A NOTE ON THE SEQUENTIAL t-TEST[1]

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The purpose of this note is to show how the results and methods of [2] can be used to deduce the asymptotic behavior of the expected sample size for the sequential t-test as the bounds get large. Implicit in this deduction is a proof that the expected sample size is finite, a result also obtained by Ifram [1] who uses somewhat different methods.

We shall only consider the one-sided sequential t-test of $H_0 : \mu = 0, \sigma > 0$ against $H_1 : \mu = \sigma, \sigma > 0$. It is easy to see that the same ideas will work when the alternative hypothesis is $\mu = \delta\sigma$ where δ is some positive number and also if the alternative hypothesis is $|\mu| = \delta\sigma$. We define the sequential t-test with bounds A, A^{-1} (we choose symmetric bounds for convenience) for the problem under consideration as follows: Put $f_\sigma(x_1, \cdots, x_n) = (1/\sigma^n) \exp\{-(1/\sigma^2) \sum_1^n (x_i - \sigma)^2\}$, $g_\sigma(x_1, \cdots, x_n) = (1/\sigma^n) \exp\{-(1/2\sigma^2) \sum_1^n x_i^2\}$. Take an $(n + 1)$th observation if

$$(1) \qquad \int_0^\infty f_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma) < A \int_0^\infty g_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma)$$

and if

$$(2) \qquad \int_0^\infty f_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma) > A^{-1} \int_0^\infty g_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma),$$

stop and accept H_0 if (2) is violated, and stop and accept H_1 if (1) is violated. The similarities between this procedure and those discussed in [2] (see, in particular, Section 2 of [2]) are obvious; the only difference is that the *a priori* distribution in this case has infinite variation and it is this difference which prevents us from drawing the desired conclusions immediately.

Let N be the number of observations required by this sequential t-test to terminate. We shall concern ourselves with EN when $\mu = 1, \sigma = 1$ and we will show that $EN \sim \log A/(\frac{1}{2} \log 2)$ as $A \to \infty$; the dependence of the distribution of N only on μ/σ then will yield the same asymptotic value for EN for any μ, σ with $\mu = \sigma$ i.e. for any point in H_1. We obtain the asymptotic value of EN by relating the sequential t-test to one of the kind considered in [2]. To accomplish this we first show that there exists $\epsilon > 0$ and $0 < a < 1$ such that

$$(3) \qquad C_n = \frac{\int_0^a g_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma) + \int_{1/a}^\infty g_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma)}{\int_a^{1/a} g_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma)} \leqq 1$$

whenever $2 - \epsilon \leqq \sum_1^n x_i^2/n \leqq 2 + \epsilon$, and such that

$$(4) \qquad D_n = \frac{\int_0^a f_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma) + \int_{1/a}^\infty f_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma)}{\int_a^{1/a} f_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma)} \leqq 1$$

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whenever $2 - \epsilon \leqq \sum_1^n x_i^2/n \leqq 2 + \epsilon$ and $1 - \epsilon \leqq \sum_1^n x_i/n \leqq 1 + \epsilon$. To see that (3) is so put $V_n = \sum_1^n x_i^2/n$ and note that

$$\int_0^a g_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma) \leqq \max_{0<\sigma\leqq a} (1/\sigma^{n+1}) \exp[-(1/2\sigma^2)V_n] \int_0^a d\sigma$$
$$= (1/a^n) \exp[-(1/2a^2)V_n] = K_n \quad \text{(say)}$$

whenever $a \leqq V_n/(n + 1)$;

$$\int_{1/a}^\infty g_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma) \leqq \max_{\sigma \geqq 1/a} (1/\sigma^{n-1}) \exp[-(1/2\sigma^2)V_n] \int_{1/a}^\infty (d\sigma/\sigma^2)$$
$$= a^n \exp(-\tfrac{1}{2}a^2 V_n) = L_n \quad \text{(say)}$$

whenever $1/a \geqq V_n/(n - 1)$;

$$\int_b^c g_\sigma(x_1, \cdots, x_n)(d\sigma/\sigma)$$
$$\geqq \min\{(1/b^n) \exp[-(1/2b^2)V_n], (1/c^n) \exp[-(1/2c^2)V_n]\} \log(c/b)$$

whenever $b \leqq V_n/n \leqq c$. By choosing $b = 2 - \epsilon$, $c = 2 + \epsilon$, and a sufficiently close to 0 it is easy to see that (3) is satisfied. Similar tactics work in establishing (4) where the extra condition on $\sum_1^n x_i/n$ is needed.

Let N_1 be the first time that (1) is violated. Then $N \leqq N_1$. Let ν_a be the first n such that

$$\int_a^{1/a} f_\sigma(d\sigma/\sigma) > A \int_0^\infty g_\sigma(d\sigma/\sigma) = A \int_a^{1/a} g_\sigma(d\sigma/\sigma)\{1 + C_n\}.$$

Then $N_1 \leqq \nu_a$. Let M_a be the first n such that

$$\int_a^{1/a} f_\sigma(d\sigma/\sigma) > 2A \int_a^{1/a} g_\sigma(d\sigma/\sigma)$$

and let T be the *last* time $\sum_1^n X_i^2/n > 2 + \epsilon$ or $\sum_1^n X_i^2/n < 2 - \epsilon$. If a and ϵ are chosen so that (3) holds then from

$$P\{\nu_a \geqq k\} \leqq P\{T \geqq k\} + P\{T < k, \nu_a \geqq k\}$$
$$\leqq P\{T \geqq k\} + P\{T < k, M_a \geqq k\}$$
$$\leqq P\{T \geqq k\} + P\{M_a \geqq k\}$$

we conclude that

$$EN \leqq E\nu_a \leqq ET + EM_a. \tag{5}$$

When $\mu = 1, \sigma = 1$ so that $EX_1^2 = 2$, it is known that $ET < \infty$ (see for example Theorem D in the Appendix of [2]) and does not depend on A. To obtain an upper bound on M_a we can cite the result of Lemma 2 in [2] (the *a priori* distribution which is relevant is the one with density $1/\sigma 2 |\log a|$ for $a \leqq \sigma \leqq 1/a$ and 0 elsewhere, and we replace the c of [2] by $1/2A$) which yields

$$EM_a \leqq [1 + o(1)] \log 2A/\inf_{\sigma>0} E_{\mu=1,\sigma=1} \log[f_1(X_1)/g_\sigma(X_1)] \tag{6}$$
$$= [1 + o(1)] \log 2A/\tfrac{1}{2} \log 2$$

where the $o(1)$ term goes to 0 as $A \to \infty$. (5) and (6) yield

(7) $$\limsup_{A\to\infty} (EN/\log A) \leqq 2/\log 2.$$

To obtain a bound in the other direction let ν_a' be the first n such that

$$A^{-1} \int_a^{1/a} g_\sigma(d\sigma/\sigma)\{1 + C_n\} < \int_a^{1/a} f_\sigma(d\sigma/\sigma)\{1 + D_n\} < A \int_a^{1/a} g_\sigma(d\sigma/\sigma)$$

is violated; let M_a' be the first n such that

$$2A^{-1} \int_a^{1/a} g_\sigma(d\sigma/\sigma) < \int_a^{1/a} f_\sigma(d\sigma/\sigma) < \tfrac{1}{2}A \int_a^{1/a} g_\sigma(d\sigma/\sigma)$$

is violated ; and let T' be the smallest integer m such that $2 - \epsilon < \sum_1^n X_i^2/n < 2 + \epsilon$ and $1 - \epsilon < \sum_1^n X_i/n < 1 + \epsilon$ for all $n \geqq m$. Then, by use of (3) and (4), $M_a' \leqq \max(\nu_a', T')$ so that $EN \geqq E\nu_a' \geqq EM_a' - ET'$. ET' is finite and independent of A (see Theorem D in the Appendix of [2]) so that using the Corollary to Theorem 1 in [2] we obtain

(8) $$\liminf_{A\to\infty} (EN/\log A) \geqq 2/\log 2.$$

(7) and (8) then yield

$$\lim_{A\to\infty} (EN/\log A) = 2/\log 2$$

all μ, σ in H_1. The same result also holds when $\mu = 0$.

REMARKS. 1. Some further computation can be made which will show that $Ee^{tN} < \infty$ for t in some neighborhood of 0. In particular, it isn't too hard to show that $P\{T \geqq k\} \leqq \rho^k$ for some $0 < \rho < 1$ and inspection of the proof of Lemma 2 of [2] and some additional computation will yield $P\{M_a \geqq k\} \leqq \alpha^k$ for some $0 < \alpha < 1$.

2. An upper bound for EN when $\mu/\sigma \neq 0$ or 1 can be obtained from the above except when $\mu/\sigma = \frac{1}{2}$ in which case we are unable to make the computation but we are led to believe that the same phenomenon occurs as discussed in Sections 3 and 4 of [2], namely, that the sequential t-test has bad asymptotic properties when $\mu/\sigma = \frac{1}{2}$ and that a modification of the test as discussed for related tests in [2] would be in order.

3. When $\mu/\sigma = 0$ or 1 it can be shown along the lines of the argument leading to Theorem 2 of [2] that the probability of error is $o(\log A/A)$. We are unable to verify whether $o(\log A/A)$ can be replaced by $O(A^{-1})$.

REFERENCES

[1] IFRAM, A. (1965). On the sample size of the ordinary, asymptotic and simplified SPRT's based on a sequence of some generality. *Ann. Math. Stat.* To be published.

[2] KIEFER, J. and SACKS, J. (1963). Asymptotically optimum sequential inference and design. *Ann. Math. Statist.* **34** 705–750.