CONVERGENCE RATES FOR THE LAW OF LARGE NUMBERS FOR LINEAR COMBINATIONS OF EXCHANGEABLE AND *-MIXING STOCHASTIC PROCESSES[1]

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1. Introduction and summary. Let $\xi = \{\xi_k \mid k = 0, \pm 1, \cdots\}$ be a sequence of real valued random variables with $E\xi_k \equiv 0$ and let $\{a_{n,k} \mid n = 1, 2, \cdots; k = 0, \pm 1, \cdots\}$ be a doubly indexed sequence of real numbers such that

$$(1) \qquad \sum_k |a_{n,k}| \leqq 1 \quad \text{for} \quad n = 1, 2, \cdots,$$

and

$$(2) \qquad \lambda(n) = \max_k |a_{n,k}| \to 0 \quad \text{as} \quad n \to \infty.$$

(In (1) we have replaced the upper bound $A < \infty$ of Condition 2 [3] by 1 which clearly entails no loss of generality.)

The following result was established in [3]:

THEOREM A. *If ξ is an independent sequence of random variables and if the moment generating functions, $f_k(t)$, of the ξ_k's exist and satisfy the condition;*

(3) *for every $\beta > 0$ there exists $T_\beta > 0$ such that $|t| \leqq T_\beta$ implies*

$$f_k(t) \leqq \exp(\beta|t|) \text{ uniformly in } k,$$

then the random variables $S_n = \sum_{k=-\infty}^{\infty} a_{n,k}\xi_k$ are defined almost surely as limits of the partial sums for each n and for every $\epsilon > 0$ there exists $\rho < 1$ (which depends on ϵ and T_β but not on the particular $a_{n,k}$'s) such that

$$P[|S_n| \geqq \epsilon] \leq 2\rho^{1/\lambda(n)}.$$

This result was used to establish exponential convergence rates for the law of large numbers for arbitrary subsequences of linear stochastic processes with absolutely convergent coefficients and for the convergence in probability to zero of Toeplitz means of independent random variables, extending previous results in [1] and [4].

In the present paper we investigate upper bounds on $P[|S_n| \geqq \epsilon]$ for two types of discrete parameter stochastic processes, ξ, closely related to independent sequences; exchangeable processes and *-mixing processes. In Section 2 we establish a basic theorem for exchangeable processes, analogous to Theorem A, which enables us to state conditions leading to upper bounds which tend to

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zero with n at virtually any sub-exponential rate. In Section 3 these bounds are shown to be sharp in certain important special cases by exhibiting a mixture of normal random variables which actually attains them under the given conditions. In Section 4, the exponential bounds obtained in [3] for independent sequences is shown to carry over to *-mixing processes, thus extending a result in [2].

2. Convergence rates for exchangeable processes. Let ξ be a process of exchangeable random variables over a probability space $(\Omega, \mathcal{A}, P)$, i.e., the joint distribution of $\xi_{i_1}, \cdots, \xi_{i_n}$ is equal to that of $\xi_1, \cdots, \xi_n$ for any selection of distinct integers $i_1, \cdots, i_n$ and any $n = 1, 2, \cdots$. Then ([5], p. 365) there exists a sub σ-field $\mathcal{B}$ of $\mathcal{A}$ such that the ξ_k's are conditionally independent and identically distributed given $\mathcal{B}$. Moreover, ([5], p. 363) the conditional distribution of ξ_0 can be assumed regular. Conditional expectation relative to this distribution will be denoted by $E^{\mathcal{B}}$.

We assume that $E\xi_0 = 0$. This implies that $\mu(\omega) = E^{\mathcal{B}}\xi_0(\omega)$ exists a.e. and, centering the ξ_k's at their conditional expectations, we can assume that

$$\mu(\omega) = 0 \quad \text{a.e.}. \tag{4}$$

Let

$$f^{\mathcal{B}}(t)(\omega) = E^{\mathcal{B}}e^{t\xi_0}(\omega).$$

This conditional moment generating function will exist in some symmetric (possibly degenerate), closed interval $[-T(\omega), T(\omega)]$ about the origin for almost all ω. We will also make the following assumption which is the analog of Condition (3):

for every β, $0 < \beta \leqq 1$, let $T_\beta(\omega)$ be the largest value of $|t|$ such that

$$f^{\mathcal{B}}(t)(\omega) \leqq e^{\beta|t|}. \tag{5}$$

We assume that $P[T_\beta(\omega) > 0] = 1$ *for all* β, $0 < \beta \leqq 1$.

Note that $P[T_\beta(\omega) \leqq T_{\beta'}(\omega)] = 1$ whenever $0 < \beta < \beta' \leqq 1$. In case $T_\beta(\omega) = \infty$ on a set of positive probability for $\beta_0 \leqq \beta \leqq 1$, we can and will redefine these random variables so that $P[T_\beta(\omega) \leqq T_{\beta'}(\omega) < \infty] = 1$ for all $0 < \beta < \beta' \leqq 1$.

The following lemma guarantees the existence of $S_n = \sum_{-\infty}^{\infty} a_{n,k}\xi_k$ and provides an expression for its conditional moment generating function.

LEMMA 1. *Let $\{a_{n,k} : k = 0, \pm 1, \cdots ; n = 1, 2, \cdots\}$ be a sequence of real numbers satisfying Conditions* (1) *and* (2), *and let ξ be a process of exchangeable random variables satisfying Conditions* (4) *and* (5). *Then there exists a random variable $T(\omega)$ with $P[0 < T(\omega) < \infty] = 1$ such that*

$$g_n^{\mathcal{B}}(t)(\omega) = \lim_{a\to-\infty, b\to\infty} \prod_{k=a}^{b} f^{\mathcal{B}}(a_{n,k}t)(\omega)$$

exists for all n and $|t| < T(\omega)$ a.e.. *The random variable*

$$S_n = \lim_{a\to-\infty, b\to\infty} \sum_{k=a}^{b} a_{n,k}\xi_k$$

exists as an almost sure limit and has conditional moment generating function $g_n^{\mathcal{B}}(t)(\omega)$. *Moreover, if* ξ_0 *possesses all moments of order* $\leqq p$, *so does* S_n. *In particular,* $ES_n = 0$.

PROOF. Define $T(\omega) = T_1(\omega)$. Then, by Assumption (5), $P[0 < T(\omega) < \infty] = 1$. Now the proof of the lemma in Section 2 of [3] with $\beta = 1$ can be carried over for almost all ω to establish that $P^{\mathcal{B}}(C_n)(\omega) = 1$ a.e., where C_n is the convergence set of the partial sums $\{\sum_{k=a}^{b} a_{n,k}\xi_k\}$, and that $E^{\mathcal{B}} \exp(tS_n)(\omega) = g_n^{\mathcal{B}}(t)(\omega)$ for $|t| < T(\omega)$ a.e. . Then $P(C_n) = EP^{\mathcal{B}}(C_n)(\omega) = 1$ which establishes the first part of the lemma.

Let

$$\mathfrak{M}_p = \|\xi_0\|_p = [E|\xi_0|^p]^{1/p}$$

for $p \geqq 1$. If $\mathfrak{M}_p < \infty$, then for $c < a \leqq -N$ and $N \leqq b < d$ we have

$$\|\textstyle\sum_a^b a_{n,k}\xi_k - \sum_c^d a_{n,k}\xi_k\|_p \leqq [(\sum_c^{a-1} + \sum_{b+1}^{d})|a_{n,k}|]\mathfrak{M}_p \to 0 \quad \text{as} \quad N \to \infty$$

by Assumption 1. By the completeness of $\mathcal{L}_p(P)$ there exists $S_n^{(p)}$ with $\|S_n^{(p)}\|_p < \infty$ such that

$$\|\textstyle\sum_a^b a_{n,k}\xi_k - S_n^{(p)}\|_p \to 0$$

as $a \to -\infty$, $b \to \infty$. Then, since S_n is defined as the almost sure limit of the partial sums, S_n and $S_n^{(p)}$ are equivalent random variables for all $p \geqq 1$. Since we have assumed $E\xi_0 = 0$, we see that $\|\xi_0\|_1 < \infty$ and

$$ES_n = \lim_{a\to-\infty, b\to\infty} E\textstyle\sum_a^b a_{n,k}\xi_k = 0.$$

The following theorem is the analog of Theorem A for exchangeable processes.

THEOREM 1. *Let* $T_\beta(\omega)$ *satisfy Condition* (5). *Then the moment generating function,* $g_\beta(t)$, *of* $T_\beta(\omega)$ *exists for all* $t \leqq 0$, *and for every* $\epsilon > 0$ *there exists* $\beta > 0$ *and* $t^* < 0$ *such that*

$$P[|S_n| \geqq \epsilon] \leqq 2g_\beta(t^*[\lambda(n)]^{-1}).$$

PROOF. Since $P[T_\beta(\omega) > 0] = 1$, $g_\beta(t) = E\exp[tT_\beta(\omega)] \leqq 1$ for all $t \leqq 0$.

Fix $\epsilon > 0$ and let $\beta = \epsilon/2$. Then, using the inequality $P[X \geqq 0] \leqq Ee^{tX}$ for $t \geqq 0$ and the fact that $P[T_\beta(\omega) \leqq T(\omega)] = 1$, it follows from Lemma 1 that

$$\begin{aligned} P[\pm S_n \geqq \epsilon] &= EP^{\mathcal{B}}[\pm S_n \geqq \epsilon](\omega) \\ &\leqq EE^{\mathcal{B}} \exp\{(\pm S_n - \epsilon)[\lambda(n)]^{-1}T_\beta(\omega)\} \\ &= E\exp\{-\epsilon T_\beta(\omega)[\lambda(n)]^{-1}\}\textstyle\prod_{k=-\infty}^{\infty} f^{\mathcal{B}}(\pm a_{n,k}[\lambda(n)]^{-1}T_\beta(\omega)) \\ &\leqq E\exp\{(-\epsilon + \beta\textstyle\sum|a_{n,k}|)[\lambda(n)]^{-1}T_\beta(\omega)\} \\ &\leqq g_\beta((\beta - \epsilon)[\lambda(n)]^{-1}) = g_\beta(t^*[\lambda(n)]^{-1}), \end{aligned}$$

where $t^* = -\epsilon/2$.

Since $P[|S_n| \geqq \epsilon] \leqq P[S_n \geqq \epsilon] + P[-S_n \geqq \epsilon]$, the theorem is proved.

This theorem leads to a wide range of convergence rates depending on the concentration of the probability of $T_\beta(\omega)$ near zero. For illustration, we provide the following examples.

COROLLARY 1. *If for every β, $0 < \beta \leqq 1$, there exists $T_\beta > 0$ such that $P[T_\beta(\omega) \geqq T_\beta] = 1$, then the moment generating function of S_n exists for $|t| \leqq T_1$ and for every $\epsilon > 0$ there exists $\rho < 1$ depending only on ϵ and T_β such that*

$$P[|S_n| \geqq \epsilon] \leqq 2\rho^{1/\lambda(n)}.$$

PROOF. From Lemma 1 and the condition of this corollary, the conditional moment generating function of S_n exists for $|t| \leqq T_1$. Thus,

$$g_n(t) = Eg_n^{\mathcal{B}}(t)(\omega),$$

the unconditional moment generating function of S_n, exists for $|t| \leqq T_1$.

Moreover, with t^* defined in Theorem 1,

$$g_\beta\{t^*[\lambda(n)]^{-1}\} = E \exp\{t^* T_\beta(\omega)[\lambda(n)]^{-1}\} \leqq \exp\{t^* T_\beta[\lambda(n)]^{-1}\} = \rho^{1/\lambda(n)},$$

where $\rho = \exp(t^* T_\beta)$.

This is the only case in which an exponential bound can be obtained from Theorem 1 as is shown by the following lemma. (The case $P[T_\beta(\omega) = 0] > 0$ is clearly excluded.)

LEMMA 2. *Suppose $0 < \beta \leqq 1$. If $P[0 < T_\beta(\omega) < \eta] > 0$ for every $\eta > 0$, then for each ρ, $0 < \rho < 1$ and $M > 0$ there exists $T = T(\rho, M) > 0$ such that for all $t \geqq T$,*

$$g_\beta(-t) > M\rho^t.$$

PROOF. Let $\epsilon = -\log \rho$ and $0 < \eta < \epsilon$. Set

$$S_{n,\eta} = [\eta/2^n \leqq T_\beta(\omega) < \eta/2^{n-1}].$$

Then $\bigcup_{n=1}^\infty S_{n,\eta} = [0 < T_\beta(\omega) < \eta]$ which implies $P(S_{n,\eta}) > 0$ for some n. Let $\delta = \eta/2^{n-1}$ and $I = [\frac{1}{2}\delta \leqq T_\beta(\omega) < \delta]$. Then, if F_β is the distribution function of $T_\beta(\omega)$,

$$\begin{aligned} g_\beta(-t) &= \int_0^\infty e^{-tx}\, dF_\beta(x) \\ &\geqq \int_I e^{-tx}\, dF_\beta(x) \geqq e^{-t\delta} \int_I dF_\beta(x) \\ &= \rho^t e^{(\epsilon-\delta)t} P[\tfrac{1}{2}\delta \leqq T_\beta(\omega) < \delta] \end{aligned}$$

by definition of ϵ. The lemma now follows, since it is clear that the multiplier of ρ^t can be made arbitrarily large for sufficiently large t.

The next two examples are special cases of the following result.

LEMMA 3. *Let $p(x) = e^{f(x)}h(x)$ be a probability density on $(0, \infty)$ where*

$$\int_0^\infty h(x) = M > 0$$

and $f(x)$ is a twice differentiable function on $(0, \infty)$ such that

(i) $f(0^+) = -\infty$,
(ii) $f'(x) \geqq 0$ on $(0, \infty)$ with $f'(0^+) = \infty$,
(iii) $f''(x) < 0$ on $(0, \infty)$.

Then there exists $T \geqq 0$ *and a continuous function* $r(t)$ *(explicitly constructed in the proof) with* $r(t) \to -\infty$ *as* $t \to \infty$ *such that for all* $t \geqq T$,

$$\int_0^\infty e^{-tx} p(x)\, dx \leqq M e^{r(t)}.$$

Proof. Let $R(x, t) = f(x) - tx$. Then, if $T = \inf_x f'(x)$, the equation $R_x(x, t) = f'(x) - t = 0$ possesses a continuous solution, $x(t) \geqq 0$, for all $t \geqq T$ by the implicit function theorem. Since $R_{xx}(x, t) = f''(x) < 0$, this solution is unique and maximizes $R(x, t)$ as a function of x. Let $r(t) = R(x(t), t)$. Since, by (ii), $x(t) \to 0$ as $t \to \infty$, $r(t) \to -\infty$ as $t \to \infty$ because of (i). Moreover, for $t \geqq T$,

$$\int_0^\infty e^{-tx} p(x)\, dx = \int_0^\infty e^{R(x,t)} h(x)\, dx \leqq M e^{r(t)},$$

as was to be shown.

It is possible for the exponential bound to be approached closely even when $P[0 < T_\beta(\omega) < \eta] > 0$ for every $\eta > 0$:

Corollary 2. *Suppose* $T_\beta(\omega)$ *is absolutely continuous with probability density*

$$p_\beta(x) = \exp\{-1/x^s\} h_\beta(x), \qquad s > 0$$

where

$$\int_0^\infty h_\beta(x)\, dx = M_\beta < \infty$$

for $0 < \beta \leqq 1$. *Then, for every* $\epsilon > 0$ *there exists* ρ, $0 < \rho < 1$, *such that*

$$P[|S_n| \geqq \epsilon] \leqq 2 M_\beta \rho^{(1/\lambda(n))^{s/s+1}}.$$

Proof. $h_\beta(x)$ and $f(x) = -1/x^s$ satisfy the conditions of Lemma 3 for $0 < \beta \leqq 1$, and a simple computation yields

$$r(t) = -[s^{-s/s+1} + s^{1/s+1}] t^{s/s+1}.$$

Thus, by Theorem 1, $P[|S_n| \geqq \epsilon] \leqq 2 M_\beta \rho^{(1/\lambda(n))^{s/s+1}}$ where $\log \rho = r(-t^*)$.

Another application of Lemma 3 yields an algebraic bound:

Corollary 3. *If* $T_\beta(\omega)$ *is absolutely continuous with probability density*

$$p_\beta(x) = x^\nu h_\beta(x), \qquad \nu > 0$$

where

$$\int_0^\infty h_\beta(x)\, dx = M_\beta < \infty$$

for $0 < \beta \leqq 1$, *then for every* $\epsilon > 0$ *there exists* $M > 0$ *such that*

$$P[|S_n| \geqq \epsilon] \leqq M[\lambda(n)]^\nu.$$

Proof. The conditions of Lemma 3 are satisfied with $h(x) = h_\beta(x)$ and $f(x) = \nu \log x$. Now, $r(t) = \nu \log(\nu/t) - \nu$ and the corollary follows from

Theorem 1 and Lemma 3 with

$$M = 2M_\beta e^{\nu}(-t^*/\nu)^{\nu}.$$

In the case of algebraic bounds, a slightly better result can be obtained for commonly occurring damping functions, $h_\beta(x)$, such as $h_\beta(x) = a(\beta)$ exp $\{-b(\beta)x\}$ by using Watson's lemma ([6], p. 37). This is illustrated in the following corollary:

COROLLARY 4. *If $T_\beta(\omega)$ is absolutely continuous with probability density of the form given in Corollary 3 where, in addition, $h_\beta(x)$ has a power series expansion for all $x \geqq 0$ with $0 < h_\beta(0) < \infty$, then for every $\epsilon > 0$ there exists $M > 0$ such that*

$$P[|S_n| \geqq \epsilon] \leqq M[\lambda(n)]^{\nu+1}.$$

PROOF. We can write

$$p_\beta(x) = \sum_{j=0}^{\infty} a_j x^{j+\nu}$$

and, because $h_\beta(x)$ is integrable, the conditions of Watson's lemma are satisfied. Since $p_\beta(x)$ is a probability density it follows that $a_0 > 0$ and, from Watson's lemma,

$$[g_\beta(-t) - (a_0\Gamma(\nu + 1)/t^{\nu+1})]/t^{-(\nu+1)} = o(1) \qquad \text{as} \quad t \to \infty.$$

Thus, for $\delta > 0$ and $M' = a_0\Gamma(\nu + 1) + \delta$ there exists T_δ such that $t \geqq T_\delta$ implies $g_\beta(-t) \leqq M't^{-(\nu+1)}$.

The result now follows from Theorem 1 by selecting M sufficiently large so that $M[\lambda(n)]^{\nu+1} \geqq 1$ for all n such that $|t^*/\lambda(n)| < T_\delta$.

We note, without proof, the following extensions:

1. Many rates other than those given in Corollaries 1 through 4 are possible. In fact it is not difficult to establish a converse to Lemma 3 wherein functions $r(t)$ can be specified and the corresponding densities yielding the desired rates constructed.

2. The assumption of absolute continuity in Lemma 3 and Corollaries 2 and 3 can be relaxed to admit arbitrary distributions for which the distribution function satisfies

$$F_\beta(x) \leqq \int_0^x p_\beta(y)\, dy$$

for $0 < \beta \leqq 1$ and all x and in some neighborhood of the origin.

3. The equality of the conditional distributions of the ξ_k's is a consequence of the exchangeability of the process ξ and is not an essential requirement of the theory. If the ξ_k's are conditionally independent given $\mathcal{B}$ but have different conditional moment generating functions, $f_k^{\mathcal{B}}(t)(\omega)$, in order to carry over the results of Lemma 1 and Theorem 1, it suffices to redefine $T_\beta(\omega)$ in Condition 5 as the largest value of $|t|$ for which

$$f_k^{\mathcal{B}}(t)(\omega) \leqq e^{\beta|t|}, \qquad \text{uniformly in} \quad k.$$

As before, it is assumed that $E^{\mathcal{B}}\xi_k = 0$ a.e.

In this context, the theory is applicable to compound experiments in which an experimental unit is selected from a population according to a given distribution and a sequence of independent but not necessarily identically distributed measurements are made on it. Then Theorem 1 provides a bound for the tail probabilities of linear combinations of the measurements averaged over the selection distribution for the population. We illustrate this application by the following easily derivable result:

THEOREM 2. *Let* $\mathbf{p} = \{p_j \mid j = 1, 2, \cdots\}$ *be a probability distribution on the positive integers and for each* $j = 1, 2, \cdots$, *let* $\xi_j = \{\xi_{jk} : k = 0, \pm 1, \cdots\}$ *be a sequence of independent random variables for which* $E\xi_{jk} = 0$ *for all* k. *For each* j *and* $0 < \beta \leqq 1$, *let* $T_{\beta,j}$ *be the largest value of* $|t|$ *such that*

$$E \exp (t\xi_{jk}) \leqq e^{\beta|t|} \quad \textit{uniformly in} \quad k.$$

Furthermore, let $\{a_{n,k} \mid n = 1, 2, \cdots; k = 0, \pm 1, \cdots\}$ *be a double sequence of real numbers satisfying Conditions* 1 *and* 2.

Now, if $\xi = \{\xi_k \mid k = 0, \pm 1, \cdots\}$ *is selected from* $\{\xi_j \mid j = 1, 2, \cdots\}$ *according to the distribution* $\mathbf{p}$ (i.e. $P[\xi = \xi_j] = p_j, j = 1, 2, \cdots$) *and if*

$$S_n = \sum_{k=-\infty}^{\infty} a_{n,k}\xi_k ,$$

then for every $\epsilon > 0$, *there exist numbers* $t^* > 0$ *and* $0 < \beta \leqq 1$ *such that*

$$P[|S_n| \geqq \epsilon] \leqq 2 \sum_{j=1}^{\infty} p_j \exp \{-t^* T_{\beta,j}[\lambda(n)]^{-1}\}.$$

3. Example of an exchangeable process for which the bounds are attained. An exchangeable process will be constructed as follows: Let α be a non-negative random variable with distribution function $G(\alpha)$. Then, for fixed α, the random variables ξ_k are taken to be independently and identically distributed with zero mean and varance α. The joint distribution of $\xi_{i_1}, \cdots, \xi_{i_n}$,

$$F_{\xi_{i_1},\cdots,\xi_{i_n}}(x_1, \cdots, x_n) = \int_0^\infty \prod_{i=1}^n \Phi_\alpha(x_i)\, dG(\alpha),$$

where

$$\Phi_\alpha(x) = (2\pi\alpha)^{-\frac{1}{2}} \int_{-\infty}^x e^{-y^2/2\alpha}\, dy,$$

is that of an exchangeable process. We will show that if $\sum a_{n,k}^2 \geqq \delta\lambda(n)$ for n sufficiently large and some $\delta > 0$, then the exponential and algebraic convergence rates of Corollaries 1 and 4 are the best rates that hold for this mixture of normal probabilities under the specified conditions on the distribution of $T_\beta(\omega)$ given in the corollaries. For this purpose we will require the following lemma:

LEMMA 4. *Let* ξ *be the above defined normal mixture, and let*

$$A_n = \sum_{j=-\infty}^{\infty} a_{n,j}^2 .$$

Then for every $\epsilon > 0$ *and* β, $0 < \beta \leqq 1$,

$$P[|S_n| \geqq \epsilon] = \tfrac{1}{2}\pi^{-\frac{1}{2}} \int_{\epsilon^2/4\beta A_n}^\infty x^{-\frac{1}{2}} \int_0^\infty e^{-xy} y^{\frac{1}{2}}\, dH_\beta(y)\, dx,$$

where $H_\beta(y)$ *is the probability distribution function of* $T_\beta(\omega)$.

PROOF. The conditional moment generating function of ξ_0 is $\exp(\alpha t^2/2)$. Thus, from Condition (5), $T_\beta(\alpha) = 2\beta/\alpha$. Moreover,

$$S_n = \sum_{k=-\infty}^{\infty} a_{n,k}\xi_k$$

is conditionally normally distributed with zero mean and variance $A_n\alpha$. Thus, if $G(\alpha)$ is the distribution function of α,

$$P[|S_n| \geqq \epsilon] = \int_0^\infty \int_{|t|\geqq \epsilon/(A_n)^{\frac{1}{2}}} (2\pi\alpha)^{-\frac{1}{2}} e^{-t^2/2\alpha}\, dt\, dG(\alpha).$$

The result now follows by making the transformation $y = 2\beta/\alpha$, substituting $dH_\beta(y) = dG(2\beta/y)(2\beta/y^2)$ and then making the transformation $x = t^2/4\beta$.

THEOREM 3. *If ξ is the above defined normal mixture and if, for some $\delta > 0$ we have $A_n \geqq \delta\lambda(n)$ for all $n \geqq N_1 > 0$, then the condition*

(6) *for every* $\beta, 0 < \beta \leqq 1$ *there exists* $T_\beta > 0$ *such that* $P[T_\beta(\omega) \geqq T_\beta] = 1$,

implies that for every $\epsilon > 0$, there exist numbers $M > 0$, $0 < \rho < 1$ and $N > 0$ such that for $n \geqq N$,

$$P[|S_n| \geqq \epsilon] > M\rho^{1/\lambda(n)}.$$

PROOF. Fix β and let T_β be the essential infimum of $T_\beta(\omega)$. Then $T_\beta > 0$ by Condition 6. If $P[T_\beta(\omega) = T_\beta] = p_\beta > 0$, then setting

$$I_\beta(x) = \int_0^\infty e^{-xy} y^{\frac{1}{2}}\, dH_\beta(y),$$

it follows that

$$I_\beta(x) \geqq p_\beta(T_\beta)^{\frac{1}{2}} \exp(-T_\beta x) \qquad \text{for all} \quad x \geqq 0.$$

If $P[T_\beta(\omega) = T_\beta] = 0$, then

$$I_\beta(x) > (T_\beta)^{\frac{1}{2}} \exp(-T_\beta x) \int_0^\infty e^{-xy}\, dH_\beta(y + T_\beta),$$

and by Lemma 2, for each $\gamma > 0$ there exists $X_\gamma > 0$ such that

$$I_\beta(x) > \exp[-(T_\beta + \gamma)x] \qquad \text{for all} \quad x \geqq X_\gamma.$$

Fix γ and $\theta > 0$. Then in both of the above cases, since $x^{\frac{1}{2}}e^{-\theta x} \to 0$ there exists X_θ' such that $x \geqq X_\theta'$ implies

$$x^{-\frac{1}{2}} I_\beta(x) > \exp[-(T_\beta + \gamma + \theta)x].$$

Thus, if $N_2 = \min\{n \mid A_n \geqq \epsilon^2/(4\beta X_\theta')\}$ and $N = \max(N_1, N_2)$, by Lemma 4 it follows that

$$P[|S_n| \geqq \epsilon] > M\rho^{1/\lambda(n)} \qquad \text{for} \quad n \geqq N,$$

where $M = 1/(T_\beta + \gamma + \theta)$ and $\rho = \exp\{-T_\beta + \gamma + \theta)\epsilon^2/(4\beta\delta)\}$.

THEOREM 4. *If ξ is the above defined normal mixture and if for some $\delta > 0$ we have $A_n \geqq \delta\lambda(n)$ for all $n \geqq N_1 > 0$, then the condition*

(7) *$T_\beta(\alpha)$ is absolutely continuous with probability density*

$$p_\beta(x) = \sum_{j=0}^\infty a_j x^{j+\nu} \qquad \textit{for all} \quad x \geqq 0 \quad \textit{and some} \quad \nu > 0,$$

implies that for every $\epsilon > 0$ *there exist numbers* $M > 0$ *and* $N > 0$ *such that for* $n \geqq N$,

$$P[|S_n| \geqq \epsilon] > M[\lambda(n)]^{\nu+1}.$$

PROOF. By Watson's lemma, the asymptotic expansion of

$$I(x) = \int_0^\infty e^{-xy} y^{\frac{1}{2}} p_\beta(y)\, dy \qquad \text{is} \qquad \sum_{j=0}^\infty a_j \Gamma(j + \nu + \tfrac{1}{2}) x^{-j-\nu-\frac{1}{2}}.$$

Thus, for $0 < \gamma < a_0\Gamma(\nu + \frac{3}{2})$ there exists $X_\gamma > 0$ such that $x \geqq X_\gamma$ implies

$$I(x) \geqq [a_0\Gamma(\nu + \tfrac{3}{2}) - \gamma] x^{-\nu-\frac{3}{2}}.$$

Consequently,

$$\int_r^\infty I(x) x^{-\frac{1}{2}}\, dx \geqq [(a_0\Gamma(\nu + \tfrac{3}{2}) - \gamma)/(\nu + 1)] r^{-(\nu+1)}, \qquad \text{for } r \geqq X_\gamma .$$

Now, let $N_2 = \min \{n \mid A_n \geqq \epsilon^2/(4\beta X_\gamma)\}$ and $N = \max (N_1, N_2)$. Then the result follows from Lemma 4 with

$$M = [(a_0\Gamma(\nu + \tfrac{3}{2}) - \gamma)/(\nu + 1)](4\beta\delta/\epsilon^2)^{\nu+1}.$$

4. Exponential convergence rates for *-mixing processes. Let $\{\xi_n : n = 0, \pm 1, \cdots\}$ be a sequence of real valued random variables on a probability space (Ω, Σ, P). For each integer n let Σ_n be the smallest σ-field with respect to which ξ_n is measurable and let Σ^n be the smallest σ-field with respect to which the ξ_k's are measurable for all $k \leqq n$.

DEFINITION. $\{\xi_n\}$ is called *-mixing if there exists a positive integer N and a real valued function g defined for integers $n \geqq N$ such that

(i) g is non-increasing,
(ii) $\lim_{n\to\infty} g(n) = 0$,
(iii) if $n \geqq N$, $A \varepsilon \Sigma^m$, and $B \varepsilon \Sigma_{n+m}$ then

$$|P(AB) - P(A)P(B)| \leqq g(n)P(A)P(B).$$

In [2] the strong law of large numbers is proved for *-mixing sequences $\{\xi_n\}$ and it is proved that the strong law holds with exponential convergence rate when the common moment generating function of the ξ_n's exists in a neighborhood of the origin. The method used to prove the latter theorem is quite crude and can not be used when one is working with sums of the form considered in this paper. However, using a slightly different technique we will establish the following theorem:

THEOREM 5. *Let* $\{\xi_n : n = 0, \pm 1, \cdots\}$ *be a real valued* *-mixing *stochastic process such that*

$$E\xi_n = 0 \qquad \text{for all} \quad n$$

and the moment generating functions f_n *of the* ξ_n's *exist and satisfy Condition* (3). *If the* $a_{n,k}$'s *satisfy Conditions* (1) *and* (2), *then*

$$S_n = \lim_{\alpha\to-\infty,\beta\to\infty} \sum_{k=\alpha}^\beta a_{n,k}\xi_k$$

exists almost everywhere for each n, and for every $\epsilon > 0$ *there exist positive numbers* A *and* $\rho < 1$ *(depending on* ϵ *but not on the* $a_{n,k}$*'s) such that*

$$P\{|S_n| \geqq \epsilon\} \leqq A\rho^{1/\lambda(n)}.$$

We will break up the proof of the theorem into two parts. In the first part we will show that no matter how $\alpha \to -\infty$ and $\beta \to \infty$ (monotonely), a limit, S_n, exists a.e. and all possible such limits agree a.e. In the second part we complete the proof of the theorem.

PROOF OF THEOREM 5, PART 1. If $\{\gamma_n\}$ is a sequence of numbers such that $\lim_{n\to\infty} \sum_{k=1}^{n} \gamma_k = A$ and $\lim_{m\to\infty} \sum_{k=-m}^{-1} \gamma_k = B$ where A and B are finite, then $\lim_{\alpha\to-\infty,\beta\to\infty} \sum_{k=\alpha}^{\beta} \gamma_k = A + B$ no matter how $\alpha \to -\infty$ and $\beta \to \infty$. To complete the first portion of the proof it is thus sufficient to show that $\lim_{\beta\to\infty} \sum_{k=1}^{\beta} a_{n,k}\xi_k$ and $\lim_{\alpha\to-\infty} \sum_{k=\alpha}^{-1} a_{n,k}\xi_k$ both exist and are finite a.e. We will prove the former; the proof of the latter is identical and will be omitted. We see that $\lim_{\beta\to\infty} \sum_{k=1}^{\beta} a_{n,k}\xi_k$ exists and is finite a.e. if $\lim_{\beta\to\infty} \sum_{1\leqq k\leqq\beta, k\equiv m (\text{mod } M)} a_{nk}\xi_k$ exists and is finite a.e. for each $m = 0, \cdots, M - 1$. The proof that the limit exists is the same for each m. We choose M so that $g(M)$ exists (set $\delta = g(M)$) and prove the existence of the limit for $m = 0$. For notational convenience we assume $M = 1$ and drop the subscript n. We want to show that

$$\lim_{\beta\to\infty} \sum_{k=1}^{\beta} a_j\xi_j \tag{8}$$

exists a.e. when $g(1) = \delta$. This limit exists a.e. if and only if for every $\epsilon > 0$

$$\lim_{m\to\infty} \lim_{n\to\infty} P\{\max_{m\leqq k\leqq n} \sum_{j=m}^{k} a_j\xi_j \leqq \epsilon\} = 1 \tag{9}$$

and

$$\lim_{m\to\infty} \lim_{n\to\infty} P\{\min_{m\leqq k\leqq n} \sum_{j=m}^{k} a_j\xi_j \geqq -\epsilon\} = 1. \tag{10}$$

The proofs of (9) and (10) are essentially the same. We will prove (9).

Suppose $\eta > 0$ and let F_n be the distribution function of ξ_n. Define $d_n = d(\xi_n, \eta)$ to be a number such that $(1 + \eta)F_n(d_n) \geqq \eta$ and $(1 + \eta)F_n(d_n^-) \leqq \eta$. Define $F_{n[\eta]}$ by

$$\begin{aligned} F_{n[\eta]}(x) &= 0, & x &< d_n \\ &= (1 + \eta)F_n(x) - \eta, & d_n &\leqq x \end{aligned} \tag{11}$$

and let $\{\xi_{n[\eta]}\}$ be an independent sequence of random variables with distribution functions $\{F_{n[\eta]}\}$.

It follows from Condition (3) that there exists $C < \infty$ such that $E\xi_n^2 \leqq C$ for all n. Thus $E\xi_{n[\eta]}^2 \leqq (1 + \eta)C$ for all n. It follows from standard arguments and from Condition (1) that $\lim_{\beta\to\infty} \sum_{k=1}^{\beta} a_k\xi_{k[\eta]}$ exists and is finite a.e. Thus for every $\epsilon > 0$

$$\lim_{m\to\infty} \lim_{n\to\infty} P\{\max_{m\leqq k\leqq n} \sum_{j=m}^{k} a_j\xi_{j[\eta]} \leqq \epsilon\} = 1.$$

Now set $\eta = \delta = g(1)$. Since the sets involved are complements of set of the type used in Lemma 3 of [2], it follows immediately from Lemmas 3 and 4 of

[2] that

$$P\{\max_{m \leqq k \leqq n} \sum_{j=m}^{k} a_j \xi_{j[\eta]} \leqq \epsilon\} \leqq P\{\max_{m \leqq k \leqq n} \sum_{j=m}^{k} a_j \xi_j \leqq \epsilon\}.$$

One immediately obtains (9) and Part 1 of the proof of Theorem 5 is complete.

For Part 2 of the proof we will need the following lemmas.

LEMMA 5. *If* $E|X| < \infty$, *then if* $Ee^{-|X|t}$ *exists*, $1 - Ee^{-|X|t} \leqq e^{tE|X|} - 1$.

PROOF. $Ee^{-|X|t}$ exists on an interval containing $[0, \infty]$. On this interval define $\phi(t) = Ee^{-|X|t} + Ee^{tE|X|} - 2$. Note that $\phi'(t) = -E|X|e^{-|X|t} + E|X|Ee^{tE|X|}$ and $\phi''(t) = EX^2e^{-|X|t} + (E|X|)^2Ee^{tE|X|}$. We see that $\phi(0) = \phi'(0) = 0$, $\lim_{t\to\infty} \phi(t) \geqq 0$, and $\phi''(t) \geqq 0$. It follows that $\phi(t) \geqq 0$ where it is defined.

LEMMA 6. $Ee^{|X|t} \leqq Ee^{Xt} + Ee^{-Xt} + e^{tE|X|} - 2$.

PROOF. By Lemma 5, $e^{tE|X|} - 2 \geqq -Ee^{-|X|t}$. Thus

$$\begin{aligned} Ee^{Xt} + Ee^{-Xt} + e^{tE|X|} - 2 &\geqq Ee^{Xt} + Ee^{-Xt} - Ee^{-|X|t} \\ &= Ee^{|X|t}. \end{aligned}$$

LEMMA 7. *For each* $a > 0$ *there exists a positive integer* M *and* $c > 0$ *such that*

$$E \exp\{t\xi_{n[b]}\} \leqq 1 + at$$

for all n, $0 \leqq t \leqq c$, *and* $b = g(M)$.

PROOF. For $0 \leqq t \leqq T_\beta$ from Condition (3) we have

$$E \exp\{t\xi_{n[b]}\} = (1 + b)[f_n(t) - \textstyle\int_{-\infty}^{d_n^*} e^{t\xi}\, dF_n(\xi)]$$

where the * indicates that the integral may include part of the mass at d_n. Then

$$E \exp\{t\xi_{n[b]}\} \leqq 1 + (1 + b)[f_n(t) - 1] + b \textstyle\int_{-\infty}^{d_n^*} (1 - e^{t\xi})\, d[(1 + b)F_n(\xi)/b].$$

Applying Lemma 5 to the integral on the right we see that

$$E \exp\{t\xi_{n[b]}\} \leqq 1 + (1 + b)[f_n(t) - 1] + b\{\exp[((1 + b)t/b) \textstyle\int_{-\infty}^{d_n^*} |\xi|\, dF_n] - 1\}.$$

Because the f_n's satisfy Condition (3), by taking b and β small enough we can make $(1 + b)[f_n(t) - 1]$ less than $\frac{1}{2}at$ for $0 \leqq t \leqq T_\beta$. For t small enough we can make

$$b\{\exp[((1 + b)t/b) \textstyle\int_{-\infty}^{d_n^*} |\xi|\, dF_n] - 1\} \leqq b(1 + \gamma)[((1 + b)t/b) \textstyle\int_{-\infty}^{d_n^*} |\xi|\, dF_n(\xi)]$$

where γ is arbitrary. The t interval depends on b and $\int_{-\infty}^{d_n^*} |\xi|\, dF_n(\xi)$. However if $\int_{-\infty}^{d_n^*} |\xi|\, dF_n(\xi)$ can be made uniformly small, then by requiring it to be small and picking fixed numbers $b = g(M)$ and γ which are small enough we can obtain a t interval for which the inequality holds and such that

$$t(1 + \gamma)(1 + b) \textstyle\int_{-\infty}^{d_n^*} |\xi|\, dF_n(\xi) \leqq \frac{1}{2}at$$

uniformly in n.

It follows from Condition (3) that $\int_{|\xi| \geqq d} |\xi|\, dF_n(\xi) \to 0$ uniformly in n as $d \to \infty$.

Suppose $\epsilon > 0$. Choose d such that $\int_{|\xi| \geqq d} |\xi| \, dF_n(\xi) \leqq \epsilon/2$ uniformly in n. Choose M such that $g(M) \leqq \epsilon/2d$. Then

$$\int_{-\infty}^{d_n^*} |\xi| \, dF_n(\xi) \leqq \int_{|\xi| \geqq d} |\xi| \, dF_n(\xi) + d \int_{-\infty}^{d_n^*} dF_n(\xi)$$
$$\leqq (\epsilon/2) + dg(M) \leqq \epsilon.$$

Thus $\int_{-\infty}^{d_n^*} |\xi| \, dF_n(\xi)$ can be made uniformly small by choosing M large enough. This completes the proof of Lemma 7.

LEMMA 8. *Theorem 5 is true if the S_n's are finite sums.*

PROOF. Since the existence of positive numbers A and $\rho < 1$ such that

$$P\{S_n \geqq \epsilon\} \leqq A\rho^{1/\lambda(n)}$$

and the existence of positive numbers A_0 and $\rho_0 < 1$ such that

$$P\{S_n \leqq -\epsilon\} \leqq A_0 \rho_0^{1/\lambda(n)}$$

are proved in the same way, we will omit the proof of the latter.

Define

$$S_{n,m}^M = \sum_{j \equiv m (\mathrm{mod}\, M)} a_{n,j} \xi_j$$

and

$$A_{n,m}^M = \sum_{j \equiv m (\mathrm{mod}\, M)} |a_{n,j}|.$$

Then since

$$P\{S_n \geqq \epsilon\} \leqq \sum_{m=0}^{M-1} P\{S_{n,m}^M \geqq (\epsilon/2)(M^{-1} + A_{n,m}^M)\}$$

it follows that it is sufficient to find $M, A_0, \cdots, A_{M-1}, \rho_0 < 1, \cdots, \rho_{M-1} < 1$ such that

$$P\{S_{n,m}^M \geqq (\epsilon/2)(M^{-1} + A_{n,m}^M)\} \leqq A_m \rho_m^{1/\lambda(n)}$$

for each $m = 0, \cdots, M-1$.

We have for every $t > 0$

$$\begin{aligned} &P\{S_{n,m}^M \geqq (\epsilon/2)(M^{-1} + A_{n,m}^M)\} \\ &\qquad \leqq E \exp \{t[S_{n,m}^M - (\epsilon/2)(M^{-1} + A_{n,m}^M)]\} \\ &\qquad \leqq \exp(-\epsilon t/2M) \exp[(-t\epsilon/2) A_{n,m}^M] E \prod_{j \equiv m (\mathrm{mod}\, M)} \exp(t a_{n,j} \xi_j). \end{aligned}$$

Now the random variables $\{e^{t a_{n,j} \xi_j}\}$ are *-mixing with the same g and because the function $\phi_t(x) = e^{tx}$ is monotone strictly increasing for $t > 0$ we see that for $t > 0$

$$[e^{t\xi_j}]_{[g(M)]} = \exp\{t\xi_{j[g(M)]}\}.$$

Thus,

$$\begin{aligned} &P\{S_{n,m}^M \geqq (\epsilon/2)(M^{-1} + A_{n,m}^M)\} \\ &\quad \leqq \exp(-\epsilon t/2M) \exp[(-t\epsilon/2) A_{n,m}^M] \prod_{j \equiv m (\mathrm{mod}\, M)} E \exp\{t|a_{n,j}|(\xi_j \operatorname{sgn} a_{n,j})_{[g(M)]}\}. \end{aligned}$$

We see that Lemma 7 holds for $E \exp [t(-\xi_n)_{[b]}]$ as well as $E \exp [t\xi_{n[b]}]$. Choose M and c so that $a = \epsilon/2$. Then for $0 \leqq t\lambda(n) \leqq c$

$$\begin{aligned} P\{S_{n,m}^M \geqq (\epsilon/2)(M^{-1} + A_{n,m}^M)\} \\ &\leqq \exp(-\epsilon t/2M) \exp[(-t\epsilon/2)A_{n,m}^M] \textstyle\prod_{j \equiv m (\mathrm{mod}\, M)} [1 + t|a_{n,j}|(\epsilon/2)] \\ &\leqq \exp(-\epsilon t/2M). \end{aligned}$$

Letting $t = c/\lambda(n)$ and $\rho = e^{-\epsilon c/2M}$ gives the desired result.

Since

$$S_n = \lim_{\alpha \to -\infty, \beta \to \infty} \textstyle\sum_{k=\alpha}^{\beta} a_{n,k}\xi_k \qquad \text{a.e.}$$

it follows that

$$P\{|\textstyle\sum_{k=\alpha}^{\beta} a_{n,k}\xi_k| \geqq \epsilon\} \to P\{|S_n| \geqq \epsilon\}.$$

But

$$P\{|\textstyle\sum_{k=\alpha}^{\beta} a_{n,k}\xi_k| \geqq \epsilon\} \leqq A\rho^{1/\lambda(n)}$$

for all α and β by Lemma 8. So $P\{|S_n| \geqq \epsilon\} \leqq A\rho^{1/\lambda(n)}$, proving the theorem.

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REFERENCES

[1] BAUM, LEONARD E., KATZ, MELVIN and READ, ROBERT R. (1962). Exponential convergence rates for the law of large numbers. *Trans. Amer. Math. Soc.* **102** 187–199.

[2] BLUM, J. R., HANSON, D. L. and KOOPMANS, L. H. (1963). On the strong law of large numbers for a class of stochastic processes. *Z. Wahrscheinlichkeitstheorie* **2** 1–11.

[3] HANSON, D. L. and KOOPMANS, L. H. (1965). On the convergence rate of the law of large numbers for linear combinations of independent random variables. *Ann. Math. Statist.* **36** 559–564.

[4] KOOPMANS, L. H. (1961). An exponential bound on the strong law of large numbers for linear stochastic processes with absolutely convergent coefficients. *Ann. Math. Statist.* **32** 583–586.

[5] LOÈVE, M. (1960). *Probability Theory*, 2nd ed. Van Nostrand, New York.

[6] SCHÄFKE, F. W. (1963). *Einführung in die Theorie der Speziellen Funktionen der Mathematischen Physik.* Springer-Verlag, Berlin.