## EQUIVALENCE AND SINGULARITY FOR FRIEDMAN URNS[1]

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**1. Introduction.** $W_0$ {respectively, $W_0'$} and $B_0$ {$B_0'$} are positive real numbers, $\alpha$ {$\alpha'$} and $\beta$ {$\beta'$} non-negative real numbers with $\alpha + \beta > 0$ {$\alpha' + \beta' > 0$}. At time 0, urn $U$ {$U'$} contains $W_0$ {$W_0'$} white and $B_0$ {$B_0'$} black balls. At time $n$, a ball is drawn at random from $U$ {$U'$} and replaced, together with $\alpha$ {$\alpha'$} balls of the same color and $\beta$ {$\beta'$} of the opposite. If the $n$th draw from $U$ {$U'$} is white, $X_n$ {$X_n'$} is 1; otherwise, 0. The distribution of $X_1, X_2, \cdots$ {$X_1', X_2', \cdots$} is $D$ {$D'$}, a probability on the space $\Omega$ of sequences of 0'r and 1's. Let $\rho = (\alpha - \beta)/(\alpha + \beta)$ {$\rho' = (\alpha' - \beta')/(\alpha' + \beta')$}. The object of this note is to prove

(1) Theorem. *$D \equiv D'$ or $D \perp D'$ according as $\rho = \rho'$ or $\rho \neq \rho'$.*

If $\rho = \rho' = 1$, then (1) follows from De Finetti's theorem; if $\rho < \rho' = 1$, then (1) follows from Reference [2], Lemma 2.1 and Theorems 2.2, 3.1.

**2. Generalities.** Let $\mathfrak{F}_n$ be the $\sigma$-field of subsets of $\Omega$ spanned by the first $n$ coordinates. If $\Pi$ is a probability on $\Omega$, let $\Pi(n + 1, i)$ be the conditional $\Pi$-probability that the $n + 1$st coordinate is $i$, given $\mathfrak{F}_n$. If $\omega \in \Omega$, let

$$S_n(\omega) = \omega(1) + \cdots + \omega(n) \tag{2}$$

and

$$E_n = n^{-1}S_n - \tfrac{1}{2}. \tag{3}$$

If $\rho < 1$, by [2],

$$E_n \to 0 \quad \text{with } D\text{-probability } 1. \tag{4}$$

Since

$$D(n + 1, 1) = [W_0 + \beta n + (\alpha - \beta)S_n]/[W_0 + B_0 + (\alpha + \beta)n], \tag{5}$$

it follows from (4) that when $\rho < 1$,

$$D(n, 1) \to \tfrac{1}{2} \quad \text{with } D\text{-probability } 1. \tag{6}$$

This may help to motivate the next result.

Let $P$ {respectively, $P'$} be the probability on the two-point set {0, 1} assigning measure $p$ {$p'$} to 1. Let $\epsilon = p' - p$.

(7) Lemma. *If $p$ and $p'$ converge to $\frac{1}{2}$, then the $P$-expectation of* $\log(dP'/dP)$ *is* $-2\epsilon^2 + o(\epsilon^2)$, *and the $P$-expectation of* $(\log(dP'/dP))^2$ *is* $4\epsilon^2 + o(\epsilon^2)$.

Proof. Expand $\log(1 + u)$ in powers of $u$. ∎

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Let $\Pi$ and $\Pi'$ be probabilities on $\Omega$, assigning positive measure to non-empty $\mathfrak{F}_n$-sets for all $n$. When the $n$th coordinate is $i$, let $L_n(\Pi', \Pi) = \log \Pi'(n, i) - \log \Pi(n, i)$; and let $L(\Pi', \Pi) = \sum L_n(\Pi', \Pi)$.

(8) LEMMA. (i) *The following two statements are equivalent:* $\Pi' \equiv \Pi$; $L(\Pi', \Pi)$ *is finite with* $\Pi$*-probability* 1 *and* $\Pi'$*-probability* 1.

(ii) *The following three statements are equivalent:* $\Pi' \perp \Pi$; $L(\Pi', \Pi) = -\infty$ *with* $\Pi$*-probability* 1; $L(\Pi', \Pi) = \infty$ *with* $\Pi'$*-probability* 1.

PROOF. Standard martingale argument.∎

**3. Proof of** (1) **when** $\rho = \rho' < 1$. There is no loss in supposing $\alpha = \alpha'$, $\beta = \beta'$, which simplifies the computation. From (5) and (7), the conditional $D$-expectation of $L_{n+1}(D', D)$ given $\mathfrak{F}_n$ is $O(1/n^2)$, as is the conditional $D$-expectation of $[L_{n+1}(D', D)]^2$ given $\mathfrak{F}_n$. Of course, (7) applies by (6). Similarly for $D'$. Now use (8) (i), and, for example, (10) of [1].

**4. If** $\rho \neq 0$, $D \perp$ **fair coin tossing.** The proof of the singularity part of (1) uses (15) below, which I could not prove more directly. Let $F$ be the probability on $\Omega$ under which the coordinates are independent and 1 with probability $\frac{1}{2}$. Recall (2) and (3).

(9) LEMMA. $\sum E_n^2 = \infty$ *with* $F$*-probability* 1.

PROOF. Let $f_n$ be the number of $j = 1, \cdots, n$ with $E_j \geqq j^{-\frac{1}{2}}$. Then

$$\limsup n^{-1} f_n = 1$$

with $F$-probability 1. Indeed, $\limsup n^{-1} f_n$ is constant $F$-almost surely, by the Hewitt-Savage 0-1 Law. The distribution of $n^{-1} f_n$ converges to the distribution of the Lebesgue measure $L$ of $\{t: 0 < t < 1, B(t) \geqq t^{\frac{1}{2}}\}$, where $B(\cdot)$ is standard Brownian motion, by the Invariance Principle. If $\lambda < 1$, $L$ exceeds $\lambda$ with positive probability, so $\limsup n^{-1} f_n \geqq \lambda$.∎

(10) REMARK. (i) If $\rho = 0$ and $W_0 = B_0$, then $D = F$.

(ii) If $\rho = 0$, but $W_0 \neq B_0$, by (i) and Section 3, $D \equiv F$.

(11) LEMMA. *If* $\rho \neq 0$, $D \perp F$.

PROOF. Only $\rho < 1$ needs proof. By Section 3, there is no loss in supposing $W_0 = B_0$. Then, from (4) and (5),

$$D(n + 1, 1) = \tfrac{1}{2} + \rho E_n + o(E_n). \tag{12}$$

From (7) and (12), the conditional $F$-expectation of $L_{n+1}(D, F)$ given $\mathfrak{F}_n$ is $-2\rho^2 E_n^2 + o(E_n^2)$; and the conditional $F$-expectation of $[L_{n+1}(D, F)]^2$ given $\mathfrak{F}_n$ is $4\rho^2 E_n^2 + o(E_n^2)$. By (8) of [1], and (9),

$$\frac{[L_2(D, F) + \cdots + L_{n+1}(D, F) + 2\rho^2(E_1^2 + \cdots + E_n^2)]}{4\rho^2(E_1^2 + \cdots + E_n^2)} \tag{13}$$

$$\to 0 \quad \text{with } F\text{-probability 1}.$$

Therefore,

$$(14)\quad [L_2(D, F) + \cdots + L_{n+1}(D, F)]/(E_1^2 + \cdots + E_n^2) \to -\tfrac{1}{2} \text{ with } F\text{-probability } 1.$$

Using (9) again, $L(D, F) = -\infty$ with $F$-probability 1. Apply (8) (ii).∎

REMARK. In contrast with (11), for $\rho < 1$, $D$-almost all $\omega$ are normal, that is, each $k$-block of 0's and 1's occurs with relative frequency $2^{-k}$. This follows easily from (4) with the help of Levy's martingale strong law of large numbers. For a discussion of this theorem, see [1].

(15) LEMMA. *If* $\rho < 1$, $\sum E_n^2 = \infty$ *with D-probability* 1.

PROOF. If $\rho = 0$, use (9) and (10). If $\rho \neq 0$, from (7) and (12), the conditional $D$-expectation of $L_{n+1}(D, F)$ given $\mathfrak{F}_n$ is $2\rho^2 E_n^2 + o(E_n^2)$; and the conditional $D$-expectation of $[L_{n+1}(D, F)]^2$ given $\mathfrak{F}_n$ is $4\rho^2 E_n^2 + o(E_n^2)$. From (10) of [1], $L(D, F)$ is finite $D$-almost surely on the set where $\sum E_n^2 < \infty$. Use (8) (ii) and (11).∎

**5. Proof of** (1) **when** $\rho \neq \rho'$, **both less than** 1. By Section 3, suppose $W_0 = B_0$ and $W_0' = B_0'$. From (7) and (12), the conditional $D$-expectation of $L_{n+1}(D', D)$ given $\mathfrak{F}_n$ is $-2(\rho - \rho')^2 E_n^2 + o(E_n^2)$, and the conditional $D$-expectation of $[L_{n+1}(D', D)]^2$ is $4(\rho - \rho')^2 E_n^2 + o(E_n^2)$. The case $\rho = 0$ has been observed in the proof of (11). For the same reason as (14), but using (15) instead of (9),

$$(16)\quad [L_2(D', D) + \cdots + L_{n+1}(D', D)]/(E_1^2 + \cdots + E_n^2) \to -\tfrac{1}{2} \text{ with } D\text{-probability } 1.$$

Use (15) again and (8) (ii).∎

**6. Conjecture.** There is a Borel function $f$ from $\Omega$ to the real line, such that for all $W_0$, $B_0$, $\alpha$, $\beta$: with $D$-probability 1, $f = \rho$.

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REFERENCES

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