SOME TESTS FOR THE INTRACLASS CORRELATION MODEL

By M. S. Srivastava

University of Toronto

1. Introduction. Let $X^{(i)}$ $(i = 1, 2, \cdots, k)$ be independent normal random p-vectors with mean vectors μ_i and nonsingular covariance matrices Σ_i. The problems with which we are concerned here are related to the comparison of dispersion matrices Σ_i's, when each dispersion matrix takes the intraclass correlation form, i.e.,

$$\Sigma_i = \sigma_i^2[(1 - \rho_i)I + \rho_i ee'], \tag{1}$$

where $e' = (1, 1, \cdots, 1)$, and I is an identity matrix of order $p \times p$. For this model, the problem of comparing dispersion matrices Σ_i reduces to that of comparing σ_i's and ρ_i's of k populations. For some related results on this model, refer to Wilks [8], Geisser [1], Votaw [7], and Selliah [6].

2. Problems. With the help of Roy's union-intersection principle [4], we propose in this paper test procedures for the following problems:

(i) To test the hypothesis $H: \rho_1 = \cdots = \rho_k; \sigma_1 = \cdots = \sigma_k$, against the alternative $A: \rho_i \neq \rho_j; \sigma_i \neq \sigma_j, i, j = 1, 2, \cdots, k, i \neq j$.

(ii) To test the hypothesis $H: \rho_i = 0, i = 1, 2, \cdots, k$, against the alternative $A: \rho_i \neq 0, i = 1, 2, \cdots, k$.

(iii) To test the hypothesis $H: \rho_1 = \cdots = \rho_k$, against the alternative $A: \rho_i \neq \rho_j, i, j = 1, 2, \cdots, k, i \neq j$.

3. Reduction to canonical form. Let S, a $p \times p$ matrix, have the Wishart distribution with mean $n\Sigma$ and degrees of freedom n, $n = N - 1$, and let X be a normal random p-vector with mean vector μ and covariance matrix Σ. In addition, let S be independent of X. For the intraclass correlation model Σ, there exists an orthogonal matrix Γ with first row $e'/p^{\frac{1}{2}}$ such that $\Gamma\Sigma\Gamma' = \text{diag}(\alpha, \beta, \cdots, \beta)$, where

$$\alpha = \sigma^2[1 + (p - 1)\rho], \qquad \beta = \sigma^2(1 - \rho). \tag{2}$$

Let $W = \Gamma S \Gamma'$. Then the pdf of W is

$$p(W) = \text{Const.}\ \alpha^{-n/2}\beta^{-m/2}|W|^{(n-p-1)/2} \exp\{-\tfrac{1}{2}w_{11}/\alpha - \tfrac{1}{2}\textstyle\sum_{r=2}^{p} w_{rr}/\beta\}, \tag{3}$$

where $m = n(p - 1)$. Let $Z = \Gamma X$, $\eta = \Gamma\mu$. Then z_1 is $N(\eta_1, \alpha)$, and z_r is $N(\eta_r, \beta)$, $r = 2, \cdots, p$. The z_r's are independently distributed for all $r = 1, 2, \cdots, p$.

Following Olkin and Pratt [3], we can obtain statistics u and v sufficient for α and β and distributed independently as $\alpha\chi_a^2$ and $\beta\chi_b^2$ where the degrees of freedom a and b depend on our knowledge of μ. For μ completely unknown,

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$$u = w_{11}, \qquad v = \textstyle\sum_{r=2}^{p} w_{rr}, \tag{4}$$

$$\mathcal{L}(u) = \mathcal{L}(\alpha\chi_n^2), \qquad \mathcal{L}(v) = \mathcal{L}(\beta\chi_m^2). \tag{5}$$

In the subsequent investigation we assume that the set of sufficient statistics $(Z^{(i)}, u_i, v_i)$ based on N_i observations are given for the set of unknown parameters $(\mu_i, \alpha_i, \beta_i)$.

4. Solution of Problem (i).

4.1. *Test criterion (equal numbers of observations).* By reduction of the covariance matrix in the canonical form, the problem of testing H against A is equivalent to:

$$H': \alpha_1 = \alpha_2 = \cdots = \alpha_k; \qquad \beta_1 = \beta_2 = \cdots = \beta_k,$$

$$A': \alpha_i \neq \alpha_j; \qquad \beta_i \neq \beta_j, \qquad\qquad i, j = 1, 2, \cdots, k, \ i \neq j.$$

Let G be the group of transformations $(u_i, v_i) \rightarrow (au_i, bv_i)$, $a > 0$, $b > 0$, $i = 1, 2, \cdots, k$. This group leaves the problem invariant, since the induced transformation of the parameter space is $(\alpha_i, \beta_i) \rightarrow (a\alpha_i, b\beta_i)$, $i = 1, 2, \cdots, k$. It is easily seen that $(u_i/u_j, v_i/v_j)$, $i \neq j = 1, 2, \cdots, k$, is a maximal invariant.

To test the hypothesis H' against A', we notice that H' is equivalent to the totality of all the sub-hypotheses $H'_{ij}: \alpha_i = \alpha_j$; $\beta_i = \beta_j$ (against $H'_{ij} \neq A_{ij}$), $i \neq j$. For testing $\alpha_i = \alpha_j$, the likelihood ratio criterion (LRC) is a monotone function of (u_i/u_j), and for testing $\beta_i = \beta_j$, the LRC is a monotone function of (v_i/v_j). It seems reasonable to use $(u_i/u_j)(v_i/v_j)$ for testing H'_{ij}. Hence, accept H'_{ij} iff (if and only if) for any given i and j, $i \neq j$,

$$K_{\epsilon_1} < (u_i/u_j)(v_i/v_j) < K_{\epsilon_2}, \tag{6}$$

where K_{ϵ_1} and K_{ϵ_2} are so chosen that the probability of (6) under H'_{ij} is $1 - \epsilon_1 - \epsilon_2$. But, since H' is equivalent to the totality of all sub-hypotheses H'_{ij} $(i \neq j = 1, 2, \cdots, k)$, we accept H' iff for all i and j, $K_{\epsilon_1} < (u_i/u_j)(v_i/v_j) < K_{\epsilon_2}$, i.e., iff

$$K_{\epsilon_1} < \max_{i,j,i\neq j} (u_i/u_j)(v_i/v_j) < K_{\epsilon_2}. \tag{7}$$

Hence, if equal numbers of observations are taken from each population, the test statistic is $F_{\max}^{(n)}F_{\max}^{(m)}$, which is the product of two Hartley's $F_{\max}$ statistics [2]; one based on n degrees of freedom and the other on m degrees of freedom (the superscript of F has been used to indicate this).

4.2. *Asymptotic distribution of* $F_{\max}^{(n)}F_{\max}^{(m)}$. We have

$$F_{\max}^{(n)}F_{\max}^{(m)} = \max_{i\neq j} (u_i/u_j)(v_i/v_j).$$

Taking the logarithm of both sides, we get

$$\log F_{\max}^{(n)}F_{\max}^{(m)} = \max_{i\neq j} [(\log u_i + \log v_i) - (\log u_j + \log v_j)].$$

We know that $\log u_i$ and $\log v_i$ are independently and asymptotically normally distributed with variances $2/n$ and $2/m$ respectively. If $w_i = \log u_i + \log v_i$, then $w_1, \cdots, w_k$ are independently and asymptotically normally distributed with variance $2(m + n)/mn$ and $\log F^{(n)}_{\max} F^{(m)}_{\max} = \max_{i \neq j} (w_i - w_j)$. Therefore, the approximate distribution of $\log F^{(n)}_{\max} F^{(m)}_{\max}$ is that of a range of k independent samples from a normal population with variance $2(m + n)/mn$. Hence, Hartley's table can be used for a test of significance.

However, for unequal sample sizes, there do not exist functions $c_i u_i$ and $d_i v_i$ such that the variances of the asymptotic distributions of $\log c_i u_i$ and $\log d_i v_i$ are independent of n_i; c_i and d_i are functions of n_i. Hence, for unequal sample sizes, we propose another test.

4.3. *Test criterion (unequal sample sizes).* We know that $\log (u_i/n_i) \sim N(\log \alpha_i, 2/n_i)$, and $\log (v_i/m_i) \sim N(\log \beta_i, 2/m_i)$. Hence for testing $\alpha_1 = \alpha_2 = \cdots = \alpha_k$, we get approximately a χ_1^2 (chi-square) statistic on $k - 1$ degrees of freedom. Similarly for testing $\beta_1 = \beta_2 = \cdots = \beta_k$, we get a χ_2^2 (chi-square) statistic on $k - 1$ degrees of freedom. Thus for testing H', we can take as the test criterion, either the sum of χ_1^2 amd χ_2^2, or the product of χ_1^2 and χ_2^2. In the latter case, we need the distribution and the percentage points of the product of two chi-squares, which will be given elsewhere.[1]

5. Solution of Problem (ii). The problem we are concerned with here is:

$$H: \rho_1 = \rho_2 = \cdots = \rho_k = 0,$$

$$A: \rho_i \neq 0, \qquad i = 1, 2, \cdots, k.$$

The group of transformations $(u_i, v_i) \to a_i(u_i, v_i)$; $a_i > 0$, $i = 1, 2, \cdots, k$ leaves the problem invariant since the group of induced transformations of the parameter space is $(\alpha_i, \beta_i) \to a_i(\alpha_i, \beta_i)$. Hence it is easily seen that $(Y_1, \cdots, Y_k) \equiv (u_1/v_1, \cdots, u_k/v_k)$ is a maximal invariant.

If we define $\tau_i = [1 + (p - 1)\rho_i]/(1 - \rho_i)$ then $\mathcal{L}\{(p - 1)Y_i\} = \mathcal{L}(\tau_i F_{n_i, m_i})$. Hence, the hypothesis H and the alternative A are equivalent to:

$$H': \tau_1 = \tau_2 = \cdots = \tau_k = 1,$$

$$A': \tau_i \neq 1.$$

To test the hypothesis H': $(\tau_1 = \cdots = \tau_k = 1)$, we notice that H' is equivalent to the totality of all sub-hypotheses $H_i': \tau_i = 1$ $(i = 1, 2, \cdots, k)$ (against $A_i' \neq H_i'$). The test procedure for testing H_i' is as follows: Accept H_i' iff (if and only if) for any given i, $K_{\epsilon_1} < F_i = (p - 1)Y_i < K_{\epsilon_2}$, where $\epsilon_1 + \epsilon_2 = \epsilon$ is the probability of an error of the first kind. Equivalently, accept H_i' against A_i' iff for any given i, $K'_{\epsilon_1} < F_i^{n_i^{1/2}} < K'_{\epsilon_2}$. But since H' is equivalent to the totality of all sub-hypotheses H_i' $(i = 1, 2, \cdots, k)$, we accept H' iff

$$K'_{\epsilon_1} < \max_i F_i^{n_i^{1/2}} < K'_{\epsilon_2}. \tag{8}$$

[1] The author notes that the distribution of the product of two chi-squares is given by Wells, Anderson and Cell, "The distribution of the product of two central or non-central chi-square variates," *Ann. Math. Statist.* **33** (1962) 1016–1020.

The asymptotic distribution of $\max_{1 \leq i \leq k} (n_i)^{\frac{1}{2}} \log F_i$ under the null hypothesis is that of the largest observation of a sample of k independent observations from a normal distribution with mean 0 and variance $2p/(p-1)$; for the hypothesis H', $n_i^{\frac{1}{2}} \log \tau_i = 0$.

For equal numbers of observations, the above test statistic is replaced by

$$\max_i F_i . \tag{9}$$

The distribution of max F is that of the maximum of the k independent samples from an F distribution with n and m degrees of freedom. It can be shown that this test is uniformly most powerful amongst all the tests based on max F, and its power function has the monotonicity property.

NOTE. Suppose we are testing the hypothesis that $\mu_1 = \cdots = \mu_k = \gamma e$, γ unknown against the alternative that $\mu_i \neq \gamma e$, $i = 1, 2, \cdots, k$. If we define $F_i = n_i \sum_{r=2}^{p} z_r^{(i)2} / \sum_{r=2}^{p} w_{rr}^{(i)} = n_i \sum_{r=2}^{p} z_r^{(i)2}/v_i$, we obtain the test statistic to be $\max_i F_i^{n_i^{\frac{1}{2}}}$ for unequal numbers of observations and $\max_i F_i$ for equal numbers of observations; F_i is distributed like an F-distribution with $p-1$ and $n_i(p-1)$ degrees of freedom.

6. Solution of Problem (iii).

6.1. *Test criterion* (*equal sample sizes*). The problem of testing H against A is equivalent to the problem of testing

$$H': \tau_1 = \tau_2 = \cdots = \tau_k = \tau \quad \text{(say)}$$

against the alternative A': $\tau_i \neq \tau_j$, $i, j = 1, 2, \cdots, k$, $i \neq j$, where $\tau_i = [1 + \overline{p-1}\rho_i]/(1-\rho_i)$.

The hypothesis H' can be split up into p_{c_2} hypotheses $H'_{ij} : \tau_i = \tau_j = \tau$ (say) against the alternative $A'_{ij} : \tau_i \neq \tau_j$. We have shown in Section 5 that $(Y_1, \cdots, Y_k)$ is a maximal invariant statistic and that $\mathcal{L}\{(p-1)Y_i\} = \mathcal{L}(\tau_i F_{n,m})$. The maximum likelihood estimate of τ_i is $\hat{\tau}_i = (p-1)Y_i$ and that of τ (under H'_{ij}) is given by

$$\hat{\tau} = [(p-2)(Y_i + Y_j)/4] + [\{(p-2)(Y_i+Y_j)/4\}^2 + (p-1)Y_iY_j]^{\frac{1}{2}}, \tag{10}$$

which for large p can be approximated by $\frac{1}{2}p(Y_i + Y_j)$. For large p, $\hat{\tau}_i = pY_i$.

The likelihood ratio statistic for testing H'_{ij} against A'_{ij}, for large p, is

$$C\, Y_i^{n/2-1} Y_j^{n/2-1}/(Y_i + Y_j)^n = C Q_{ij}^{n/2-1}/(1+Q_{ij})^n, \tag{11}$$

where $Q_{ij} = Y_i/Y_j$, the ratio of two F variables.

Hence we accept H'_{ij} iff, $K_{\epsilon_1} < Q_{ij} < K_{\epsilon_2}$, where K_{ϵ_1}, K_{ϵ_2} are so chosen as to make the probability of an error of the first kind $= \epsilon_1 + \epsilon_2$. Since H' is equivalent to the totality of all sub-hypotheses H'_{ij}, we accept H' iff for all i, j $(i \neq j)$, $K_{\epsilon_1} < Q_{ij} < K_{\epsilon_2}$.

Hence the test statistic is

$$Q_{\max} = Y_{\max}/Y_{\min} . \tag{12}$$

NOTE. An alternative test procedure can be obtained if we view H' as equivalent to the totality of all the subhypotheses $H'_{ik} : \tau_i = \tau_k$. The test statistic is

(13) $$Y_{\max}/Y_k = \max_{1 \leq i \leq k-1} (Y_i/Y_k).$$

The distribution and percentage points of this distribution will be given elsewhere.

6.2. *Asymptotic distribution of* $Q_{\max}$. We have

$$\log_e Q_{\max} = \log_e Y_{\max} - \log_e Y_{\min}.$$

For large n, $\log_e (p - 1)Y_i$, $i = 1, 2, \cdots k$, are independently and asymptotically normally distributed with mean $\log_e \tau_i$ and variance $2p/n(p - 1)$. Hence, under the hypothesis H', the approximate distribution of $\log_e Q_{\max}$ is that of the range of k independent samples from a normal population with mean 0 and variance $2p/n(p - 1)$. The percentage points can therefore be obtained from Hartley's table.

6.3. *Test criterion (unequal sample sizes).* We know from above that $\log_e (p - 1)Y_i$'s are independently and asymptotically normally distributed with means $\log \tau_i$ and variances $2p/n_i(p - 1)$. Hence, we apply the χ^2 test for testing H'.

6.4. *Two populations case.* For $k = 2$, the problem is that of testing $H: \tau_1 = \tau_2$ against $A: \tau_1 \neq \tau_2$. The likelihood ratio statistic is, for large p, a monotone function of $Q = Y_1/Y_2$. The pdf of Q and its percentage points are given by Schumann and Bradley [5], for equal sample size case ($n_1 = n_2$, $m_1 = m_2$).

However, for large n_1, and n_2, we may make use of the log-transformation given in Section 6.3. For large and equal numbers of observations, we may make use of the log-transformation given in Section 6.2.

REMARK. For the general alternative A: not H, the same tests are obtained.

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