A NOTE ON MINIMUM DISCRIMINATION INFORMATION[1]

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This note contains a simple proof of the minimum discrimination information theorem in Kullback (1959), pp. 36–39 and an affirmative answer to a suggestion in a personal communication from Dr. I. J. Good that the theorem could be applied even to random elements of a Banach space.

Let X be a space of points x, S a σ-field of sets of X, and P_2 a probability measure on S. Let $T(x)$ be a real valued S-measurable function such that

$$M_2 = \int_X \exp(T(x))\, dP_2 < \infty \tag{1}$$

and let the probability measure P^* be defined by

$$P^*(A) = \int_A (\exp(T(x))/M_2)\, dP_2, \qquad \text{for } A \,\varepsilon\, S. \tag{2}$$

Suppose that $T(x)$ is P^*-integrable, and let

$$\theta = \int_X T(x)\, dP^*. \tag{3}$$

Now let P_1 be an arbitrary probability measure on S. If $P_1 \ll P_2$ define

$$I(P_1, P_2) = \int_X [\log(dP_1/dP_2)]\, dP_1, \tag{4}$$

otherwise define $I(P_1, P_2) = \infty$. It is clear that $I(P^*, P_2)$ is finite; in fact, from (2) and (3),

$$I(P^*, P_2) = \theta - \log M_2. \tag{5}$$

Theorem. *If P_1 is a probability measure on S such that T is P_1-integrable and*

$$\int_X T(x)\, dP_1 = \theta, \tag{6}$$

then

$$I(P_1, P_2) \geqq I(P^*, P_2) = \theta - \log M_2 \tag{7}$$

with equality if and only if $P_1 = P^*$ on S.

Proof. If $I(P_1, P_2) = \infty$ there is nothing to prove. Suppose then that $I(P_1, P_2) < \infty$. In this case $P_1 \ll P_2$, and we write $f(x) = dP_1/dP_2$. Then

$$I(P_1, P_2) = \int_X f(x) \log f(x)\, dP_2 \tag{8}$$

and

$$I(P^*, P_2) = \int_X f^*(x) \log f^*(x)\, dP_2 \tag{9}$$

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where $f^* = dP^*/dP_2 = e^T/M_2$ by (2). In view of (3), (6) and f^* as just defined, we have that

$$(10) \qquad \int_X f(x) \log f^*(x)\, dP_2 = \int_X f^*(x) \log f^*(x)\, dP_2 .$$

We see from (8), (9) and (10) that it suffices to show that the left-hand side of (10) cannot exceed the right-hand side of (8), that is, that

$$(11) \qquad \int_X f(x) \log [f^*(x)/f(x)]\, dP_2 \leqq 0.$$

Since $\log z \leqq z - 1$ for all $z \geqq 0$ with equality only for $z = 1$, and since

$$(12) \qquad \int_X f\, dP_2 = \int_X f^*\, dP_2 = 1,$$

it follows that (11) holds with equality only if $P_1 \equiv P^*$.

The foregoing applies to *any* probability space (X, S, P_2) and any statistic T. Suppose now that

(a) X is a real separable Banach-space;

(b) X^* is the dual space of X consisting of all the continuous linear functionals $x^*(x)$ on X;

(c) S is the σ-algebra generated by the continuous linear functionals on X;

(d) m_1, m_2 are elements of X;

(e) P_2 is a probability measure defined on S with the mean value m_2 defined via a Pettis integral, that is, $x^*(x)$ is P_2-integrable for each x^* and m_2 is a (necessarily unique) element in X such that $x^*(m_2) = \int_X x^*(x)\, dP_2$ for all x^* in X^*.

We write $m_2 = E_2(x)$ [cf. Grenander (1963), p. 128, Mourier (1953), p. 164; (1956), p. 231, Pettis (1938)]. Let x^* be a fixed continuous linear functional and take $T(x) \equiv x^*(x)$ and write M_2 in (1) as $M_2(x^*)$. Let P_1 be a probability measure and m_1 an element of X such that

$$(13) \qquad \int_X y^*(x)\, dP_1 = y^*(m_1) = \int_X (y^*(x) \exp x^*(x)/M_2(x^*))\, dP_2$$

for all continuous linear functionals y^* in X^*, that is, $m_1 = E_1(x)$. Then (13) holds in particular when $y^* = x^*$, that is, P_1 satisfies (6) with $\theta = x^*(m_1)$ defined by (3), and we have (7) with $\theta - \log M_2 = x^*(m_1) - \log M_2(x^*)$.

REFERENCES

GRENANDER, U. (1963). *Probabilities on Algebraic Structures*. Wiley, New York.

KULLBACK, S. (1959). *Information Theory and Statistics*. Wiley, New York.

MOURIER, E. (1953), Éléments aléatoires dans un espace de Banach. *Ann. de l'Inst. Henri Poincaré* **13** 161–244.

MOURIER, E. (1956). L-random elements and L^*-random elements in Banach spaces. *Proc. Third Berkeley Symp. Math. Statist. Prob.* **2** 231–242. Univ. of California Press.

PETTIS, B. J. (1938). On integration in vector spaces. *Trans. Amer. Math. Soc.* **44** 277–304.