THE RELATIONSHIP ALGEBRA AND THE ANALYSIS OF VARIANCE OF A PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN

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0. Summary. The analysis of variance of a partially balanced incomplete block design is investigated in connection with its relationship algebra. This gives a somewhat clearer insight into the structure of the partition of the total sum of squares than before. A. T. James [5] dealt with the same problem with the balanced incomplete block design and this article should be regarded as a generalization of his work to the partially balanced incomplete block design.

The definitions and necessary notations concerning a partially balanced incomplete block design (PBIBD) and its association algebra are briefly given in Section 1. Although the properties of the association algebra have been known in another expression [4], they are presented in Section 2 in the form fitting to our discussions on the relationship algebra. In Section 3, the definition and properties of the relationship algebra of a PBIBD are given and these are believed to be new. In Section 4, the analysis of variance of a PBIBD is considered and the partition of the sum of squares due to treatments adjusted pertinent to the association under consideration is given. Finally, Section 5 is devoted to the analysis of variance of a PBIBD of triangular type and a numerical illustration.

1. A partially balanced incomplete block design and its association algebra. For the sake of the reader's convenience, we give a brief description of a partially balanced incomplete block design and its association algebra. Reference should be made to [1], [2], [3] and [7].

Given v treatments $\phi_1, \phi_2, \cdots, \phi_v$, a relation among them satisfying the following three conditions is called an association with m associate classes:

(a) any two treatments are either 1st, or 2nd, $\cdots$, or mth associates,

(b) each treatment has n_i ith associates, $i = 1, 2, \cdots, m$, and

(c) for each pair of treatments which are ith associates, there are p^i_{jk} $(i, j, k = 1, 2, \cdots, m)$ treatments which are jth associates of the one treatment of the pair and at the same time kth associates of the other.

We have a partially balanced incomplete block design—PBIBD in short—if there are b blocks each containing k experimental units in such a way that

(1) each block contains k $(\leqq v)$ different treatments,

(2) each treatment occurs in r blocks, and

(3) any two treatments which are ith associates occur together in λ_i blocks, $i = 1, 2, \cdots, m$.

In a degenerate case when $m = 1$, a PBIBD reduces to a balanced incomplete block design. PBIBD with 2 associate classes have been useful in practical applications.

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Parameters describing the association are

$$v,\ n_i,\ p^i_{jk}, \qquad i, j, k = 1, 2, \cdots, m$$

and additional design parameters are

$$b,\ r,\ k,\ \lambda_i, \qquad i = 1, 2, \cdots, m.$$

It should be noted that

$$(1.1) \qquad n_1 + \cdots + n_m = v - 1$$

and

$$(1.2) \qquad p^i_{jk} = p^i_{kj} \quad \text{(symmetry with respect to the subscripts).}$$

Further it can be shown that

$$(1.3) \qquad \textstyle\sum_{k=1}^m p^i_{jk} = n_j - \delta_{ij}$$

and

$$(1.4) \qquad n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij},$$

where δ_{ij} stands for the Kronecker delta.

There are $r(k-1)$ treatments occurring in the blocks in which a fixed treatment ϕ occurs and they are classified into m associate classes with respect to ϕ. On the other hand, since there are n_i ith associates of ϕ occurring in λ_i blocks, it follows that

$$(1.5) \qquad n_1\lambda_1 + n_2\lambda_2 + \cdots + n_m\lambda_m = r(k-1).$$

A treatment may be regarded as the 0th associate of itself. Thus we add the following notational conventions:

$$(1.6) \qquad n_0 = 1, \qquad \lambda_0 = r, \qquad p^0_{jk} = \delta_{jk} n_j, \qquad p^i_{0k} = p^i_{k0} = \delta_{ik}.$$

Under these notations, we have the following relations

$$(1.7) \qquad \textstyle\sum_{i=0}^m n_i = v;$$

$$(1.8) \qquad \textstyle\sum_{k=0}^m p^i_{jk} = n_j;$$

$$(1.9) \qquad \textstyle\sum_{i=0}^m n_i\lambda_i = rk.$$

Let $\mathbf{A}_0$ be the unit matrix of degree v. This represents the 0th association. Also let $\mathbf{A}_i$ be a symmetric matrix of degree v such that its elements $a^{\beta}_{\alpha i}$ in the αth row and in the βth column is 1 if ϕ_α and ϕ_β are ith associates and is 0 otherwise. $\mathbf{A}_0, \mathbf{A}_1, \cdots, \mathbf{A}_m$ are called the association matrices. It can be seen that

$$(1.10) \qquad \textstyle\sum_{i=0}^m \mathbf{A}_i = \mathbf{G}_v$$

where $\mathbf{G}_v$ stands for the square matrix of degree v whose elements are all unity. Hence $\mathbf{A}_0, \mathbf{A}_1, \cdots, \mathbf{A}_m$ are linearly independent with respect to the field of all real numbers. Furthermore one can show that

$$\mathbf{A}_j\mathbf{A}_k = \textstyle\sum_{i=0}^{m} p_{jk}^{i}\mathbf{A}_i . \tag{1.11}$$

Thus the linear closure of the set $\mathbf{A}_0, \mathbf{A}_1, \cdots, \mathbf{A}_m$ with respect to the field of all real numbers is a linear associative and commutative algebra $\mathfrak{A}$, called the associative algebra. The abstract counterpart of this matrix algebra $\mathfrak{A}$ is denoted by $\mathfrak{a}$.

2. Properties of the association algebra. Let

$$\mathcal{P}_k = \begin{Vmatrix} p_{0k}^{0} & p_{0k}^{1} & \cdots & p_{0k}^{m} \\ p_{1k}^{0} & p_{1k}^{1} & \cdots & p_{1k}^{m} \\ \vdots & & & \\ p_{mk}^{0} & p_{mk}^{1} & \cdots & p_{mk}^{m} \end{Vmatrix}, \qquad k = 0, 1, \cdots, m, \tag{2.1}$$

then it can be shown that the mappings

$$\mathbf{A}_k \rightarrow \mathcal{P}_k , \qquad k = 0, 1, \cdots, m, \tag{2.2}$$

generate the regular representation $(\mathfrak{a})$ of the association algebra $\mathfrak{a}$. The abstract algebra $\mathfrak{a}$ is completely reducible in the field of all rational numbers [9], and hence it is completely reducible in any number field. On the other hand, Schur's lemma ([9] Lemma (3.1.A), p. 81) shows us that any irreducible representation of a commutative algebra in an algebraically closed field must be linear. From the general theory of algebra [9], one knows that any representation of a completely reducible algebra decomposes into irreducible representations, each of which is equivalent to one of the irreducible constituents of the regular representation of the algebra. Since $\mathfrak{a}$ is the abstract counterpart of the matrix algebra $\mathfrak{A}$ generated by symmetric matrices and is of rank $m + 1$, the regular representation $(\mathfrak{a})$ decomposes into $m + 1$ inequivalent and linear representations in the field of all real numbers.

On account of the fact that

$$\mathbf{A}_i\mathbf{G}_v = \mathbf{G}_v\mathbf{A}_i = n_i\mathbf{G}_v , \qquad i = 0, 1, \cdots, m, \tag{2.3}$$

one can choose a non-singular real matrix $\mathbf{C}$ of degree $m + 1$, being of the form

$$\mathbf{C} = \begin{Vmatrix} c_{00} & c_{01} & \cdots & c_{0m} \\ c_{10} & c_{11} & \cdots & c_{1m} \\ \vdots & & & \\ c_{m0} & c_{m1} & \cdots & c_{mm} \end{Vmatrix} \tag{2.4}$$

with $c_{00} = c_{01} = \cdots = c_{0m} = 1$ in such a way that

$$\mathbf{C}\mathcal{P}_u\,\mathbf{C}^{-1} = \begin{Vmatrix} z_{0u} & & & \mathbf{0} \\ & z_{1u} & & \\ & & \ddots & \\ \mathbf{0} & & & z_{mu} \end{Vmatrix}, \qquad z_{0u} = n_u ,\ u = 0, 1, \cdots, m. \tag{2.5}$$

One can construct $m + 1$ mutually orthogonal idempotent matrices belonging to $\mathfrak{A}$ as follows:

$$\mathbf{A}_u^{\#} = \left(\sum_{t=0}^{m} c_{ut}\mathbf{A}_t\right) \Big/ \left(\sum_{t=0}^{m} c_{ut} z_{ut}\right), \qquad u = 0, 1, \cdots, m, \tag{2.6}$$

with respective ranks $\alpha_0, \alpha_1, \cdots, \alpha_m$. It is noticed that $\mathbf{A}_0^{\#} = v^{-1}\mathbf{G}_v$, $\alpha_0 = 1$, and $\sum_{u=0}^{m} \mathbf{A}_u^{\#} = \mathbf{I}_v$.

The ranks $\alpha_0, \alpha_1, \cdots, \alpha_m$ of the orthogonal idempotents are determined by the linear equations

$$\begin{aligned} \alpha_0 + \alpha_1 + \cdots + \alpha_m &= v, \\ \alpha_0 n_1 + \alpha_1 z_{11} + \cdots + \alpha_m z_{m1} &= 0, \\ &\vdots \\ \alpha_0 n_m + \alpha_1 z_{1m} + \cdots + \alpha_m z_{mm} &= 0. \end{aligned} \tag{2.7}$$

As an example, the association algebra of the association of triangular type is explained below.

The number of treatments is $v = n(n - 1)/2$, where n is a positive integer. We take an $n \times n$ square, and fill the $n(n - 1)/2$ positions above the main diagonal by different $v = n(n - 1)/2$ treatments, taken in order (see Figure 1). The positions in the main diagonal are left blank, while the positions below the main diagonal are filled so that the scheme is symmetrical with respect to the main diagonal. The two treatments in the same column are 1st associates, whereas two treatments which do not occur in the same column are 2nd associates. Hence

$$n_1 = 2n - 4, \qquad n_2 = (n - 2)(n - 3)/2. \tag{2.8}$$

The regular representation of the association algebra in this case is given by

$$\begin{aligned} &\mathbf{A}_0 \to \mathcal{P}_0 = \mathbf{I}_3, \\ &\mathbf{A}_1 \to \mathcal{P}_1 = \begin{Vmatrix} 0 & 1 & 0 \\ 2n-4 & n-2 & 4 \\ 0 & n-3 & 2n-8 \end{Vmatrix}, \\ &\mathbf{A}_2 \to \mathcal{P}_2 = \begin{Vmatrix} 0 & 0 & 1 \\ 0 & n-3 & 2n-8 \\ (n-2)(n-3)/2 & (n-3)(n-4)/2 & (n-4)(n-5)/2 \end{Vmatrix}. \end{aligned} \tag{2.9}$$

	1	2	3	4
1		5	6	7
2	5		8	9
3	6	8		10
4	7	9	10	

FIG. 1. $n = 5$, $v = 10$

Transforming by the non-singular matrix

$$\mathbf{C} = \begin{Vmatrix} 1 & 1 & 1 \\ 2n-4 & n-4 & -4 \\ -(n-2)(n-3) & n-3 & -2 \end{Vmatrix},$$

one gets

$$\mathbf{C}\mathcal{P}_1\mathbf{C}^{-1} = \begin{Vmatrix} 2n-4 & 0 & 0 \\ 0 & n-4 & 0 \\ 0 & 0 & -2 \end{Vmatrix},$$

$$\mathbf{C}\mathcal{P}_2\mathbf{C}^{-1} = \begin{Vmatrix} (n-2)(n-3)/2 & 0 & 0 \\ 0 & -(n-3) & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

Therefore

$$\sum_{t=0}^{2} c_{1t}z_{1t} = (2n-4)\cdot 1 + (n-4)\cdot(n-4) + (-4)\cdot[-(n-3)] = n(n-2),$$

$$\sum_{t=0}^{2} c_{2t}z_{2t} = -(n-2)(n-3) + (n-3)\cdot(-2) + (-2)\cdot 1 = -(n-1)(n-2).$$

Hence one obtains the three mutually orthogonal idempotent matrices

$$\begin{aligned} \mathbf{A}_0^{\#} &= [2/n(n-1)] [\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2], \\ \mathbf{A}_1^{\#} &= [1/n(n-2)] [(2n-4)\mathbf{A}_0 + (n-4)\mathbf{A}_1 - 4\mathbf{A}_2], \\ \mathbf{A}_2^{\#} &= [1/(n-1)(n-2)][(n-2)(n-3)\mathbf{A}_0 - (n-3)\mathbf{A}_1 + 2\mathbf{A}_2], \end{aligned} \tag{2.10}$$

with respective ranks $\alpha_0 = \operatorname{tr} \mathbf{A}_0^{\#} = 1$, $\alpha_1 = \operatorname{tr} \mathbf{A}_1^{\#} = n - 1$, $\alpha_2 = \operatorname{tr} \mathbf{A}_2^{\#} = n(n - 3)/2$.

3. The relationship algebra of a partially balanced incomplete block design. We now define the so called relationship matrices of a PBIBD. There are $w = bk = vr$ experimental units altogether and they are numbered from one through w in any way once and for all.

(1) Identity relation: corresponding to this relation, we take $\mathbf{I} = \mathbf{I}_w$, i.e., the unit matrix of degree w.

(2) Universal relation: corresponding to this relation, we take $\mathbf{G} = \mathbf{G}_w$, i.e., the matrix of degree w whose elements are all unity.

(3) Block relation: let the incidence matrix of blocks be

$$\mathbf{\Psi} = \|\mathbf{n}_1\mathbf{n}_2 \cdots \mathbf{n}_b\|,$$

where

$$\mathbf{n}_\alpha' = (\eta_{\alpha 1}, \eta_{\alpha 2}, \cdots, \eta_{\alpha w}) \text{ with } \eta_{\alpha f} = 1, \quad \text{if the } f\text{th unit belongs to the } \alpha\text{th block}, \quad = 0, \quad \text{otherwise},$$

then the block relation is represented by a $w \times w$ matrix

$$\mathbf{B} = \mathbf{\Psi}\mathbf{\Psi}'. \tag{3.1}$$

(4) Treatment relation: let the incidence matrix of treatments be

$$\mathbf{\Phi} = \|\zeta_1\zeta_2 \cdots \zeta_v\|,$$

where

$$\zeta_\alpha' = (\zeta_{\alpha 1}, \zeta_{\alpha 2}, \cdots, \zeta_{\alpha w}) \text{ with } \zeta_{\alpha f} = 1, \quad \text{if the } \alpha\text{th treatment occurs at the } f\text{th unit,}$$
$$= 0, \quad \text{otherwise,}$$

then the treatment relation is represented by the following $m + 1$ matrices degree w:

(3.2) $$\mathbf{T} = \mathbf{T}_0, \mathbf{T}_1, \cdots, \mathbf{T}_m$$

where

(3.3) $$\mathbf{T}_u = \|t_{fg}^u\| = \mathbf{\Phi}\mathbf{A}_u\mathbf{\Phi}', \qquad u = 0, 1, \cdots, m.$$

One can see immediately that

(3.4) $$\sum_{u=0}^m \mathbf{T}_u = \mathbf{G},$$

also

(3.5) $$\mathbf{G}^2 = w\mathbf{G}, \qquad \mathbf{BG} = \mathbf{GB} = k\mathbf{G}, \qquad \mathbf{B}^2 = k\mathbf{B}$$

and

(3.6) $$\mathbf{GT}_u = \mathbf{T}_u\mathbf{G} = rn_u\mathbf{G}, \qquad u = 0, 1, \cdots, m.$$

Let $\mathbf{N} = \|n_{\alpha a}\|$ be the incidence matrix of the design, then, since $\mathbf{N} = \mathbf{\Phi}'\mathbf{\Psi}$ and

(3.7) $$\mathbf{NN}' = \sum_{u=0}^m \lambda_u\mathbf{A}_u$$

it follows that

(3.8) $$\mathbf{NN}' = \sum_{u=0}^m \rho_u\mathbf{A}_u$$

where

(3.9) $$\rho_0 = \sum_{i=0}^m n_i\lambda_i = rk, \qquad \rho_u = \sum_{j=0}^m z_{uj}\lambda_j, \qquad u = 1, 2, \cdots, m.$$

(3.8) is the spectral decomposition of the matrix $\mathbf{NN}'$ and ρ_u, $u = 0, 1, \cdots, m$ are the characteristic roots of the matrix with respective multiplicities α_u, $u = 0, 1, \cdots, m$.

The design is said to be regular, if all the ρ_u's are positive. We shall be concerned with regular PBIBD's. It can be shown that

(3.10) $$\mathbf{TBT} = \mathbf{\Phi NN}'\mathbf{\Phi}' = \sum_{u=0}^m \lambda_u\mathbf{T}_u,$$

(3.11) $$\mathbf{T}_u\mathbf{BT}_w = \sum_{t=0}^m \left(\sum_{k,l=0}^m \lambda_k p_{uk}^l p_{lw}^t\right)\mathbf{T}_t$$

and

(3.12) $$\mathbf{T}_u\mathbf{T}_w = \sum_{t=0}^m p_{uw}^t\mathbf{T}_t.$$

Hence in the regular case, the linear closure with respect to the field of all real numbers of the set of the following $4m + 3$ linearly independent matrices

$$\mathbf{I},\ \mathbf{G},\ \mathbf{B},\ \mathbf{T}_u,\ \mathbf{T}_u\mathbf{B},\ \mathbf{B}\mathbf{T}_u,\ \mathbf{B}\mathbf{T}_u\mathbf{B}, \qquad u = 1, 2, \cdots, m, \tag{3.13}$$

is a linear associative algebra $\mathfrak{R}$, which is called the relationship algebra of the PBIBD.

The relationship algebra $\mathfrak{R}$ contains a commutative subalgebra $\mathfrak{J}$ generated by $\mathbf{T}_u$, $u = 0, 1, \cdots, m$, which is isomorphic to the association algebra $\mathfrak{A}$.

As a special case, one gets the relationship algebra of a balanced incomplete block design—BIBD in short—as the linear closure [**I**, **G**, **B**, **T**, **TB**, **BT**, **BTB**]. This algebra was investigated by A. T. James [5] in detail. H. B. Mann [6] exploited a more general algebra which is associated with the analysis of variance of testing linear hypotheses. The relationship algebra $\mathfrak{R}$ of a PBIBD is located inbetween the James algebra and the Mann algebra in its generality, so to speak.

The relationship algebra $\mathfrak{R}$ of a PBIBD is generated by symmetric relationship matrices, it is completely reducible. Hence all irreducible representations of $\mathfrak{R}$ should be obtained by reducing its regular representation.

[**G**], the totality of the multiples of **G**, is a one-dimensional two-sided ideal of $\mathfrak{R}$ and $\mathbf{G}^2 = w\mathbf{G}$, $\mathbf{BG} = \mathbf{GB} = k\mathbf{G}$, $\mathbf{T}_u\mathbf{G} = \mathbf{G}\mathbf{T}_u = rn_u\mathbf{G}$. Hence we obtain a linear representation $\mathfrak{R}_G^{(1)}$ induced by [**G**] as follows:

$$\mathfrak{R}_G^{(1)}: \mathbf{I} \to 1,\ \mathbf{G} \to w,\ \mathbf{B} \to k,\ \mathbf{T}_u \to rn_u . \tag{3.14}$$

Next we consider the factor algebra $\mathfrak{R}/[\mathbf{G}]$, i.e., consider the algebra $\mathfrak{R}$ mod **G**. To this end, it is convenient to change the basis of $\mathfrak{R}$ into [$\mathbf{I}, \mathbf{G}, \mathbf{B}, \mathbf{T}_u^*, \mathbf{T}_u^*\mathbf{B}, \mathbf{B}\mathbf{T}_u^*, \mathbf{B}\mathbf{T}_u^*\mathbf{B}$, $u = 1, 2, \cdots, m$]. Since

$$\mathbf{T}_u^* = \textstyle\sum_{j=0}^m c_{uj}\mathbf{T}_j , \tag{3.15}$$

it follows that

$$\begin{aligned}\mathbf{T}_u^*\mathbf{T}_w^* &= \mathbf{\Phi}\mathbf{A}_u^*\mathbf{\Phi}'\mathbf{\Phi}\mathbf{A}_w^*\mathbf{\Phi}' = r\mathbf{\Phi}\mathbf{A}_u^*\mathbf{A}_w^*\mathbf{\Phi}' \\ &= r\left(\textstyle\sum_{i=0}^m c_{ui}z_{ui}\right)\delta_{uw}\mathbf{T}_u^*\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\mathbf{T}_u^*\mathbf{B}\mathbf{T}_w^* &= \mathbf{\Phi}\mathbf{A}_u^*\mathbf{\Phi}'\mathbf{\Psi}\mathbf{\Psi}'\mathbf{\Phi}\mathbf{A}_w^*\mathbf{\Phi}' = \mathbf{\Phi}\mathbf{A}_u^*\mathbf{N}\mathbf{N}'\mathbf{A}_w^*\mathbf{\Phi}' \\ &= \left(\textstyle\sum_{i=0}^m z_{ui}\lambda_i\right)\mathbf{\Phi}\mathbf{A}_u^*\mathbf{A}_w^*\mathbf{\Phi}' = \rho_u\left(\textstyle\sum_{i=0}^m c_{ui}z_{ui}\right)\delta_{uw}\mathbf{T}_u^*.\end{aligned} \tag{3.17}$$

The following m subalgebras

$$[\mathbf{T}_u^*, \mathbf{B}\mathbf{T}_u^*, \mathbf{T}_u^*\mathbf{B}, \mathbf{B}\mathbf{T}_u^*\mathbf{B}] \bmod \mathbf{G}, \qquad u = 1, 2, \cdots, m \tag{3.18}$$

are two-sided ideals of $\mathfrak{R}$ annihilating each other and also they are annihilated by **G**. In fact, for instance

$$\mathbf{T}_u^*\mathbf{B}\mathbf{T}_w^*\mathbf{B} = \mathbf{T}_u^*\mathbf{B}\mathbf{T}_w^*\mathbf{B} = \rho_u\left(\textstyle\sum_{i=0}^m c_{ui}z_{ui}\right)\delta_{uw}\mathbf{T}_u^*\mathbf{B}.$$

If $\rho_u = 0$, then $\mathbf{B}\mathbf{T}_u^* = \mathbf{T}_u^*\mathbf{B} = \mathbf{B}\mathbf{T}_u^*\mathbf{B} = \mathbf{0}$ and consequently the subalgebra reduces to [$\mathbf{T}_u^*$]. Thus in the regular case, there are m inequivalent irreducible representations of the 2nd degree, each of which being induced by a left-sided ideal of the above two-sided ideals.

Now by direct calculations, one obtains

$$\mathbf{T}_i[\mathbf{T}_u^*, \mathbf{B}\mathbf{T}_u^*, \mathbf{T}_u^*\mathbf{B}, \mathbf{B}\mathbf{T}_u^*\mathbf{B}]$$

$$= [\mathbf{T}_u^*, \mathbf{B}\mathbf{T}_u^*, \mathbf{T}_u^*\mathbf{B}, \mathbf{B}\mathbf{T}_u^*\mathbf{B}] \begin{Vmatrix} rz_{ui} & \rho_u z_{ui} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & rz_{ui} & \rho_u z_{ui} \\ 0 & 0 & 0 & 0 \end{Vmatrix},$$

(3.19) $$\mathbf{B}[\mathbf{T}_u^*, \mathbf{B}\mathbf{T}_u^*, \mathbf{T}_u^*\mathbf{B}, \mathbf{B}\mathbf{T}_u^*\mathbf{B}]$$

$$= [\mathbf{T}_u^*, \mathbf{B}\mathbf{T}_u^*, \mathbf{T}_u^*\mathbf{B}, \mathbf{B}\mathbf{T}_u^*\mathbf{B}] \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & k \end{Vmatrix}.$$

Thus the m irreducible representations of the 2nd degree are given by

(3.20) $$\mathfrak{R}_u^{(2)}: \quad \begin{array}{l} \mathbf{I} \to \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \mathbf{G} \to \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}, \quad \mathbf{B} \to \begin{Vmatrix} 0 & 0 \\ 1 & k \end{Vmatrix}, \\ \mathbf{T}_i \to \begin{Vmatrix} rz_{ui} & \rho_u z_{ui} \\ 0 & 0 \end{Vmatrix}, \qquad i = 1, 2, \cdots, m. \end{array}$$

Other irreducible representations of $\mathfrak{R}$ are obtained by considering the factor algebra

$$\mathfrak{R}/[\mathbf{G}]/[\mathbf{T}_u^*, \mathbf{B}\mathbf{T}_u^*, \mathbf{T}_u^*\mathbf{B}, \mathbf{B}\mathbf{T}_u^*\mathbf{B}, u = 1, 2, \cdots, m]/[\mathbf{G}].$$

They are given by

(3.21) $$\mathfrak{R}_0^{(1)}: \mathbf{I} \to 1, \mathbf{G} \to 0, \mathbf{B} \to 0, \mathbf{T}_i \to 0, \quad i = 1, 2, \cdots, m,$$

and

(3.22) $$\mathfrak{R}_1^{(1)}: \mathbf{I} \to 1, \mathbf{G} \to 0, \mathbf{B} \to k, \mathbf{T}_i \to 0, \quad i = 1, 2, \cdots, m.$$

Since $1^2 + 1^2 + 1^2 + m2^2 = 4m + 3$ there can be no other irreducible representations of $\mathfrak{R}$.

We shall show that

(3.23) $$\mathfrak{R} \sim (w - b - v + 1)\mathfrak{R}_0^{(1)} + (b - v)\mathfrak{R}_1^{(1)} + \mathfrak{R}_G^{(1)} + \textstyle\sum_{u=1}^m \alpha_u \mathfrak{R}_u^{(2)},$$

i.e., $\mathfrak{R}$ is equivalent to the right hand side if the design is regular and therefore $v \leqq b$. If $v > b$, then $\mathbf{NN}'$ must be singular and hence at least one of the ρ_u's should vanish.

Let

$$\mathfrak{R} \sim \gamma_0 \mathfrak{R}_0^{(1)} + \gamma_1 \mathfrak{R}_1^{(1)} + \gamma_G \mathfrak{R}_G^{(1)} + \textstyle\sum_{u=0}^m \beta_u \mathfrak{R}_u^{(2)}$$

then one gets the following equations:

$$\begin{aligned} \operatorname{tr} \mathbf{I} &= w = \gamma_0 + \gamma_1 + \gamma_G + 2\textstyle\sum_{u=1}^m \beta_u , \\ \operatorname{tr} \mathbf{G} &= w = w\gamma_G , \\ \operatorname{tr} \mathbf{B} &= w = k\gamma_1 + k\gamma_G + k\textstyle\sum_{u=1}^m \beta_u , \\ \operatorname{tr} \mathbf{T}_i &= 0 = rn_i \gamma_G + r\textstyle\sum_{u=1}^m \beta_u , \qquad (i = 1, 2, \cdots, m). \end{aligned} \tag{3.24}$$

From the first three equations of (3.24), one gets $\gamma_G = 1$, $\gamma_0 = w - b - \sum_{u=1}^m \beta_u$. Comparing the last m equations of (3.24) with those of (2.7), one gets $\beta_u = \alpha_u$, $u = 1, 2, \cdots, m$. Consequently it follows that $\gamma_0 = w - b - v + 1$.

Let

$$\mathbf{T}_u^{\#} = \mathbf{\Phi} \mathbf{A}_u^{\#} \mathbf{\Phi}', \qquad u = 1, 2, \cdots, m \tag{3.25}$$

then it is clear that

$$\mathbf{T}_u^{\#}\, \mathbf{T}_w^{\#} = r\delta_{uw} \mathbf{T}_u^{\#}. \tag{3.26}$$

Now let us consider the following m matrices:

$$\mathbf{V}_u = (\mathbf{T}_u^{\#} - k^{-1}\mathbf{B}\mathbf{T}_u^{\#})(\mathbf{T}_u^{\#} - k^{-1}\mathbf{T}_u^{\#}\mathbf{B}), \qquad u = 1, 2, \cdots, m. \tag{3.27}$$

One can show that $\mathbf{V}_u\mathbf{V}_w = \delta_{uw} r(r - \rho_u/k)\mathbf{V}_u$ and $\operatorname{tr} \mathbf{V}_u = r(r - \rho_u/k)\alpha_u$. Thus the m matrices

$$\mathbf{V}_u^{\#} = [k/r(rk - \rho_u)](\mathbf{T}_u^{\#} - k^{-1}\mathbf{B}\mathbf{T}_u^{\#})(\mathbf{T}_u^{\#} - k^{-1}\mathbf{T}_u^{*}\mathbf{B}) \tag{3.28}$$

are mutually orthogonal idempotent matrices of respective ranks α_u, $u = 1, 2, \cdots, m$. Hence the following expression

$$\mathbf{I} = w^{-1}\mathbf{G} + (k^{-1}\mathbf{B} - w^{-1}\mathbf{G}) + \textstyle\sum_{u=1}^m \mathbf{V}_u^{\#} + (\mathbf{I} - k^{-1}\mathbf{B} - \sum_{u=1}^m \mathbf{V}_u^{\#}) \tag{3.29}$$

is a decomposition of the unit $\mathbf{I}$ of $\mathfrak{R}$ into mutually orthogonal idempotents, and this will be shown to be useful in the analysis of variance of the design.

4. The analysis of variance of a PBIBD. We are concerned with the linear model which is often called the intra-block model, i.e.,

$$\mathbf{x} = \gamma \mathbf{j} + \mathbf{\Phi}\boldsymbol{\tau} + \mathbf{\Psi}\boldsymbol{\beta} + \mathbf{e}$$

where $\mathbf{x}' = (x_1, \cdots, x_w)$ stands for the observations of size w, γ is the general mean, $\mathbf{j}' = (1, 1, \cdots, 1)$, $\boldsymbol{\tau}' = (\tau_1, \cdots, \tau_v)$ and $\boldsymbol{\beta}' = (\beta_1, \cdots, \beta_b)$ are treatment and block effects being subjected to the restrictions $\sum_{\alpha=1}^v \tau_\alpha = \sum_{a=1}^b \beta_a = 0$, respectively, and finally $\mathbf{e}' = (e_1, \cdots, e_w)$ is the error vector being distributed as $N(0', \sigma^2\mathbf{I})$.

We have the adjusted normal Equation [2]

$$[r(1 - k^{-1})\mathbf{A}_0 - (\lambda_1/k)\mathbf{A}_1 - \cdots - (\lambda_m/k)\mathbf{A}_m]\mathbf{t} = \mathbf{Q}$$

or

$$\textstyle\sum_{u=1}^m [(rk - \rho_u)/k]\mathbf{A}_u^{\#}\mathbf{t} = \mathbf{Q}, \tag{4.1}$$

where $\mathbf{t}' = (t_1, \cdots, t_v)$ is the best linear estimate of $\boldsymbol{\tau}'$ and $\mathbf{Q}' = (Q_1, \cdots, Q_v)$ is the treatment sum adjusted by blocks.

Let the linearly independent column vectors of $\mathbf{A}_u^{\#}$ be

$$\mathbf{a}_{\nu_u+1}^{(u)}, \mathbf{a}_{\nu_u+2}^{(u)}, \cdots, \mathbf{a}_{\nu_u+\alpha_u}^{(u)}, \qquad \nu_u = 1 + \alpha_1 + \cdots + \alpha_{u-1},$$

then, by multiplying $\mathbf{a}_{\nu_u+\alpha}^{(u)\prime}$ to both sides of the (4.1) from the left, one obtains

$$[(rk - \rho_u)/k]\mathbf{a}_{\nu_u+\alpha}^{(u)\prime}\mathbf{t} = \mathbf{a}_{\nu_u+\alpha}^{(u)\prime}\mathbf{Q}$$

or

$$\mathbf{a}_{\nu_u+\alpha}^{(u)\prime}\mathbf{t} = [k/(rk - \rho_u)]\mathbf{a}_{\nu_u+\alpha}^{(u)\prime}\mathbf{Q}. \tag{4.2}$$

It can be seen that

$$E(\mathbf{a}_{\nu_u+\alpha}^{(u)\prime}\mathbf{t}) = \mathbf{a}_{\nu_u+\alpha}^{(u)\prime}\boldsymbol{\tau}, \tag{4.3}$$

$$\mathrm{Var}\,(\mathbf{a}_{\nu_u+\alpha}^{(u)\prime}\mathbf{t}) = [k/(rk - \rho_u)]\sigma^2 \cdot \mathbf{A}_{\nu_u+\alpha,\nu_u+\alpha}^{(u)} .$$

Thus a complete balance is achieved over the set of the normalized contrasts

$$\mathbf{a}_{\nu_u+\alpha}^{(u)\prime}\mathbf{t}/(\mathbf{a}_{\nu_u+\alpha,\nu_u+\alpha}^{(u)})^{\frac{1}{2}}, \qquad \alpha = 1, 2, \cdots, \alpha_u, [8].$$

Since $\mathbf{x}'\mathbf{V}_u^{\#}\mathbf{x} = [k/(rk - \rho_u)]\mathbf{Q}'\mathbf{A}_u^{\#}\mathbf{Q} = \mathbf{t}'\mathbf{A}_u^{\#}\mathbf{Q}$ and $\sum_{u=1}^m \mathbf{A}_u^{\#} = \mathbf{I}_v - \mathbf{A}_0^{\#}$ one can see that

$$\sum_{u=1}^m \mathbf{x}'\mathbf{V}_u^{\#}\mathbf{x} = \mathbf{t}'\mathbf{Q} = s_t^2 \text{: sum of squares due to treatments adjusted.} \tag{4.4}$$

Under the present model, one gets

$$\mathbf{x}'\mathbf{V}_u^{\#}\mathbf{x} = \mathbf{e}'\mathbf{V}_u^{\#}\mathbf{e} + 2\boldsymbol{\tau}'\mathbf{A}_u^{\#}(\boldsymbol{\Phi}' - k^{-1}\mathbf{N}\boldsymbol{\Psi}')\mathbf{e} + (r - \rho_u/k)\boldsymbol{\tau}'\mathbf{A}_u^{\#}\boldsymbol{\tau}, \tag{4.5}$$

and therefore

$$\chi_u^2 = \mathbf{x}'\mathbf{V}_u^{\#}\mathbf{x}/\sigma^2 \tag{4.6}$$

is distributed as the non-central chi-square distribution of degrees of freedom α_u with the non-centrality parameter

$$\boldsymbol{\Delta}_u = [(rk - \rho_u)/\sigma^2 k]\boldsymbol{\tau}'\mathbf{A}_u^{\#}\boldsymbol{\tau} . \tag{4.7}$$

The sum of squares due to error s_e^2 is given by

$$s_e^2 = \mathbf{x}'(\mathbf{I} - k^{-1}\mathbf{B} - \textstyle\sum_{u=1}^m \mathbf{V}_u^{\#})\mathbf{x} = \mathbf{e}'(\mathbf{I} - k^{-1}\mathbf{B} - \textstyle\sum_{u=1}^m \mathbf{V}_u^{\#})\mathbf{e} \tag{4.8}$$

and therefore

$$\chi_e^2 = s_e^2/\sigma^2 \tag{4.9}$$

is distributed as the central chi-square distribution of degrees of freedom $w - b - v + 1$. The variates $\chi_1^2, \cdots, \chi_m^2, \chi_e^2$ are mutually independent in stochastic sense.

Hence under the null-hypothesis

$$H_0^{(u)}: \mathbf{A}_u^{\#}\boldsymbol{\tau} = \mathbf{0} \tag{4.10}$$

the test statistic

$$F_u = [(w - b - v + 1)/\alpha_u]\mathbf{x}'\mathbf{V}_u^{\#}\mathbf{x}/s_e^2 \tag{4.11}$$

is distributed as the central F-distribution of degrees of freedom $(\alpha_u, w - b - v + 1)$.

5. A numerical example. We shall present the analysis of variance of a PBIBD of triangular type in detail with a numerical example.

In this case, we have

$$\mathbf{A}_0^{\#} = [2/n(n-1)][\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2], \qquad \alpha_0 = 1,$$

$$\mathbf{A}_1^{\#} = [1/n(n-1)][(2n-4)\mathbf{A}_0 + (n-4)\mathbf{A}_1 - 4\mathbf{A}_2], \qquad \alpha_1 = n-1,$$

$$\mathbf{A}_2^{\#} = [1/(n-1)(n-2)][(n-2)(n-3)\mathbf{A}_0 - (n-3)\mathbf{A}_1 + 2\mathbf{A}_2], \qquad \alpha_2 = n(n-3)/2.$$

We assume the following inner structure of the treatment effects:

$$n = 5, \qquad v = n(n-1)/2 = 10.$$

	$\theta_1 + \theta_2 + \pi_{12}$	$\theta_1 + \theta_3 + \pi_{13}$	$\theta_1 + \theta_4 + \pi_{14}$	$\theta_1 + \theta_5 + \pi_{15}$
$\theta_2 + \theta_1 + \pi_{21}$		$\theta_2 + \theta_3 + \pi_{23}$	$\theta_2 + \theta_4 + \pi_{24}$	$\theta_2 + \theta_5 + \pi_{25}$
$\theta_3 + \theta_1 + \pi_{31}$	$\theta_3 + \theta_2 + \pi_{32}$		$\theta_3 + \theta_4 + \pi_{34}$	$\theta_3 + \theta_5 + \pi_{35}$
$\theta_4 + \theta_1 + \pi_{41}$	$\theta_4 + \theta_2 + \pi_{42}$	$\theta_4 + \theta_3 + \pi_{43}$		$\theta_4 + \theta_5 + \pi_{45}$
$\theta_5 + \theta_1 + \pi_{51}$	$\theta_5 + \theta_2 + \pi_{52}$	$\theta_5 + \theta_3 + \pi_{53}$	$\theta_5 + \theta_4 + \pi_{54}$	

The inner parameters are subjected to the restrictions

$$\textstyle\sum_{i=1}^{n} \theta_i = 0, \quad \pi_{ij} = \pi_{ji}, \quad \sum_{j(\neq i)} \pi_{ij} = 0, \quad i = 1, 2, \cdots, n \tag{5.1}$$

so that these are just $v - 1$ independent parameters.

By direct calculations, we obtain the following expression:

$$\mathbf{A}_1^{\#}\boldsymbol{\tau} = n^{-1} \cdot \left\{\begin{array}{llllll} (n-2)\theta_1 + (n-2)\theta_2 & -2\theta_3 - \cdots & & -2\theta_{n-1} & -2\theta_n \\ (n-2)\theta_1 & -2\theta_2 + (n-2)\theta_3 - \cdots & & -2\theta_{n-1} & -2\theta_n \\ & \cdots\cdots & & & \\ (n-2)\theta_1 & -2\theta_2 & -2\theta_3 - \cdots + (n-2)\theta_{n-1} & & -2\theta_n \\ (n-2)\theta_1 & -2\theta_2 & -2\theta_3 - \cdots & -2\theta_{n-1} + (n-2)\theta_n & \\ -2\theta_1 + (n-2)\theta_2 + (n-2)\theta_3 - \cdots & & & -2\theta_{n-1} & -2\theta_n \\ -2\theta_1 + (n-2)\theta_2 & -2\theta_3 + \cdots & & -2\theta_{n-1} & -2\theta_n \\ & \cdots\cdots & & & \\ -2\theta_1 + (n-2)\theta_2 & -2\theta_3 - \cdots & & -2\theta_{n-1} + (n-2)\theta_n & \\ & \cdots\cdots & & & \\ -2\theta_1 & -2\theta_2 & -2\theta_3 - \cdots & -2\theta_{n-1} + (n-2)\theta_n & \end{array}\right\} \tag{5.2}$$

and the

$$\alpha\text{th component of } \mathbf{A}_2^{\#}\boldsymbol{\tau}$$

$$= [(n-1)(n-2)]^{-1}[(n-1)(n-3)\cdot\text{interaction } \pi_{ij}(i < j) \text{ of } \tau_\alpha$$

$$(5.3) \qquad - (n-3)\cdot \textstyle\sum \text{ interactions corresponding to the 1st associates of } \phi_\alpha$$

$$+ 2\cdot \textstyle\sum \text{ interactions corresponding to the 2nd associates of } \phi_\alpha].$$

Thus the vector $\mathbf{A}_1^{\#}\boldsymbol{\tau}$ represents the contrasts between the main effects $\boldsymbol{\theta}$ and the vector $\mathbf{A}_2^{\#}\boldsymbol{\tau}$ represents the contrasts between interactions π_{ij}.

Now we show a numerical example of the analysis of variance of a PBIBD of triangular type (Table 1).

There are n kind ingredients $I_1, I_2, \cdots, I_n$ which are known to be efficient

TABLE 1

A design of triangular type

Block Treatments	1	2	3	4	5	6	7	8	9	10	Treatment Total
1 = (1, 2)			2.31		2.81	1.65			2.58		9.40
2 = (1, 3)		2.51					1.41	1.90	3.06		8.88
3 = (1, 4)	2.89		2.29					1.95		2.04	9.16
4 = (1, 5)				2.54		2.09	2.36			2.03	9.02
5 = (2, 3)	2.28			2.81					2.20	2.07	9.36
6 = (2, 4)		1.77	2.49	2.31			3.02				9.59
7 = (2, 5)	2.72	2.29				1.57		2.60			9.18
8 = (3, 4)				2.81	2.99	2.28		2.44			10.52
9 = (3, 5)	2.54		2.44		2.23		2.12				9.33
10 = (4, 5)		1.54			2.87				2.77	2.09	9.27
Block totals	10.43	8.11	9.53	10.47	10.95	7.59	8.91	8.89	10.61	8.22	93.71

TABLE 2

Association

Treatment	1st Associates	2nd Associates
1	2, 3, 4, 5, 6, 7	8, 9, 10
2	1, 3, 4, 5, 8, 9	6, 7, 10
3	1, 2, 4, 6, 8, 10	5, 7, 9
4	1, 2, 3, 7, 9, 10	5, 6, 8
5	1, 2, 6, 7, 8, 9	3, 4, 10
6	1, 3, 5, 7, 8, 10	2, 4, 9
7	1, 4, 5, 6, 9, 10	2, 3, 8
8	2, 3, 5, 6, 9, 10	1, 4, 7
9	2, 4, 5, 7, 8, 10	1, 3, 6
10	3, 4, 6, 7, 8, 9	1, 2, 5

in gaining weights of hogs if added in the feed stuff. We are interested to know whether there are interactions between any two of the ingredients when the mixtures of the two are added in the feed stuff.

We make $v = n(n - 1)/2$ mixtures of the possible pairs (I_i, I_j), $i \neq j$. The main effects of the n original ingredients are denoted by θ_i, $i = 1, 2, \cdots, n$ and the interactions between I_i and I_j is denoted by π_{ij}. Then the inner-parametric representations of the mixtures are given by $\theta_\alpha = \theta_i + \theta_j + \pi_{ij}$ if the αth treatment is the mixture of I_i and I_j for $\alpha = 1, 2, \cdots, v = n(n - 1)/2$. Hence in this situation, the association scheme of the triangular type is naturally defined among the v treatments (Table 2).

Suppose by taking ten litters of 4 hogs each as blocks, a PBIBD of triangular type with parameters $n = 5, v = b = 10, r = k = 4, \lambda_1 = 1, \lambda_2 = 2$ is adopted yielding the following results. Observations are the gains of weights of hogs in pounds after feeding the mixtures of ingredients 3 months. This experiment is a hypothetical one and the data are borrowed from R. C. Bose and T. Shimamoto [3] and therefore this example should be regarded as a purely illustrative one.

TABLE 3

Adjusted treatment tables and related sums

	Q	A_1Q	A_2Q	$A_1^{\#}Q$	$A_2^{\#}Q$
1	−0.2700	0.0525	0.2175	−0.1625	−0.1075
2	−0.2500	−0.3075	0.5575	−0.2692	0.0192
3	−0.1075	0.8800	−0.7725	0.2217	−0.3292
4	0.2225	−1.0300	0.8075	−0.1950	0.4175
5	−0.5725	0.6600	−0.0875	−0.1617	−0.4108
6	0.3350	0.3175	−0.6505	0.3292	0.0058
7	0.4250	−1.1125	0.6875	−0.0875	0.5125
8	1.0450	−1.4225	0.3775	−0.2225	0.8225
9	−0.6250	0.6675	−0.0425	−0.1992	−0.4308
10	−0.2025	1.2950	−1.0925	0.2967	−0.4992
Total........	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 4

Analysis of variance

Sources of Variation	Sum of Squares	d.f.	m.s.s.	Variance Ratio
Blocks........................	3.2284	9		
Treatment eliminating blocks.	0.7467	9	0.08295	
Main effect....................	0.1343	4	0.03357	0.230
Interactions..................	0.6124	5	0.12248	0.841
Errors........................	3.0585	21	0.14550	
Total.......................	7.0336	39		

Now in this case, since

$$\rho_1 = z_{10}\lambda_0 + z_{11}\lambda_1 + z_{12}\lambda_2 = 4 + 1\cdot 1 - 2\cdot 2 = 1,$$

$$\rho_2 = z_{20}\lambda_0 + z_{21}\lambda_1 + z_{22}\lambda_2 = 4 - 2\cdot 1 + 1\cdot 2 = 4$$

and

$$\mathbf{A}_1^{\#} = \tfrac{1}{15}(6\mathbf{A}_0 + \mathbf{A}_1 - 4\mathbf{A}_2), \qquad \alpha_1 = 4,$$

$$\mathbf{A}_2^{\#} = \tfrac{1}{12}(6\mathbf{A}_0 - 2\mathbf{A}_1 + 2\mathbf{A}_2), \qquad \alpha_2 = 5,$$

it follows that

$$\mathbf{A}_1^{\#}\mathbf{Q} = \tfrac{1}{15}(6\mathbf{Q} + \mathbf{A}_1\mathbf{Q} - 4\mathbf{A}_2\mathbf{Q}),$$

$$\mathbf{A}_2^{\#}\mathbf{Q} = \tfrac{1}{6}(3\mathbf{Q} - \mathbf{A}_1\mathbf{Q} + \mathbf{A}_2\mathbf{Q}).$$

There is a relation $\mathbf{A}_1^{\#}\mathbf{Q} + \mathbf{A}_2^{\#}\mathbf{Q} = \mathbf{Q}$. Finally the sum of squares due to main-effects and interactions is given by

$$\mathbf{x}'\mathbf{V}_1^{\#}\mathbf{x} = [k/(rk - \rho_1)]\mathbf{Q}'\mathbf{A}_1^{\#}\mathbf{Q} = \tfrac{4}{15}\mathbf{Q}'\mathbf{A}_1^{\#}\mathbf{Q}$$

and

$$\mathbf{x}'\mathbf{V}_2^{\#}\mathbf{x} = [k/(rk - \rho_2)]\mathbf{Q}'\mathbf{A}_2^{\#}\mathbf{Q} = \tfrac{1}{3}\mathbf{Q}'\mathbf{A}_2^{\#}\mathbf{Q},$$

respectively, satisfying the relation

$$\tfrac{4}{15}\mathbf{Q}'\mathbf{A}_1^{\#}\mathbf{Q} + \tfrac{1}{3}\mathbf{Q}'\mathbf{A}_2^{\#}\mathbf{Q} = \mathbf{t}'\mathbf{Q}.$$

Thus we get the following table of the analysis of variance by use of the auxiliary Tables 3 and 4.

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