ON HOTELLING'S WEIGHING DESIGNS UNDER AUTOCORRELATION OF ERRORS

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0. Summary. Some results are indicated in this note to show the efficiencies of weighing designs when least square estimates are used and errors are correlated.

1. Introduction. Hotelling's weighing design problem [2] may be described as follows: Let it be required to determine the individual weights of p objects and let there be N weighing operations in all on a balance without bias. Results of weighings would then fit into the model $Y = X\beta + e$, where Y is an $N \times 1$ random observed vector of the recorded results of weighings; X $(X = (x_{ij}),$ $i = 1, 2, \cdots, N; j = 1, 2, \cdots, p)$ is an $N \times p$ matrix of known quantities with $x_{ij} = +1, -1,$ or 0, if, in the ith weighing operation, the jth object is placed respectively in the left pan, right pan, or in none; β is a $p \times 1$ vector $(p \leqq N)$ representing the weights of the objects; e is an $N \times 1$ unobserved random vector such that $E(e) = 0$ and $E(ee') = \sigma^2 I$. X represents the weighing design matrix. When X is of full rank, the least square estimates of the weights are given by $\hat{\beta} = (X'X)^{-1}X'Y$, and the covariance matrix by Cov $(\hat{\beta}) = B = \sigma^2(X'X)^{-1} = \sigma^2 C$. C_{ii}, which is the ith diag element of C, represents the variance factor for the ith object. In weighing designs, we search for the elements x_{ij} such that C_{ii} is the least for each i.

2. Definition for the efficiency of a weighing design. A weighing design has been called the best if

(i) each variance factor C_{ii} is the least, or

(ii) the average trace of C, that is, $\sum_{i=1}^{p} C_{ii}/p$ is the least, or

(iii) the det $|C|$ is the least. In some situations [2], [3], (i), (ii) and (iii) would lead to an equivalent measure of efficiency. For the purpose of this paper, we shall adopt definition (ii), and shall denote the efficiency as trace efficiency.

3. Autocorrelation. We are not aware of any recorded literature indicating whether the efficiencies of the standard weighing designs would alter, if at all, when the errors are autocorrelated. Some results in this direction are indicated here. We shall take the error structure to take the form, $E(ee') = \sigma^2 V$, where V is given by

$$V = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{N-1} \\ \rho & 1 & \rho & \cdots & \rho^{N-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \cdots & 1 \end{bmatrix},$$

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and shall suppose further that ρ is not known. Under this modified model, the least square estimates are still given by $\hat{\beta} = C\ X'Y$. Cov $(\hat{\beta})$ is different, and is given by

$$\text{Cov}\ (\hat{\beta}) = B = \sigma^2 CX'VXC. \tag{3.1}$$

From (3.1), we shall have tr B as given by

$$\text{tr}\ B = \sigma^2\ \text{tr}\ (MM'V), \tag{3.2}$$

where $M = XC$. In particular, when $p = N$, X is of dimensions $N \times N$. (3.1) would then reduce to

$$B = \sigma^2 X^{-1} V X'^{-1} \tag{3.3}$$

and tr B to

$$\text{tr}\ B = \sigma^2\ \text{tr}\ [(XX')^{-1}V]. \tag{3.4}$$

4. Chemical balance problem. Hotelling [2] has shown that the best weighing design for a chemical balance is given by an orthogonal matrix with ± 1 as its elements. (The design matrix is orthogonal in the sense that $X'X$ is diagonal.) In this situation, $p = N$, and the weighing design comes out as the best by any of the three criteria of efficiency as referred to in Section 2, when the errors are uncorrelated.

Mood [3] has pointed out that the best chemical balance design is connected with the Hadamard determinant problem. Hadamard matrix H_N exists when $N \equiv 0\ (\text{mod}\ 4)$ with the exception of $N = 2$. Plackett and Burman [4] have constructed all Hadamard matrices of order less than or equal to 100 (excepting 92).

When, however, the errors are correlated, trace efficiency of the optimum chemical balance design would be obtained as σ^2/N by (3.4). It would further be clear from (3.3) that an individual variance factor may be a function of ρ, but the sum of the variance factors is independent of ρ. In other words, the average trace does not alter even when the errors are autocorrelated.

When the errors are not correlated, orthogonal chemical balance design is the best also by definition (iii) for the criterion of efficiency, as the value of $|C|$ is $1/N^N$ which is the least. Under autocorrelation, the value of the corresponding det $= (1 - \rho^2)^{N-1}/N^N$. These two are the same only when $\rho = 0$.

5. Spring balance problem. In a spring balance design, the elements x_{ij} are restricted to be $+1$ or 0. The efficient designs for spring balance are given by the designs P_K of Mood [3]. Banerjee [1] has indicated that a balanced incomplete block design (BIBD), used as a spring balance weighing design, would give the same efficiency as the P_K of Mood by definition (ii). (Definition (iii) cannot be adopted to measure efficiency in this case, as the dimensions of the design matrices in the two cases are not the same.) BIBD's represent [1] some efficient submatrices of P_K of Mood [3].

When $p = N$, the designs L_N of Mood are the most efficient by any definition. In L_N, $N = 4k + 3$, k being any positive integer. It has been shown in [1] that the designs L_N of Mood are given by a special class of BIBD's (an orthogonal series). We shall indicate the efficiencies of these two classes of BIBD's (BIBD's in general, and the subclass as given by L_N) under autocorrelation of errors.

6. Efficiency of BIBD (general) under autocorrelation. BIBD's are characterised by the parametric relationships, $bk = vr$, $\lambda(v - 1) = r(k - 1)$, where the parameters b, k, v, r and λ have their usual meanings. In weighing designs, v (the number of varieties or treatments) takes the place of p, the number of objects to be weighed, and b (the number of blocks) takes the place of N, the number of weighing operations that can be made. For such a design, $(X'X)$ and $(X'X)^{-1}$ take, respectively the following forms:

$$(X'X) = \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda & \lambda & \lambda & \cdots & r \end{bmatrix},$$

(6.1)

$$(X'X)^{-1} = \begin{bmatrix} r^* & -\lambda^* & -\lambda^* & \cdots & -\lambda^* \\ -\lambda^* & r^* & -\lambda^* & \cdots & -\lambda^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\lambda^* & -\lambda^* & -\lambda^* & \cdots & r^* \end{bmatrix},$$

where $r^* = [r + \lambda(p - 2)]/(r - \lambda)[r + \lambda(p - 1)]$, and

$$\lambda^* = \lambda/(r - \lambda)[r + \lambda(p - 1)].$$

When the errors are correlated, B reduces to

$$B = \sigma^2[\{(r^* + \lambda^*)I_p - \lambda^* J_{p\times p}\}X']V[\{(r^* + \lambda^*)I_p - \lambda^* J_{p\times p}\}X']'$$
$$= \sigma^2[(r^* + \lambda^*)X' - \lambda^* k J_{p\times N}]V[(r^* + \lambda^*)X' - \lambda^* k J_{p\times N}]',$$

where I_p is an identity matrix of order p and J is a matrix of elements 1, the suffixes of J denoting its dimensions.

Taking form (3.2), we get

$$\operatorname{tr} B = \sigma^2 \operatorname{tr} (MM'V), \tag{6.2}$$

where $M = [(r^* + \lambda^*)X - \lambda^* k J_{N\times p}]$.

Substituting the value of $(r^* + \lambda^*) = 1/(r - \lambda)$, and performing the required matrix multiplications, we get tr B reduced to

$$\operatorname{tr} B = \sigma^2 \operatorname{tr} [\{\mu(XX') - \nu J_{N\times N}\}V], \tag{6.3}$$

where the scalars μ and ν are positive and given by $\mu = 1/(r - \lambda)^2$,

$$\nu = [\{1/(r - \lambda)\} + 1/[r + \lambda(p - 1)]](\lambda k^2)/[(r - \lambda)\{r + \lambda(p - 1)\}].$$

The diagonal elements of (XX') in (6.3) are k, and the off-diagonal elements are

such as cannot exceed $(k-1)$, and, in this situation, k has to be equal to $(p-1)$. Admitting of this maximum possible value for these off-diagonal elements, we show as indicated below that ν is greater than $(k-1)\mu$.
For $\nu > (k-1)\mu$, if,

$$[\lambda k^2/(r-\lambda)\{r+\lambda(p-1)\}][1/(r-\lambda)+1/\{r+\lambda(p-1)\}] > (k+1)/(r-\lambda)^2$$

or if,

$$[k+(1-\lambda/r)] > (p-1), \tag{6.4}$$

(6.4) holds good, as, in this extreme situation, $k=(p-1)$.

The off-diagonal elements of (XX') will be different for different BIBD's, and these elements may not be equal to one another in all designs. Let the off-diagonal elements be equal to t $(t \leqq k-1)$. (6.3) would then reduce to

$$\operatorname{tr} B = \sigma^2[N(k\mu-\nu)-2(\nu-t\mu)R],$$

where

$$R = (N-1)\rho+(N-2)\rho^2+\cdots+\rho^{N-1}, \tag{6.5}$$

$$R = [N\rho(1-\rho)-\rho(1-\rho^N)]/(1-\rho)^2, \qquad \rho<1. \tag{6.6}$$

When $\rho = 0$, the average trace would reduce to

$$\begin{aligned}(N/p)(k\mu-\nu)\sigma^2 &= (N/p)\{k/(r-\lambda)^2-[\lambda k^2/(r-\lambda)\{r+\lambda(p-1)\}] \\ &\quad\cdot[1/\{r-\lambda\}+1/\{r+\lambda(p-1)\}]\}\sigma^2 \\ &= [\{r+\lambda(p-2)\}/(r-\lambda)\{r+\lambda(p-1)\}]\sigma^2.\end{aligned} \tag{6.7}$$

(6.7) is the same [1] as found for the average trace for such designs when errors are uncorrelated.

From the fact that $\nu > t\mu$, it is clear that, when ρ is positive, BIBD's, used as weighing designs, will have increased precision, as compared to the situation when the errors are uncorrelated.

If, on the contrary, ρ is negative, the precision will be less. A question may arise in this situation to find out if it would be possible to work out a spring balance design such that the factor $(\nu - t\mu)$ would be negative, and thus precision would be increased. It is indicated below that such a situation is not possible. This fact will be demonstrated with reference to L_N of Mood which corresponds to the special class of BIBD's, as referred to before. Although the result is shown for a special subclass, it will hold good in general.

7. Efficiency for the designs L_N of Mood. For L_N, $p=N$, $r=k=\frac{1}{2}(N+1)$, $\lambda=\frac{1}{4}(N+1)$. As X is of dimensions $p\times p$, we shall use (3.4) to find tr B. The off-diagonal elements of $(XX')^{-1}$ will be equal to λ, and the average trace will reduce to

$$\sigma^2[r^*-(2\lambda^*/N)R] = \sigma^2[4N/(N+1)^2-\{8/N(N+1)^2\}R], \tag{7.1}$$

where R is as defined in (6.5). It is noticed from (7.1) that, when $\rho = 0$, we get the same efficiency from L_N, as we would obtain in the situation when the errors are not correlated.

The extreme favorable situation would be obtained when $\rho = +1$. The average trace, substituting the value of R from (6.5), would reduce to

$$(7.2) \quad \sigma^2[4N/(N+1)^2 - 8N(N-1)/2N(N+1)^2] = 4\sigma^2/(N+1)^2.$$

(7.2) shows that the efficiency would be increased considerably in this situation. In particular, for L_3, each estimate will have a variance equal to $\sigma^2/4$, a level of precision obtainable only for the best chemical balance design for $N = 4$. (This remark is made here because it is known that a spring balance design can be no more than about $\frac{1}{4}$ as efficient as designs for a chemical balance [3].) The extreme unfavorable situation would arise when $\rho = -1$. The value of R may, in this case, be conveniently calculated from (6.6). As N is odd, R will be equal to $\frac{1}{2}(1 - N)$. Hence, the average trace would reduce to

$$(7.3) \quad \sigma^2[4N/(N+1)^2 - 8(1-N)/2N(N+1)^2] = 4\sigma^2(N^2 + N - 1)/N(N+1)^2.$$

(7.3) would be less than σ^2, pointing out that, for a spring balance, a design may be preferred to individual weighing operations even under such an extreme disadvantage.

A question now arises if, when ρ is negative, it should be possible to work out a spring balance design such that $-(\nu - t\mu)$, the factor that multiplies R should be positive. Let $p = N$, as in L_N. A reference to the construction of $(XX')^{-1}$ would make it clear that if the off-diagonal elements of $(XX')^{-1}$ are positive, the factor multiplying R will be positive. But, for the off-diagonal elements of $(XX')^{-1}$ to be positive, the off-diagonal elements of (XX') have to be negative. And, this is not possible as the elements x_{ij} of X are either 1 or 0 in a spring balance design.

It has been shown above that, when ρ is positive, a BIBD would lead to higher efficiency under autocorrelation. When ρ is positive, an overall advantage is feasible, because $\rho, \rho^2, \cdots, \rho^{N-1}$ will all be positive. But, when ρ is negative, an even power of ρ is positive, and an odd power negative. Hence, the disadvantage in a comparable situation cannot be as sweeping as from a positive ρ.

We shall indicate next with reference to the best chemical balance design if it would be possible to take advantage of the signs of ρ (if known) for working out a still better design.

8. Status of an orthogonal chemical balance design. The average trace of an orthogonal $(N \times N)$ chemical balance design, σ^2/N, remains the same as demonstrated earlier, even if the errors are correlated irrespective of the sign of ρ. It still remains to be seen if such an orthogonal chemical balance design can be further improved upon (in the sense of finding a lower trace) when at least the signs of ρ are known. If possible, let there be a transformation $Z = AX$, where X is an orthogonal matrix, and A, an $N \times N$ non-singular matrix such that the

average trace for the derived design matrix Z would be less than σ^2/N. The average trace would be obtained as

$$(8.1)\quad (1/N)\operatorname{tr} B = (\sigma^2/N)\operatorname{tr} Z^{-1}VZ'^{-1} = (\sigma^2/N)\operatorname{tr}(ZZ')^{-1}V$$
$$= (\sigma^2/N)\operatorname{tr}[\{A'^{-1}(XX')^{-1}A^{-1}\}V] = (\sigma^2/N^2)\operatorname{tr}[(AA')^{-1}V].$$

From (8.1), it would be clear that if $(AA')^{-1}$ is at least of the form

$$(8.2)\qquad \begin{bmatrix} 1 & u & 0 & \cdots & 0 & 0 \\ u & 1 & u & \cdots & 0 & 0 \\ 0 & u & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & u \\ 0 & 0 & 0 & \cdots & u & 1 \end{bmatrix} \text{ with } u = \pm 1,$$

the average trace could be made less than σ^2/N, taking $+1$ or -1 for u depending on the minus sign or plus sign of ρ. Taking, for the sake of an illustration, the dimensions of (8.2) to be 4×4 (i.e., $N = 4$), the value of the det (8.2) comes to $(1 - u^2)^2 - u^2$ which is negative for $u = \pm 1$. This is inconsistent, as $(AA')^{-1}$ is positive definite. For a fractional value of u, however, we may be able to get a positive value of the det (8.2), but the matrix A would in that case, consist, in general, of numbers of different magnitudes as its elements. Thus, it would not be possible to work out the weighing design $Z = AX$, as the elements of a weighing design matrix have to be ± 1 or 0. It appears therefore that it is not possible to work out in general a better design than an orthogonal chemical balance design.

9. Concluding remarks. Efficiencies of three classes of basic weighing designs have been discussed in this note. It has been pointed out, interalias, that an orthogonal chemical balance design is the best even when the errors are correlated. For spring balance designs, however, BIBD's (the general series or the series as given by L_N) may be safely adopted as efficient weighing designs, whatever be the sign of autocorrelation.

The special cases would, however, need special considerations.

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