ON A CLASS OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

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0. Summary. The purpose of this note is to show that the existence of any one of a particular family of four partially balanced incomplete block designs (pbibd) implies the existence of the remaining three designs. A sufficient condition for the existence of this family is given and a nonexistence result is also obtained.

1. Introduction. Following the usual notation [2] let the parameters of an m classes pbibd be denoted by v, b, r, k, λ_i, n_i and matrices $P_i = (p^i_{jk})$, $i, j, k = 1, 2, \cdots, m$. Let $N = (n_{ij})$ be the usual $v \times b$ incidence matrix of the design where $n_{ij} = 1$ or 0 according as treatment i occurs or does not occur in block j. The characteristic roots of NN' together with their multiplicities have been obtained in [5] for the cases $m = 2, 3$. Further the dual of a design with incidence matrix N is a design having N' for its incidence matrix.

For $n > 4$ a triangular association scheme T_n for $v = \binom{n}{2}$ treatments is defined as follows [3]. The treatments are numbered $1, 2, \cdots, \binom{n}{2}$ and are arranged in a symmetric square array in which the main diagonal positions are left blank (denoted by x) and the positions above the main diagonal are filled by the numbers $1, 2, \cdots, \binom{n}{2}$. Thus

$$T_5 = \begin{pmatrix} x & 1 & 2 & 3 & 4 \\ 1 & x & 5 & 6 & 7 \\ 2 & 5 & x & 8 & 9 \\ 3 & 6 & 8 & x & 10 \\ 4 & 7 & 9 & 10 & x \end{pmatrix}. \tag{1.1}$$

Two treatments in T_n are said to be 1-associates if they occur in the same row or same column; otherwise they are 2-associates. It is then easy to verify that $n_1 = 2(n-2)$, $p^1_{11} = n - 2$, $p^2_{11} = 4$. The question—when do the above values of v, n_1, p^1_{11}, p^2_{11} for a two classes association scheme imply that the association scheme is T_n—has been considered by various authors [4], [6], [9]. It has been shown [6] that if $n \neq 8$, then the above parameters imply that the association scheme is actually T_n, and for $n = 8$, there are exactly two other association schemes which are possible besides T_8.

Following Bose [1], we define a partial geometry (r, k, t) as follows. We have a system of undefined points and lines together with an incidence relation satisfying the following postulates.

P_1. Any two points are incident with not more than one line.

P_2. Each point is incident with r lines.

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P_3. Each line is incident with k points.

P_4. If the point P is not incident with the line l, there pass through P exactly t lines ($t \geqq 1$) intersecting l.

It is then known that an (r, k, t) is a pbibd with two associate classes with parameters

$$\begin{aligned} &v = k[(r-1)(k-1)+t]/t, \qquad b = r[(r-1)(k-1)+t]/t, r, k, \\ (1.2)\quad &\lambda_1 = 0, \qquad \lambda_2 = 1, \qquad n_1 = (r-1)(k-1)(k-t)/t, \\ &p_{11}^1 = n_1 + rt - 1 - r(k-1), \qquad p_{11}^2 = (r-1)(k-t)(k-t-1)/t, \\ &1 \leqq t \leqq r, \qquad 1 \leqq t \leqq k. \end{aligned}$$

Conversely any pbibd with the above parameters is a partial geometry (r, k, t). Further the dual of a partial geometry (r, k, t) is again a partial geometry (r^*, k^*, t) with $r^* = k$ and $k^* = r$, which therefore, is again a pbibd with parameters obtained from (1.2) by interchanging r and k. For the same reason the dual of (r^*, k^*, t) is (r, k, t).

2. A family of pbibd's. In this section unless otherwise stated n stands for an integer $n > 2$, $n \neq 4$.

Consider a two classes pbibd A_1 with parameters

$$\begin{aligned} (2.1)\quad A_1 : v = \binom{2n}{2}, \quad b = (2n-1)(2n-3), \quad r = 2n-3, \quad k = n, \\ \lambda_1 = 0, \quad \lambda_2 = 1, \quad n_1 = 2(2n-2), \quad p_{11}^1 = 2n-2, \quad p_{11}^2 = 4. \end{aligned}$$

Then the association scheme of A_1 is T_{2n}. Further from (1.2) it is easy to verify that for $n > 2$ A_1 is a partial geometry $(2n-3, n, n-2)$ and hence its dual A^* is again a partial geometry $(n, 2n-3, n-2)$ and hence a pbibd with parameters

$$\begin{aligned} (2.2)\quad A_1^*: v = (2n-1)(2n-3), \quad b = \binom{2n}{2}, \quad r = \mathrm{n}, \quad k = 2\mathrm{n}-3, \\ \lambda_1 = 0, \quad \lambda_2 = 1, \quad n_1 = 2(n-1)^2, \quad p_{11}^1 = p_{11}^2 = (n-1)^2, \end{aligned}$$

and that the dual of A_1^* is the design A_1. Thus for all $n > 2$ the designs A_1 and A_1^* either both exist or both do not exist.

Again for A_1 it is easy to verify that $rk - v\lambda_1 = n(r - \lambda_1)$. Hence from the result of Raghavrao [7], it follows that each block of A_1 contains $2k/2n = 1$ treatment from each row of the association scheme T_{2n}. Let the treatment in the first row of T_{2n} be $1, 2, \cdots, 2n-1$ and let U_i be the set of $(2n-3)$ blocks of A_1 each containing the treatment i, $i = 1, 2, \cdots, 2n-1$. From the fact that the treatments i and j are 1-associates and $\lambda_1 = 0$, it follows that the sets U_i for different values of i account for all the block of A_1. Now omit the treatment i from each block of U_i to obtain the set U_i' of $(2n-3)$ blocks. Then any two blocks of U_i' are disjoint and contain all the treatments of T_{2n} not lying in its 1st and the $(i+1)$th row exactly once. Let T_{2n-1} be the array obtained from T_{2n} by omitting its 1st row and 1st column, then it is obvious that U_i' contains all the treatments of T_{2n-1} not lying in its ith row exactly once. It is now easy to see that the sets U_i', $i = 1, 2, \cdots, 2n - 1$ constitute a pbibd A_2 with parameters

$$(2.3)\qquad \begin{aligned} &A_2 : v = \binom{2n-1}{2}, \quad b = (2n-1)(2n-3), \quad r = 2n-3, \\ &k = n-1, \quad \lambda_1 = 0, \quad \lambda_2 = 1, \quad n_1 = 2(2n-3), \\ &p_{11}^1 = 2n-3, \quad p_{11}^2 = 4 \end{aligned}$$

and with association scheme T_{2n-1}. It now follows that the existence of A_1 or A_1^* implies the existence of A_2.

We now show that the existence of the design A_2 implies the existence of the design A_1 and hence of A_1^*, which taken together with what has already been proved will imply the coexistence of the designs A_1, A_1^* and A_2 for $n > 2$, $n \neq 4$.

Suppose A_2 given by (2.3) is a design with treatments numbered $2n, 2n + 1, \cdots, \binom{2n}{2}$. Then its association scheme is T_{2n-1}. Let

$$(2.4)\qquad T_{2n-1} = \begin{Bmatrix} x & 2n & 2n+1 & \cdots & 4n-3 \\ & x & 4n-2 & \cdots & 6n-6 \\ & & x & & \\ & & & \ddots & \\ & & & x & \binom{2n}{2} \\ & & & & x \end{Bmatrix}.$$

Let θ be any treatment of T_{2n-2} obtained from T_{2n-1} by omitting the first row and the first column of T_{2n-1}. Then θ has two 1-associates and $2n - 4$ 2-associates from the first row of T_{2n-1}. Noting that $\lambda_1 = 0$ and $\lambda_2 = 1$, it is now obvious that through each of the treatments $4n - 2, 4n - 1, \cdots, 6n - 6$ there is a different block which contains no treatment from the first row of T_{2n-1}. Denote the set of $(2n - 3)$ blocks thus obtained by S_1'. Then each of the treatments $4n - 2$, $4n - 1, \cdots, 4n - 6$ occurs exactly once in S_1'. Let θ be any other treatment of T_{2n-2} which occurs in S_1'. Then since there is only one block through θ which contains no treatment of the first row of T_{2n-1}, it is obvious that θ occurs exactly once in S_1'. Hence the blocks of S_1' are all disjoint and they account for $(2n - 3)(n - 1)$ treatments in all from T_{2n-2}. Since the number of treatments in T_{2n-2} is again $(2n - 3)(n - 1)$, it is obvious that the set S_1' contains no treatment from the first row of T_{2n-1} and contains all the other treatments exactly once. Since the scheme T_{2n-1} is unchanged by interchanging the ith row with the first row and ith column with the first column, it is now obvious that the blocks of A_2 can be partitioned into mutually disjoint and exhaustive sets S_i' each of $(2n - 3)$ blocks with the property that all the treatments of T_{2n-1} excepting those in the ith row occur exactly once in S_i', $i = 1, 2, \cdots, 2n - 1$. We now adjoin to each block β' of S_i' a new treatment numbered i, giving the block β. Denote the set of blocks β thus obtained by S_i; $i = 1, 2, \cdots, 2n - 1$. Consider the design obtained by the sets of blocks $S_1, S_2, \cdots, S_{2n-1}$. For this design the values of v, b, r, k are the same as those of A_1. Further the $2n - 1$ new treatments $1, 2, \cdots, 2n - 1$ are such that no two of them occur in the same block and the treatment numbered i occurs together in a block with all the treatments of the association scheme of T_{2n-1} excepting those in its ith row. It is now obvious that the design thus obtained has the same parameters as that of A_1 and has the association scheme given by the symmetric array

$$(2.5) \qquad T_{2n} = \begin{Bmatrix} x & 1 & 2 & 3 & \cdots & 2n-1 \\ & x & 2n & 2n+1 & \cdots & 4n-6 \\ & & x & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \binom{2n}{2} \\ & & & & & x \end{Bmatrix}$$

obtained by adding an initial row and column containing $(x \ 1 \ 2 \ \cdots \ 2n - 1)$ to T_{2n-1}.

We have thus proved

LEMMA 1. *If $n > 2$ and $n \neq 4$, the designs A_1, A_1^* and A_2 given respectively by* (2.1), (2.2) *and* (2.3) *either all exist or all do not exist.*

We now show that the dual A_2^* of A_2 with the association scheme T_{2n-1} given by (2.4) is a pbibd with three associate classes. The parameters v, b, r, k of A_2^* are given by $v = (2n - 1)(2n - 3)$, $b = \binom{2n-1}{2}$, $r = n - 1$, $k = 2n - 3$. To prove that A_2^* is a pbibd with three associate classes, it is sufficient to establish a three classes association scheme between the blocks of A_2. We proceed to do exactly this.

We note that any two blocks of A_2 can intersect in at most one treatment and further as we have already seen, the blocks of A_2 can be partitioned into sets S_i' of $(2n - 3)$ blocks each, such that the set S_i' contains all the treatments of T_{2n-1} excepting those in its ith row exactly once. We now define the association scheme between the blocks of A_2 as follows. Two blocks of A_2 are (i) 1-associates if they intersect in one treatment, (ii) 2-associates if they belong to the same set S_i' and (iii) 3-associates otherwise. Let μ_i denote the number of treatments common to two blocks which are i-associates and m_i denote the number of i-associates of any block. Then $\mu_1 = 1$, $\mu_2 = \mu_3 = 0$ and $m_1 = 2(n - 1)(n - 2)$, $m_2 = 2n - 4$, $m_3 = 2(n - 1)^2$. Let β_1' and β_2' be two blocks of A_2 which are i-associates and let $q_{jk}^i(\beta_1', \beta_2')$ denote the number of blocks in A_2 which are simultaneously j-associates of β_1'; and k-associates of β_2'; $i, j, k = 1, 2, 3$. Then [8] it is sufficient to verify that for $i = 1, 2, 3$; $j, k = 1, 2$ the value of $q_{jk}^i(\beta_1', \beta_2') = q_{jk}^i$, a constant independent of β_1' and β_2' so long as β_1' and β_2' are i-associates.

It is obvious from the definition that $q_{22}^2 = 2n - 5$, $q_{12}^2 = q_{22}^3 = q_{22}^1 = 0$.

Now consider a block β_1', say, from S_i' and let $\beta_1' = (\theta_1, \theta_2, \cdots, \theta_{n-1})$. Then θ_1 does not occur in the ith row of T_{2n-1}. Let it occur in the jth row of T_{2n-1}, $j \neq i$. Then in S_j' there is no block containing θ_1. Since θ_1 occurs in the jth row of T_{2n-1}, $\theta_2, \theta_3, \cdots, \theta_{n-1}$ cannot occur in this row since $\lambda_1 = 0$. Hence each of these $n - 2$ treatments occurs in S_j' and further since any two of them already occur together in β_1', they occur in $n - 2$ different blocks of S_j'. Thus β_1' has $n - 2$ 1-associates in S_j'. Since the treatment θ_1 also occurs in kth row of T_{2n-1}, $k \neq i$, $k \neq j$, it follows that β_1' has also $n - 2$ 1-associates in S_k'. Similarly if θ_2 occurs in pth and qth row of T_{2n-1} then, $p \neq i$, $q \neq i$, $p \neq q$, $p \neq j$, $p \neq k$, $q \neq j$, $q \neq k$ since for two treatments in the same row of T_{2n-1} $\lambda_1 = 0$. It is now obvious by considering the remaining treatments $\theta_2, \theta_3, \cdots, \theta_{n-1}$, that β_1' has exactly $n - 2$ 1-associates from each S_j', $j \neq i$, thus accounting for the value $m_1 = 2(n - 1)(n - 2)$.

Let β_1' and β_2' be two blocks which are 1-associates. Then β_1' and β_2' belong to different sets S_i'. Then it is obvious that $q_{12}^1(\beta_1', \beta_2') = n - 3$. Similarly if β_1' and β_2' are 3-associates, then $q_{12}^3(\beta_1', \beta_2') = n - 2$.

By virtue of our mode of construction of A_2 from A_1 and A_1 from A_2, we can set up the following correspondence between blocks of A_1 and A_2. If β is a block of A_1 from the set U_i we denote by β' the block of A_2 obtained by deleting the treatment i from β. Similarly if β' is a block of A_2 from the set S_i', β will denote the block of A_1 obtained by adding the treatment i to β'.

Now let $\beta_1' = (\theta_1, \theta_2, \cdots, \theta_{n-1})$, $\beta_2' = (\phi_1, \phi_2, \cdots \phi_{n-1})$ be two blocks of A_2 which are 2-associates. Then they belong to the same set S_i'. Since the blocks of S_i' are disjoint θ_1 does not occur in β_2'. We have already shown that A_2 can be embedded in A_1. Now consider the corresponding blocks β_1 and β_2 in A_1. Since A_1 is a partial geometry with $t = n - 2$, there are $n - 2$ blocks in A_1 containing θ_1 each of which intersects the block β_2. One of these blocks is β_1 which intersects β_2 in the treatment i. Hence there are $n - 3$ other blocks $\gamma_1, \gamma_2, \cdots, \gamma_{n-3}$ in A_1 each containing θ_1 and such that γ_i and γ_j intersect β_2 in exactly one but different treatments from $\phi_1, \phi_2, \cdots, \phi_{n-1}$. The corresponding blocks $\gamma_1', \gamma_2', \cdots, \gamma_{n-3}'$ of A_2 are common 1-associates of β_1' and β_2'. We get similarly $n - 3$ common 1-associates of β_1' and β_2' by considering each of the remaining treatments $\theta_2, \theta_3, \cdots, \theta_{n-1}$. Obviously these sets of $n - 3$ blocks are all disjoint. We thus get the value $q_{11}^2(\beta_1', \beta_2') = (n - 1)(n - 3)$.

Now consider the value p_{22}^2 in A_1^*. From the relations amongst p_{jk}^i, it is easy to verify that $p_{22}^2 = (2n - 5) + (n - 1)(n - 3)$. This represents the number of blocks in A_1 with association scheme (2.5) which have one treatment in common with two blocks β_1 and β_2 of A_1 which have themselves one treatment in common. Consider the corresponding blocks β_1' and β_2' of A_2. If β_1' and β_2' belong to S_i' and S_j' respectively, then they are 1-associates in A_2. Since S_i' has $n - 2$ 1-associates of β_2', there are $n - 3$ blocks in S_i' other than β_1' which are 1-associates of β_2' and similarly $n - 3$ blocks in S_j' other than β_2' which are 1-associates of β_1'. It is easy to see that if we omit this set of $2(n - 3)$ blocks from the set of p_{22}^2 blocks in A_1 indicated above, we get $q_{11}^1(\beta_1', \beta_2') = (n - 2)^2$.

The value p_{22}^1 in A_1^* is easily seen to be $n(n - 2)$ and represents the number of blocks in A_1 which have one treatment in common with two blocks β_1 and β_2 of A_1 which themselves have no treatment in common. Then the corresponding blocks β_1' and β_2' are 3-associates in A_2, and belong to different sets S_i' and S_j'. By subtracting from this value of p_{22}^1 the number $n - 2$ of blocks in S_i' which are 1-associates of β_2' and an equal number of blocks in S_j' which are 1-associates of β_1' we are obviously left with the value $q_{11}^3(\beta_1', \beta_2') = (n-2)^2$.

It is now obvious that the dual A_2^* of A_2 is a pbibd with three associate classes and has parameters

$$
\begin{aligned}
A_2^*: v &= (2n-1)(2n-3), \quad b = \binom{2n-1}{2}, \\
r &= n-1, \quad k = 2n-3, \quad \lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 0, \\
n_1 &= 2(n-1)(n-2), \quad n_2 = 2(n-2), \\
p_{11}^1 &= (n-2)^2, \quad p_{12}^1 = n-3, \quad p_{22}^1 = 0, \\
p_{11}^2 &= (n-1)(n-3), \quad p_{12}^2 = 0, \quad p_{22}^2 = 2n-5, \\
p_{11}^3 &= (n-2)^2, \quad p_{12}^3 = n-2, \quad p_{22}^3 = 0.
\end{aligned} \tag{2.6}
$$

Now we consider the design A_2^* for $n > 2$. If N is the incidence matrix of this design, then it is easy to verify [5] that NN' has the characteristic roots $\theta_0 = (n-1)(2n-3)$, $\theta_1 = 0$, $\theta_2 = 2(n-1)$, $\theta_3 = 1$ with respectively multiplicities $\alpha_0 = 1$, $\alpha_1 = \alpha_3 = (2n-1)(n-2)$, $\alpha_2 = 2(n-1)$. Further for $A_2^* v - b = \alpha_1$. Hence from [10], it follows that the dual of A_2^* for $n > 2$ is a pbibd having the parameters of A_2.

We thus have the following lemma.

LEMMA 2. *If $n > 2$, $n \neq 4$ the designs A_2 and A_2^* given by* (2.3) *and* (2.6) *either both exist or both do not exist.*

Finally combining the two lemmas, we have the following theorem.

THEOREM. *If $n > 2$ and $n \neq 4$, then the existence of any one of the designs A_1, A_1^*, A_2, A_2^* given by* (2.1), (2.2), (2.3) *and* (2.6) *implies the existence of the other three designs.*

It is known [8] that A_2^* exists if $n = 2^m + 1$, $m \geqq 1$. Hence from the above theorem it follows that all the four designs above can be constructed for these values of n.

3. A nonexistence result. We note that for any $n > 2$, the design A_2 always has association scheme T_{2n-1}. We now give a direct construction for A_2 with $n = 3$ i.e. the design with $v = 10$, $b = 15$, $r = 3$, $k = 2$, $\lambda_1 = 0$, $\lambda_2 = 1$ and the association scheme T_5. We can then construct the other three designs for $n = 3$. With T_5 given (1.1) it is easy to see that the sets S_i' can be uniquely written as

S_1': [(5, 10) (6, 9) (7, 8)]

S_2': [(2, 10) (3, 9) (4, 8)]

S_3': [(1, 10) (3, 7) (4, 6)]

S_4': [(1, 9) (2, 7) (4, 5)]

S_5': [(1, 8) (2, 6) (3, 5)]

It is easy to verify that the above sets of blocks actually constitute the design A_2.

Recalling the proof of the theorem, it is useful to note that for any value of $n > 2$, the designs A_1 and A_1^* coexist, the designs A_1 with association scheme T_{2n} and A_2 coexist, and the existence of A_2^* implies the existence of A_2.

We now show that the design A_2 with $n = 4$ i.e. the design with $v = 21$, $b = 35$, $r = 5$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 1$ with the association scheme T_7 is impossible. If the design is to exist we should be able to write down consistent sets S_i', $i = 1, 2, \cdots, 7$ to form the design. Let

$$T_7 = \begin{pmatrix} x & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & x & 7 & 8 & 9 & 10 & 11 \\ 2 & 7 & x & 12 & 13 & 14 & 15 \\ 3 & 8 & 12 & x & 16 & 17 & 18 \\ 4 & 9 & 13 & 16 & x & 19 & 20 \\ 5 & 10 & 14 & 17 & 19 & x & 21 \\ 6 & 11 & 15 & 18 & 20 & 21 & x \end{pmatrix}$$

Then consistent with the conditions for A_2 we have the following possible sets for S_1'.

D_1. [(7, 16, 21) (8, 14, 20) (9, 15, 17) (10, 13, 18) (11, 12, 19)].
D_2. [(7, 16, 21) (8, 15, 19) (9, 14, 18) (10, 12, 20) (11, 13, 17)].
D_3. [(7, 17, 20) (8, 13, 21) (9, 14, 18) (10, 15, 16) (11, 12, 19)].
D_4. [(7, 17, 20) (8, 15, 19) (9, 12, 21) (10, 13, 18) (11, 14, 16)].
D_5. [(7, 18, 19) (8, 13, 21) (9, 15, 17) (10, 12, 20) (11, 14, 16)].
D_6. [(7, 18, 19) (8, 14, 20) (9, 12, 21) (10, 15, 16) (11, 13, 17)].

The possible sets for S_2' are $E_1, E_2, \cdots, E_6$ where E_i is obtained from D_i by replacing 7, 8, 9, 10, 11 respectively by 2, 3, 4, 5, 6. It is easy to see that D_1 is incompatible with E_1 or E_2 because otherwise the pair (16, 21) will occur twice contracting $\lambda_2 = 1$. Similarly D_1 is incompatible with E_3 because of the pair (12, 19), wih E_4 because of the pair (13, 18), with E_5 because of the pair (15, 17) and with E_6 because of the pair (14, 20). Similarly it can be verified that no other set D_i is compatible with any set E_j. This implies the nonexistence of A_2 with $n = 4$ together with the nonexistence of A_2^* and A_1 with association scheme T_8 and A_1^* with the association scheme T_8 for its dual.

Since the arguments used in this section are purely combinatorial, they can be used to either construct A_2 or prove its impossibility for higher values of n.

REFERENCES

[1] Bose, R. C. (1963). Strongly regular graphs, partial geometries and partially balanced designs. *Pacific J. Math.* **13** 389–419.

[2] Bose, R. C. and Nair, K. R. (1939). Partially balanced incomplete block designs. *Sankhyā* **4** 337–372.

[3] Bose, R. C. and Shimamoto, T. (1952). Classification and analysis of partially balanced incomplete block designs, with two associate classes. *J. Amer. Statist. Assoc.* **47** 151–184.

[4] CONNOR, W. S. (1958). The uniqueness of the triangular association scheme. *Ann. Math. Statist.* **29** 262–266.
[5] CONNOR, W. S. and CLATWORTHY, W. H. (1954). Some theorems for partially balanced designs. *Ann. Math. Statist.* **25** 100–112.
[6] HOFFMAN, A. J. (1960). On the uniqueness of triangular association scheme. *Ann. Math. Statist.* **31** 492–497.
[7] RAGHAVARAO, D. (1960). On the block structure of certain PBIB designs with two associate classes having triangular and L_2 association schemes. *Ann. Math. Statist.* **31** 787–791.
[8] RAY-CHAUDHARI, D. K. (1965). Some configurations in finite projective spaces and partially balanced incomplete block designs. *Canad. J. Math.* **17** 114–123.
[9] SHRIKHANDE, S. S. (1959). On a characterisation of the triangular association scheme. *Ann. Math. Statist.* **30** 39–47.
[10 SHRIKHANDE, S. S. and BHAGWANDAS (1965). Duals of incomplete block designs. *J. Indian Statist. Assoc.* **3** 30–37.