ON CROSSINGS OF LEVELS AND CURVES BY A WIDE CLASS OF STOCHASTIC PROCESSES[1]

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0. Summary. In this paper, upcrossings, downcrossings and tangencies to levels and curves are discussed within a general framework. The mean number of crossings of a level (or curve) is calculated for a wide class of processes and it is shown that tangencies have probability zero in these cases. This extends results of Ito [1] and Ylvisaker [7] for stationary normal processes, to non stationary and non normal cases. In particular the corresponding result given by Leadbetter and Cryer [3] for normal, non stationary processes can be slightly improved to apply under minimal conditions. An application is also given for an important non normal process.

1. Introduction. The problem of obtaining the mean number of crossings of a level in unit time by a *stationary normal* process, has received considerable attention in the literature. The interest in this problem stems from the work of Rice [5], and has now received a final solution by Ito [1] and Ylvisaker [7], who have obtained Rice's formula under minimal conditions.

For non stationary *normal* processes, a corresponding result has been given by Leadbetter and Cryer [3] under sufficient conditions which are very close to being necessary. It is however (as will be seen below) possible to improve this result slightly, and state it under the minimal conditions corresponding to those of [1] and [7] for the stationary case.

Non normal processes have been discussed to some extent in this context by Ivanov [2], who gives an explicit result for the mean number of crossings of a level in the *stationary* case. The conditions assumed by Ivanov require, in particular that the covariance function should possess a finite fourth derivative at the origin.

In seeking minimal conditions for the validity of results of this nature, it becomes necessary to distinguish between the number of "genuine" crossings of a level u in a given time, and the number of times the value u is assumed in that interval. The difference between these two quantities is (as will be discussed in the next section) the number of *tangencies* to the given level. The methods to be used provide, initially, a formula for the mean number of crossings of a level u, by a stochastic process in a given time. However if one can show that there is zero probability of the process being somewhere tangential to the level u in that time, then the formula will apply also to the mean number of times the value u

Received 15 June 1965; revised 20 September 1965.

[1] This research was supported by the Army, Navy, Air Force and NASA under a contract administered by the Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.

is assumed by the process in the time interval considered. This question was discussed completely for stationary normal processes by Ylvisaker [7] who demonstrated the absence of tangencies under general conditions, for that case.

In Section 2 we shall give the appropriate definitions for crossings and tangencies, in a general context. We shall then derive the formula for the mean number of crossings of a level under general conditions, for a wide class of stochastic processes.

In Section 3 upcrossings and downcrossings will be discussed, and it will be shown that, under conditions applying to the previous calculations, tangencies have probability zero. Hence the mean number of times a process actually assumes a value u in a given time interval may be calculated.

Finally in Section 4 these results will be applied to normal processes, and to the case of the envelope of a stationary normal process—an important non normal application.

The above remarks have concerned crossings of a level. In fact the results will apply equally well when we consider curves, instead of fixed levels. For crossings of (or tangencies to) a curve $u(t)$ by a process $\xi(t)$ are equivalent to crossings of (or tangencies to) the zero level by the process $\xi(t) - u(t)$. Hence there will be no loss of generality in restricting attention to fixed levels, and we shall usually do so.

2. The mean number of crossings of a level by a stochastic process. For a given "level" u, let G_u denote the set of continuous functions $f(t)$ on $0 \leqq t \leqq 1$ such that $f(t)$ is not identically equal to u in any interval, and $f(0) \neq u \neq f(1)$. We shall say (as in [7]) that $f(t) \varepsilon G_u$ *crosses the level* u at t_0 if, in each neighborhood of t_0 there are points t_1 and t_2 such that $[f(t_1) - u][f(t_2) - u] < 0$. From the definition of G_u it is easy to see that, if $f \varepsilon G_u$ and if $t_1 < t_2$ are two points in the unit interval such that $[f(t_1) - u][f(t_2) - u] < 0$, then $f(t)$ crosses the level u at some point between t_1 and t_2. This simple fact will be useful at various times in the sequel.

If $f(t) \varepsilon G_u$ crosses the level u at t_0, we have $f(t_0) = u$. However it is possible for $f(t)$ to assume the value u at some point t_0 without crossing the level u there. This will be termed a *tangency*, even though the function may be non-differentiable.

In this section we shall deal specifically with *crossings* of a level u by a stochastic process. Suppose then, here and throughout, that $\{\xi(t): 0 \leqq t \leqq T\}$ is a stochastic process which possesses continuous sample functions with probability one. It will further be assumed that $\xi(t)$ has an absolutely continuous distribution and, indeed, that this is true of the joint distribution of $\xi(t)$ and $\xi(s)$ for any $t \neq s$. For convenience we take $T = 1$. Then it is clear, with the above assumptions, that $\xi(t) \varepsilon G_u$ with probability one, for any level u. Write $C(u)$ for the number of crossings of the level u by $\xi(t)$ in $0 \leqq t \leqq 1$. The following lemma has been used previously in various forms (and proved by Ylvisaker [7] in this context.)

LEMMA 1. *For each* n, *let* $\xi_n(t)$ *be the process defined to coincide with* $\xi(t)$ *at points*

$t_{n,r} = r/2^n$, $r = 0, 1 \cdots 2^n$, and linear between such points. Write C_n for the number of crossings of the level u by $\xi_n(t)$ in $0 \leqq t \leqq 1$. Then $C_n \uparrow C(u)$ with probability one, as $n \to \infty$.

We note in passing that, since C_n is clearly a random variable for each n, it follows that the limit $C(u)$ is also a random variable.

Let us write now $f_{t,s}(x, y)$ for the joint density of $\xi(t)$ and $\xi(s)$ and

$$(2.1) \qquad g_{t,\tau}(x, y) = \tau f_{t,t+\tau}(x, x + \tau y).$$

That is g is the joint probability density for $\xi(t)$ and the ratio $[\xi(t + \tau) - \xi(t)]/\tau$. We then have the following general result.

THEOREM 1. *Let $\xi(t)$ have continuous sample functions, as defined, and let the function g be defined by (2.1). Then*

$$(2.2) \qquad \mathcal{E}\{C(u)\} = \lim_{n\to\infty} \int_0^1 dt[\int_0^\infty \int_{-z}^0 + \int_\infty^0 \int_0^{-z}]\Psi_{n,t}(u + 2^{-n}x, z)\, dx\, dz$$

in which, (writing $t_r = t_{n,r} = r/2^n$),

$$(2.3) \qquad \Psi_{n,t}(x, y) = g_{t_r, 2^{-n}}(x, y)$$

for $t_r \leqq t < t_{r+1}$, $r = 0, 1, \cdots, 2^n$.

PROOF. From Lemma 1 monotone convergence at once yields

$$(2.4) \qquad \mathcal{E}\{C(u)\} = \lim_{n\to\infty} \mathcal{E}\{C_n\}.$$

Now

$$(2.5) \qquad \mathcal{E}\{C_n\} = \sum_{r=0}^{2^n-1} [P\{\xi_r > u > \xi_{r+1}\} + P\{\xi_r < u < \xi_{r+1}\}],$$

in which ξ_r is written for $\xi(r/2^n)$. The first term on the right of (2.5) can be written as $\sum_{r=0}^{2^n-1} P\{\xi_r > u > \xi_r + 2^{-n}\eta_r\}$ where $\eta_r = (\xi_{r+1} - \xi_r)/2^{-n}$. But ξ_r and η_r have the joint density $g_{t_r,2^{-n}}(x, y)$ and hence this latter expression is

$$\sum_{r=0}^{2^n-1} \int_{-\infty}^0 dz \int_u^{u-2^{-n}z} g_{t_r,2^{-n}}(x, z)\, dx$$

$$= 2^{-n} \sum_{r=0}^{2^n-1} \int_{-\infty}^0 dz \int_0^{-z} g_{t_r,2^{-n}}(u + 2^{-n}x, z) dx$$

$$= \int_0^1 dt \int_{-\infty}^0 dz \int_0^{-z} \Psi_{n,t}(u + 2^{-n}x, z)\, dx.$$

This, and the corresponding result for the final term of (2.5) yield the desired conclusion.

The theorem just proved is of a very general nature. By making more specific assumptions we are able to obtain the following useful result from it.

THEOREM 2. *Suppose that $\xi(t)$ has continuous sample functions with probability one and that*

(i) *$g_{t,\tau}(x, z)$ is continuous in (t, x) for each z, τ,*

(ii) *$g_{t,\tau}(x, z) \to p_t(x, z)$ as $\tau \to 0$, uniformly in (t, x) for each z and*

(iii) *$g_{t,\tau}(x, z) \leqq h(z)$ for all t, τ, x, where $\int_{-\infty}^\infty |z|h(z)\, dz < \infty$.*

Then

$$(2.6) \qquad \mathcal{E}\{C(u)\} = \int_0^1 dt \int_{-\infty}^\infty |z|p_t(u, z)\, dz < \infty.$$

PROOF. With the notation of (2.3), if $t_r \leqq t < t_{r+1}$,

$$|\Psi_{n,t}(u + 2^{-n}x, z) - p_t(u, z)| \leqq |g_{t_r, 2^{-n}}(u + 2^{-n}x, z) - p_{t_r}(u + 2^{-n}x, z)| + |p_{t_r}(u + 2^{-n}x, z) - p_t(u, z)|.$$

The first term tends to zero as $n \to \infty$ by uniform convergence. The second term tends to zero since $t_r = t_{n,r} \to t$ and the uniform limit $p_t(y, x)$ is continuous in (t, y). Thus

$$\Psi_{n,t}(u + 2^{-n}x, z) \to p_t(u, z) \quad \text{as} \quad n \to \infty.$$

But by assumption (iii), $\Psi_{n,t}(u + 2^{-n}x, z) \leqq h(z)$ where $\int_{-\infty}^{\infty} |z|h(z)\, dz < \infty$. Thus the required result follows from (2.2) by dominated convergence.

Finally we note that the mean number of crossings of the level u by $\xi(t)$ in the interval $(0, T)$ is obtained by carrying out the t integration in (2.6) over that interval instead of the range $0 \leqq t \leqq 1$.

3. Upcrossings, tangencies and u-values. We shall say (cf. [6]) that a function $f(t) \varepsilon G_u$ has an *upcrossing* of the level u at t_0 if there exists $\epsilon > 0$ such that $f(t) \leqq u$ in $(t_0 - \epsilon, t_0)$ and $f(t) \geqq u$ in $(t_0, t_0 + \epsilon)$. We note that since $f \varepsilon G_u$, there must be points in $(t_0 - \epsilon, t_0)$ where $f(t) < u$ and points in $(t_0, t_0 + \epsilon)$ where $f(t) > u$. Write $C_U(u)$ for the number of upcrossings of the level u in $0 \leqq t \leqq 1$, by the process $\xi(t)$ defined early in the previous section.

Downcrossings of the level u are similarly defined by reversing inequalities. Write $C_D(u)$ for the number of downcrossings of the level u by $\xi(t)$ in $0 \leqq t \leqq 1$.

In general there can be crossings which are neither upcrossings nor downcrossings. However, if the conditions of Theorem 2 hold, it follows from (2.6) that $C(u)$ is finite, with probability one. Further, if $C(u) < \infty$ it is easy to show that all crossings are upcrossings or downcrossings, and hence $C(u) = C_U(u) + C_D(u)$ with probability one. Thus $C_U(u)$ is finite with probability one. In this event we may obtain a formula for $\mathcal{E}\{C_U(u)\}$ along exactly the same lines as the derivation of (2.6). In fact Lemma 1 then has an obvious modification to deal with upcrossings and, in modifying (2.5), we consider only the second of the two terms on the right. This then leads to the results that $C_U(u)$, $C_D(u)$ are random variables with means

$$\mathcal{E}\{C_U(u)\} = \int_0^1 dt \int_0^{\infty} z p_t(u, z)\, dz, \tag{3.1}$$

$$\mathcal{E}\{C_D(u)\} = \int_0^1 dt \int_{-\infty}^{0} |z| p_t(u, z)\, dz, \tag{3.2}$$

formulae which are valid under the conditions of Theorem 2.

We note in passing that from (3.1) and (3.2),

$$\mathcal{E}\{C_U(u) - C_D(u)\} = \int_0^1 dt \int_{-\infty}^{\infty} z p_t(u, z)\, dz.$$

However it may be shown that

$$\begin{aligned}\mathcal{E}\{C_U(u) - C_D(u)\} &= P\{\xi(0) < u < \xi(1)\} - P\{\xi(0) > u > \xi(1)\} \\ &= P\{\xi(0) < u\} - P\{\xi(1) < u\}.\end{aligned}$$

The fact that these two expressions for $\mathcal{E}\{C_U(u) - C_D(u)\}$ are identical (under the conditions of Theorem 2) provides quite an amusing excercise.

We turn now to the problem of showing that tangencies have probability zero. In Section 2, a tangency to the level u by $f \varepsilon G_u$ was defined as any u-value of f which is not a crossing of the level u. Equivalently one may say that $f(t) \varepsilon G_u$ has a tangency to the level u at t_0 if $f(t_0) = u$ and if there is a neighborhood of t_0 on which $f(t) - u$ does not change sign. If $f(t) \leqq u$ on this neighborhood, the tangency will be called a *tangency from below* and otherwise a *tangency from above*. Let $T(u)$, $T_A(u)$, $T_B(u)$ denote respectively the number of tangencies, tangencies from above, and tangencies from below by our process $\xi(t)$, in $0 \leqq t \leqq 1$. Then we have the following result.

LEMMA 2. *Suppose $\xi(t)$ satisfies the general conditions stated early in Section 2. Suppose also that $C(u) < \infty$, $C(u - 1/n) < \infty$, $n = 1, 2, \cdots$, with probability one. Then, with probabiliy one,*

$$C_U(u) + T_B(u) \leqq \liminf_{n\to\infty} C_U(u - 1/n).$$

PROOF. By assumption $\xi(t) \varepsilon G_{u-1/n}$ for all n and $\xi(t) \varepsilon G_u$, with probability one. For such a sample function suppose that $C_U(u) + T_B(u) \geqq m$. Then m points $t_1 \cdots t_m$ can be chosen at which ξ has an upcrossing or tangency from below of u. Surround these points by disjoint open intervals $(t_i - a, t_i + a)$. There is a point s_i in $(t_i - a, t_i)$ at which $\xi(s_i) < u$. Choose n_0 such that $f(s_i) < u - 1/n_0$, $i = 1 \cdots m$. But since by continuity $\xi(t_i) = u > u - 1/n$, it follows that there is a crossing of $u - 1/n$ in each interval (s_i, t_i), when $n \geqq n_0$. In fact there is an upcrossing in each such interval when $n \geqq n_0$. For if the first of the (finite number of) crossings in say (s_1, t_1) is at t_0, it must, by a remark made above, be an upcrossing or a downcrossing. It cannot be a downcrossing since this would imply the existence of a point in (t_1, t_0) at which $\xi(t) > u$ and hence another crossing between t_1 and t_0.

Thus each interval (s_i, t_i) contains an upcrossing of $u - 1/n$, when $n \geqq n_0$ and hence $C_U(u - 1/n) \geqq m$ for all $n \geqq n_0$. Thus $\liminf_{n\to\infty} C_U(u - 1/n) \geqq C_U(u) + T_B(u)$, whether or not the right hand side is finite, and the truth of the lemma follows.

Suppose now that the conditions of Theorem 2 are satisfied. Then in particular, so are the conditions of the above lemma, and thus, with probability one,

$$T_B(u) \leqq \liminf_{n\to\infty} C_U(u - 1/n) - C_U(u). \tag{3.3}$$

The right hand side of (3.3) is a random variable whose expectation is, by Fatou's lemma, dominated by $\liminf_{n\to\infty} \mathcal{E}\{C_U(u - 1/n)\} - \mathcal{E}\{C_U(u)\}$. But since $p_t(u, z) \leqq h(z)$ where $|z|h(z)$ is integrable, an easy application of dominated convergence to (3.1) shows that $\mathcal{E}\{C_U(u- 1/n)\} \to \mathcal{E}\{C_U(u)\}$ as $n \to \infty$. That is the expectation of the random variable on the right of (3.3) is zero, and since this random variable is non negative, it vanishes with probability one.

Hence $T_B(u) = 0$ with probability one. The same is true of $T_A(u)$ and thus also of $T(u)$. Since this is so, it follows that the number $N(u)$ of times $\xi(t)$ assumes the value u in $0 \leqq t \leqq 1$ is equal, with probability one, to the random variable $C(u)$. These facts are summarized in the following result.

THEOREM 3. *Under the conditions of Theorem 2, Equations* (3.1) *and* (3.2) *give the mean number of upcrossings and downcrossings of the level u by* $\xi(t)$ *in* $0 \leqq t \leqq 1$. *Further, the probability is zero that* $\xi(t)$ *will become tangential to the level u somewhere in* $0 \leqq t \leqq 1$ *and* $\mathcal{E}\{N(u)\}$, *the mean number of times* $\xi(t)$ *assumes the value u in* $0 \leqq t \leqq 1$ *is also given by the right hand side of* (2.6).

As already noted, we can obtain corresponding results for crossings of (and tangencies to) a curve $u(t)$ by the process $\xi(t)$, by simply considering crossings of (and tangencies to) the axis by $\xi(t) - u(t)$. Of course we shall require $u(t)$ to be continuous in order to guarantee that $\xi(t) - u(t)$ should possess continuous sample functions. However we do require more than continuity of $u(t)$ for $\xi(t) - u(t)$ to satisfy all the conditions of Theorem 2. In fact $\xi(t) - u(t)$ will satisfy these conditions if we require that $u(t)$ possess a continuous derivative $u'(t)$ in $0 \leqq t \leqq 1$ and if the function $g_{t,\tau}(x, z)$ is continuous in (t, x, z) for each τ and is such that $g_{t,\tau}(x, z) \to p_t(x, z)$ as $\tau \to 0$, uniformly in (t, x, z). In that case the mean number of crossings of the curve $u(t)$ by $\xi(t)$ in $0 \leqq t \leqq 1$ is given by

$$(3.4) \qquad \int_0^1 dt \int_{-\infty}^{\infty} |z| p_t[u(t), z + u'(t)]\, dz$$

with corresponding modifications for upcrossings and downcrossings.

4. Applications.

(i) *Normal processes.* Let $\xi(t)$ be a separable normal process with $\mathcal{E}\{\xi(t)\} = m(t)$, Cov $\{\xi(t), \xi(s)\} = r(t, s)$. Suppose that $m(t)$ has a continuous derivative $m'(t)$ in $0 \leqq t \leqq 1$ and that $r(t, s)$ has the mixed second partial derivative $r_{11}(t, s)$, continuous in $0 \leqq t, s \leqq 1$. Write $r_{01}(t, s)$ for the first partial derivative with respect to s and suppose that the "non degeneracy" condition $r(t, t) r_{11}(t, t) < r_{01}^2(t, t)$ holds for each t in $0 \leqq t \leqq 1$. It is then a straightforward exercise to show that the conditions of Theorem 2 are valid, and to obtain the formula given by Leadbetter and Cryer [3], viz.,

$$\mathcal{E}\{N(0)\} = \int_0^1 \gamma\sigma^{-1}(1 - \rho^2)^{\frac{1}{2}} \phi(m/\sigma)\{2(\phi(\eta) + \eta(2\Phi(\eta) - 1)\}\, dt,$$

where

$$\sigma^2 = r(t, t), \quad \gamma^2 = r_{11}(t, t), \quad \gamma\rho\sigma = r_{01}(t, t), \qquad \eta = [m' - \gamma\rho m/\sigma]/[\gamma(1 - \rho^2)^{\frac{1}{2}}],$$

ϕ and Φ being the standard normal density and distribution function respectively, and the dependences on t being suppressed. We note that this result holds *without* the assumption made in [3] that $\xi(t)$ possess a *continuous sample* derivative. Of course $\xi(t)$ does possess a quadratic mean derivative by virtue of the existence and continuity of $r_{11}(t, u)$.

(ii) *The envelope of a stationary normal process.* We consider now an important non normal application—that of the "envelope" of a stationary normal process. Let, then, $\xi(t)$ be a stationary normal process with mean zero, covariance function $r(\tau)$ with (for convenience) $r(0) = \text{var } \xi(t) = 1$. Let $F(\lambda)$ denote the spectrum in its real form. That is

$$r(\tau) = \int_0^\infty \cos \tau\lambda \, dF(\lambda).$$

Write, for the spectral representation of $\xi(t)$,

$$\xi(t) = \int_0^\infty \cos \lambda t \, dz_1(\lambda) + \int_0^\infty \sin \lambda t \, dz_2(\lambda),$$

where $z_1(\lambda)$, $z_2(\lambda)$ are independent processes each having independent increments and are such that $\mathcal{E}|dz_1(\lambda)|^2 = \mathcal{E}|dz_2(\lambda)|^2 = dF(\lambda)$. Define the "Hilbert transform" $\hat{\xi}(t)$ of $\xi(t)$ by

$$\hat{\xi}(t) = \int_0^\infty \sin \lambda t \, dz_1(\lambda) - \int_0^\infty \cos \lambda t \, dz_2(\lambda).$$

Then $\mathcal{E}\{\hat{\xi}(t)\hat{\xi}(t+\tau)\} = \mathcal{E}\{\xi(t)\xi(t+\tau)\} = r(\tau)$. Write also $r^*(\tau) = \mathcal{E}\{\xi(t)\hat{\xi}(t+\tau)\}$. Then $r^*(\tau) = \int_0^\infty \sin \tau\lambda \, dF(\lambda)$.

The *envelope* $R(t)$ of $\xi(t)$ may be defined (cf. Middleton [4], Section 2.2) by

$$R(t) = [\xi^2(t) + \hat{\xi}^2(t)]^{\frac{1}{2}}.$$

This definition is formally different but in fact the same as that of Rice [5]. It is motivated by the fact that it does adequately conform to the ordinary notion of an envelope in important special cases, such as that of a modulated sine wave.

Now the random variables $\xi(t)$, $\hat{\xi}(t)$, $\xi(t+\tau)$, $\hat{\xi}(t+\tau)$, have a joint, normal distribution with zero means and covariance matrix (writing $r = r(\tau)$, $r^* = r^*(\tau)$)

$$\begin{bmatrix} 1 & 0 & r & r^* \\ 0 & 1 & -r^* & r \\ r & -r^* & 1 & 0 \\ r^* & r & 0 & 1 \end{bmatrix}.$$

This matrix is easily inverted and if we write $A = 1 - r^2 - r^{*2}$ we have for the joint density of $\xi(t)$, $\hat{\xi}(t)$, $\xi(t+\tau)$, $\hat{\xi}(t+\tau)$,

$$(4\pi^2 A)^{-1} \exp \{-(1/2A)[x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2r(x_1x_3 + x_2x_4) - 2r^*(x_1x_4 - x_2x_3)]\}.$$

By writing $x_1 = R_1 \cos \theta_1$, $x_2 = R_1 \sin \theta_1$, $x_3 = R_2 \cos \theta_2$, $x_4 = R_2 \sin \theta_2$ and integrating over θ_1, θ_2 we find for the joint density $f_{t,t+\tau}(R_1, R_2) = f_\tau(R_1, R_2)$ of $R(t)$, $R(t+\tau)$,

$$f_\tau(R_1, R_2) = (R_1R_2/A) \exp[-(R_1^2 + R_2^2)/2A] I_0\{(R_1R_2/A)(r^2 + r^{*2})^{\frac{1}{2}}\},$$

where I_0 is the Bessel function of order zero with imaginary argument. (This formula has been given by Rice [5].) It thus follows that the joint density for $R(t)$, $S_t(\tau) = [R(t+\tau) - R(t)]/\tau$ is given by $g_{t,\tau}(R, S) = g_\tau(R, S)$ where

$$g_\tau(R, S) = \tau[R(R + \tau S)/A] \exp - (1/A)\{R^2 + \tau RS + \tau^2S^2/2\} \cdot I_0[R(R + \tau S)(r^2 + r^{*2})^{\frac{1}{2}}/A],$$

$R \geqq 0, R + \tau S \geqq 0$ (r, r^* and A being functions of τ). Now if we assume that the first two spectral moments $\lambda_1 = \int_0^\infty \lambda \, dF(\lambda)$, $\lambda_2 = \int_0^\infty \lambda^2 \, dF(\lambda)$ are finite, it follows that $r(\tau)$ has the expansion

$$r(\tau) = 1 - \lambda_2\tau^2/2 + o(\tau^2)$$

and that

$$r^2 + r^{*2} = 1 - \Delta\tau^2 + o(\tau^2),$$
$$A = \Delta\tau^2 + o(\tau^2)$$

as $\tau \to 0$, where $\Delta = \lambda_2 - \lambda_1^2$. Now, from the asymptotic expansion for $I_0(z)$ it follows that

$$I_0(z) \sim e^z/(2\pi z)^{\frac{1}{2}} \quad \text{as} \quad z \to \infty.$$

Using this fact it is easy to show that

$$g_\tau(u + \tau y, S) \to (2\pi\Delta)^{-\frac{1}{2}} \exp(-S^2/2\Delta) u \exp(-u^2/2)$$

as $\tau \to 0$ for any fixed u, y, S. It further follows that there are positive constants K, k such that for all $R > O$, $g_\tau(u + \tau y, S) \leqq K \exp(-kS^2)$. From these facts it is an easy application of dominated convergence to show from (2.2) that the mean number of crossings of the level u by $R(t)$ in $0 \leqq t \leqq 1$ is given by

$$\mathcal{E}\{C(u)\} = (2\pi\Delta)^{-\frac{1}{2}} u \exp(-u^2/2) \int_{-\infty}^{\infty} |S| \exp(-S^2/2\Delta) \, dS$$
$$= (2\Delta/\pi)^{\frac{1}{2}} u \exp(-u^2/2).$$

The fact that tangencies to any level by $R(t)$ have probability zero, and the formulae for the mean number of upcrossings and downcrossings, follow in the same way as for the general case. In fact we have that

$$\mathcal{E}\{C_U(u)\} = \mathcal{E}\{C_D(u)\} = \tfrac{1}{2}\mathcal{E}\{C(u)\} = (\Delta/2\pi)^{\frac{1}{2}} u \exp(-u^2/2).$$

Acknowledgments. I am very grateful to Professor Harald Cramér and Dr. N. Donald Ylvisaker for valuable conversations concerning parts of this work. I would also like to thank Mr. J. D. Cryer for his helpful comments and the referee for his suggestions for improving the organization of the material.

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