) \\ &\leqq \text{Prob}\,(T(\bar{H}) \geqq c \mid \theta \varepsilon \bar{H}) = \alpha. \end{aligned} \tag{2}$$

The equality is a consequence of (iii), the inequality of (ii).

3. Likelihood ratio tests. The lemma seems to adapt itself naturally to likelihood ratio tests. When the $T(H)$ are negative logarithm likelihood ratio sta-

Received 5 May 1964; revised 16 July 1965.

tistics and the usual regularity conditions (those given, for example, in [7], Section 13.8) assuring that the asymptotic distributions of these are chi-square are satisfied, then Condition (i) alone is sufficient for the lemma to hold asymptotically.

First, Condition (ii) is automatically satisfied since

$$\sup_{\theta \varepsilon H} dF(X, \theta)/\sup_{\theta \varepsilon \Omega} dF(X, \theta) \tag{3}$$

is non-decreasing in H and its negative logarithm non-increasing. Second, Condition (iii) holds asymptotically, the common limiting distribution for all θ in H being chi-square with degrees of freedom determined by the dimensions of H and Ω. The only change entailed in the proof of the lemma is the replacement of the probabilities by their limits.

4. Examples. The following test is equivalent to Scheffé's test [5]. X is spherically normally distributed, **G** a set of linear hypotheses about $E(X)$ for which $\bar{G}$, the intersection of all members of **G** is non-empty, i.e. the hypotheses in **G** are mutually consistent. Define **H** as the set of hypotheses generated by **G** under intersection. Members of **H** thus specify that $E(X)$ lies in some subspace of its space of possible values. The likelihood ratio tests of members of **H** are equivalent to the usual F tests. Condition (i) is satisfied by the definition of H, (ii) by the likelihood ratio statistics, (iii) since the F ratio has an F distribution under the null hypothesis.

A multinomial test. $X = (x_1, x_2, \cdots, x_k)$ is multinormially distributed, **G** a set of linear hypotheses about $E(X)$ for which $\bar{G}$ is non-empty. Define **H** as the set of hypotheses generated by **G** under intersection. The likelihood ratio tests of members of **H** lead to entropy or information statistics which are asymptotically chi-square in distribution. (For discussion of this see [3], Sections 5.5, 5.6, and Chapter 6.) Similar tests can be made with contingency tables [2], [3] Chapter 8.

REFERENCES

[1] Dunn, Olive Jean (1964). Multiple comparisons using rank sums. *Technometrics* **6** 241–252.

[2] Garner, W. R. and McGill, William J. (1956). The relation between information and variance analyses. *Psychometrika* **21** 219–228.

[3] Kullback, Solomon (1959). *Information Theory and Statistics*. Wiley, New York.

[4] Lehmann, E. L. (1957). A theory of some multiple decision problems, I. *Ann. Math. Statist.* **28** 1–25.

[5] Scheffé, H. (1953). A method for judging all contrasts in the analysis of variance. *Biometrika* **40** 87–104.

[6] Scheffé Henry (1959). *The Analysis of Variance*. Wiley, New York.

[7] Wilks, Samuel S. (1962). *Mathematical Statistics*. Wiley, New York.