SCALE PARAMETER ESTIMATION FROM THE ORDER STATISTICS OF UNEQUAL GAMMA COMPONENTS

By M. B. Wilk, R. Gnanadesikan and Elizabeth Lauh

Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey

1. Introduction. Attention has been given by numerous workers to problems of parameter estimation from order statistics, where the ordered observations are obtained from a random sample from a common distribution (see, for example, the books by Blöm (1958) and Sarhan and Greenberg (1963) and references therein). It appears that there has been no previous consideration of problems of parameter estimation based on ordered observations derived from unequal components—i.e., not all observations come from the same distribution.

Motivation for consideration of such a problem is suggested by the following example. In a general analysis of variance the various mean squares will not necessarily have equal degrees of freedom, and there may not exist any meaningful decomposition into quantities having the same degrees of freedom. On the other hand, it may be useful to conceive of a collection of such mean squares as all being relevant to estimating a common error variance. To overcome the possible bias from the inclusion of "overly large" mean squares, without a prior commitment as to which mean squares may reflect systematic effects, the estimation may be based on the smaller mean squares using an order statistics formulation. Such a point of view for the case of equal components has been discussed in Wilk, Gnanadesikan and Freeny (1963), Wilk, Gnanadesikan and Huyett (1963), and Wilk and Gnanadesikan (1961), (1964a).

Other examples wherein estimation is based on order statistics from unequal components may arise in reliability applications. For instance, on an exponential model of component failure, a system using varying (but known) numbers of components would have a gamma distribution of failure, whose shape parameter reflects the number of components but whose scale parameter does not. Thus several such systems under reliability study would generate ordered failure data corresponding to the present formulation.

The present paper is concerned with the estimation of an unknown common scale parameter based on subsets of order statistics derived from a sample of shape-scaled gamma random variables, with known shape parameters not necessarily all equal.

The method of estimation used here is that of maximizing the appropriate likelihood function. This approach has been applied in other contexts in the references mentioned above as well as in Wilk, Gnanadesikan and Huyett (1962).

In the context of unequal components, there arise conceptual and mathematical problems in connection with an order statistics formulation which can be and have been evaded in the usual equal components case. The conceptual issue relates to what is the appropriate reference set for statistical inference.

Received 2 February 1965.

As Fisher (1956) and others have argued, statistical inferences should be made conditional on distinguishable relevant subsets of possible outcomes. In the particular case of order statistics from unequal components there is a specified relationship of the populations corresponding to the order of the actual observations. To the extent that this information is available it constitutes a relevant consideration in statistical inference. Various types of conditioning may be defined.

Techniques of estimation under group conditioning (defined in Section 2) are described in the present paper. Another type of conditioning (complete conditioning) underlies the development of certain internal comparison procedures for the joint assessment of a collection of analysis of variance mean squares (Wilk and Gnanadesikan (1964b), (1964c)).

Section 2 of the paper gives a statement of the general problem, with notation, and gives examples of various types of conditioning. Sections 3, 4 and 5 deal respectively with estimation from smallest, largest and intermediate observations on shape-scaled gamma random variables, under group conditioning. Illustrative examples are given. Properties of the estimates for these examples are discussed in Section 6. Section 7 gives some concluding discussion. Certain tables to facilitate the estimation procedures described are given in an Appendix.

2. Problem and notation. Consider a sample of independent random variables, $X_1, X_2, \cdots, X_K$, where X_i has the gamma distribution with known shape parameter η_i^* and unknown scale parameter λ, $i = 1, 2, \cdots, K$. The distribution of the shape-scaled random variable $S_i^* = X_i/\eta_i^*$ is,

$$f(s_i^*; \lambda, \eta_i^*) = [(\lambda\eta_i^*)^{\eta_i^*}/\Gamma(\eta_i^*)] \exp(-\lambda\eta_i^* s_i^*) s_i^{*\eta_i^*-1}, \qquad s_i^* > 0, \lambda > 0, \eta_i^* > 0. \tag{1}$$

Let $S_1 \leqq S_2 \leqq \cdots \leqq S_K$ denote the *ordered* shape-scaled quantities and let $\eta_1, \eta_2, \cdots, \eta_K$ denote the shape parameters associated with the *ordered* S_i's.

The general joint density of $S_1, \cdots, S_K$ is given by

$$C\prod_1^K f(s_i; \lambda, \eta_i), \qquad 0 < s_1 \leqq s_2 \leqq \cdots \leqq s_K < \infty, \tag{2}$$

where the constant of proportionality is given by

$$1/C = \int_0^\infty ds_1 \int_{s_1}^\infty ds_2 \cdots \int_{s_{K-1}}^\infty ds_K \prod_1^K f(s_i; \lambda, \eta_i). \tag{3}$$

In the case of equal components, C reduces to $K!$.

The general statistical problem considered herein is to estimate λ, given some subset of $S_1, S_2, \cdots, S_K$, and having partial or complete information about the corresponding $\eta_1, \eta_2, \cdots, \eta_K$. For example, suppose $K = 6$ and that only observations s_2 and s_4 are available on S_2 and S_4, having known shapes η_2 and η_4 respectively. Additionally, suppose it is known that S_1 came from population with shape η_1, S_3 from η_3, while S_5 and S_6 came from some permutation of η_5 and η_6. The problem would then be to estimate the common unknown scale parameter using a formulation which would incorporate the observed quantities

s_2 and s_4 as well as the available information on the order relationships amongst the six populations.

In the case where one is concerned with only a subset of the complete ordered sample, alternate marginal distributions for the subset derive from different conceptions as to the conditioning imposed on the order statistics not in the subset. For example, consider the marginal distribution of S_i. If the only constraint on the basic K-dimensional sample space is that the ith order statistic is associated with the population having shape parameter η_i, then the marginal distribution of S_i is given by

$$(4) \quad f(s_i\,;\lambda,\eta_i)\sum_{(j_1,\cdots,j_{K-1})}\prod_{u=j_1}^{j_{i-1}}\int_0^{s_i} f(s_u\,;\lambda,\eta_u)\,ds_u \cdot\prod_{v=j_i}^{j_{K-1}}\int_{s_i}^{\infty} f(s_v\,;\lambda,\eta_v)\,ds_v,$$

where $(j_1, \cdots, j_{K-1})$ is a permutation of the $(K-1)$ numbers $1, 2, \cdots, i-1, i+1, \cdots, K$, and the summation is over all such permutations.

An alternate conditioning conception is one in which the sample space is constrained so that the ordered observations correspond to the populations with shape parameters $\eta_1, \eta_2, \cdots, \eta_i, \cdots, \eta_K$, respectively, though only the ith ordered value, S_i, is observed.

Then, the marginal distribution of S_i would be

$$(5) \quad f(s_i\,;\lambda,\eta_i)\int_0^{s_i}\int_0^{s_{i-1}}\cdots\int_0^{s_2}\prod_{j=1}^{i-1} f(s_j\,;\lambda,\eta_j)\,ds_j\int_{s_i}^{\infty}\int_{s_{i+1}}^{\infty}\cdots\int_{s_{K-1}}^{\infty}\prod_{j=i+1}^{K} f(s_j\,;\lambda,\eta_j)\,ds_j.$$

A third conditioning conception would be where one observed s_i on the ith ordered S_i having shape η_i and it were known only that $(i-1)$ smaller $\{S_j\}$ came from the collection of shapes $(\eta_1, \eta_2, \cdots, \eta_{i-1})$ while the $(K-i)$ larger $\{S_j\}$ came from the collection of shapes $(\eta_{i+1}, \cdots, \eta_K)$. In this case, referred to here as *group conditioning*, the marginal distribution of S_i is

$$f(s_i\,;\lambda,\eta_i)\prod_{j=1}^{i-1}\int_0^{s_i} f(s_j\,;\lambda,\eta_j)\,ds_j\prod_{j=i+1}^{K}\int_{s_i}^{\infty} f(s_j\,;\lambda,\eta_j).$$

In succeeding sections of the paper, consideration is given to the estimation of λ from contiguous subsets of the order statistics under the general conception of group conditioning. The approach used is maximum likelihood.

3. Estimation from smallest shape-scaled gammas. In the present section, the situation studied is one in which the estimation of λ is to be based on $S_1, S_2, \cdots, S_M$, with shape parameters $\eta_1, \eta_2, \cdots, \eta_M$, respectively, under the conception that the remaining $(K-M)$ populations are constrained only in that the observations from them each exceeds the observed S_M.

Thus, in this circumstance, in which it is given that the observed $S_1, \cdots, S_M$, came from populations having parameters $\eta_1, \cdots, \eta_M$, respectively, while the remaining, unobserved, $S_{M+1}, \cdots, S_K$ might come from any permutation of the populations with parameters $\eta_{M+1}, \cdots, \eta_K$, the likelihood function of λ is proportional to

$$(6) \qquad \lambda^{\eta}\exp\{-\lambda\sum_{j=1}^{M}\eta_j S_j\}\prod_{j=M+1}^{K}\int_{S_M}^{\infty} t^{\eta_j-1}e^{-\lambda\eta_j t}\,dt,$$

where $\eta = \sum_{j=1}^{K}\eta_j$.

The likelihood equation may then be reduced to the following:

$$(7) \qquad 1/\lambda = (1/\eta)[\textstyle\sum_{j=1}^{M} \eta_j S_j + \sum_{j=M+1}^{K} \eta_j E(U_j \mid U_j > S_M)],$$

where U_j is a shape-scaled gamma random variable with density as in Equation (1), having parameter values λ and η_j and where $E(U_j \mid U_j > S_M)$ denotes the conditional expectation of U_j under the constraint that U_j exceeds the observed S_M.

Note that in the case in which $M = K$ this reduces to the usual pooled estimate

$$(8) \qquad 1/\hat{\lambda} = (1/\eta) \textstyle\sum_{j=1}^{K} \eta_j S_j .$$

When $M < K$, the form of the estimate is one in which the unobserved $S_{M+1}, \cdots, S_K$, are replaced in the pooling by their conditional expectations.

For computational purposes, it is convenient to write (7) in the form

$$(9) \qquad \textstyle\sum_{j=1}^{M} \eta_j - \zeta \sum_{j=1}^{M} \eta_j S_j/\eta_M S_M = \sum_{M+1}^{K} H[\eta_j, (\eta_j/\eta_M)\zeta],$$

where

$$(10) \qquad \zeta = \lambda \eta_M S_M ,$$

$$H(a, b) = e^{-b}/J(a, b),$$

and

$$(11) \qquad J(a, b) = \textstyle\int_1^\infty t^{a-1} e^{-bt}, \qquad a > 0,\ b > 0.$$

Tables of the function $H(a, b)$ are given in Table I of the Appendix, tabulated in terms of a and a/b on a grid of values as follows:

$a = 1(1)10(2)20, \quad a/b = .1(.01).2(.02).4(.03).7(.05)1.0(.1)2.0(.2)3.0(.5)4.0,$

$a = 10(10)100, \quad a/b = .5(.01).6(.02)1.0(.05)2.0.$

To obtain values of the $H(a, b)$ function, the computational procedure for $J(a, b)$ was that described in Wilk, Gnanadesikan and Huyett (1962).

To solve Equation (9) for ζ using the tables of $H(a, b)$ one can proceed as follows: In a given problem the values of $\eta_1, \cdots, \eta_M$ and of $S_1, \cdots, S_M$ are known and so, for a series of trial values of ζ, the right hand side of Equation (9) can be evaluated using the tables of $H(a, b)$. Since the left hand side is linear in ζ, plots of the two sides of the equation can then be constructed and the root approximated by their intersection. Computer programs for the iterative solution of this equation have been prepared using a "halving" procedure.

It is useful to note that the root $\hat{\zeta}$ lies in the interval between

$$(12) \qquad B_u\{1 - \textstyle\sum_{j=M+1}^{K} H[\eta_j, (\eta_j/\eta_M)B_u]/\sum_{j=1}^{M} \eta_j\} \text{ and } B_u ,$$

where

$$B_u = \textstyle\sum_{j=1}^{M} \eta_j/[(1/\eta_M S_M) \sum_{j=1}^{M} \eta_j S_j].$$

Given the value of the root $\hat{\zeta}$, the maximum likelihood estimate of λ is then given by $\hat{\lambda} = \hat{\zeta}/\eta_M S_M$.

For the special case $K = 2$, $M = 1$, the estimating equation reduces to,

$$\eta_1 - \zeta = H[\eta_2, (\eta_2/\eta_1)\zeta], \tag{13}$$

which does not explicitly involve the observations. Hence the root of Equation (13) may be tabulated directly. Table II in the Appendix gives roots of Equation (13) for a range of values $\eta_1, \eta_2 = .5(.5)5(1)10(2)20(5)50$.

This special case may be applicable in certain circumstances involving sample sizes greater than 2. For example, if one considered estimating error variance in an analysis of variance from interaction sums of squares, the collection of interaction sums of squares might first be partitioned, according to the order of interaction, into two groups. The estimation might then be based on the smaller pooled mean square using the formulation of the special case above. This procedure has the advantage that it provides partial protection against the inadvertent inclusion of non-central sums of squares in the error variance estimate.

Another special case is when $M = 1$, in which event the estimating equation becomes,

$$\zeta = \eta_1 - \sum_{j=2}^{K} H[\eta_j, (\eta_j/\eta_1)\zeta]. \tag{14}$$

Although Equation (14) is independent of the observations, tabulation of the root is not feasible since in general it would involve K-way tables. From Equation (14) it is apparent that $\hat{\zeta}$ may be interpreted as an adjusted shape parameter; note that $\hat{\zeta}$ is always less than η_1. It is further true that $\hat{\zeta} > \eta_1 - \sum_{j=2}^{K} H(\eta_j, \eta_j)$, which follows from the monotone increasing character of $H(a, b)$ with respect to its second argument.

To illustrate the methods described above, they are applied to two sets of data. The first set of data consists of computer generated sums of squares of standard normal variables, with degrees of freedom corresponding to the usual breakdown

TABLE A

Shape-scaled Gammas $\{S_i\}$	$\{\eta_i\}$ $(\eta_i = \nu_i/2)$
0.28260	1
0.45325	2
0.45840	1
0.58520	4
1.31010	8
1.63995	2
1.68645	2
1.80910	1
1.88635	4
2.25205	2
2.55168	4
2.74430	2
3.00035	4
3.64610	2
3.75980	1

TABLE B

(*Bennett and Franklin* (1954) *pp.* 589–592)

Source	Shape-scaled Gammas $\{S_i\}$	$\{\eta_i\}$ ($\eta_i = \nu_i/2$)
$C \times T$	0.091878×10^6	4.5
Bet. reducing times (T)	0.095803×10^6	1.5
$C \times R$	0.118376×10^6	4.5
Residual	0.176795×10^6	10.5
$T \times R$	0.273896×10^6	4.5
Bet. catalyst conc. (C)	0.450850×10^6	1.5
Bet. reductants (R)	0.640221×10^6	1.5
Bet. oxidants (O)	5.494267×10^6	1.5
Bet. extractant conc. (E)	9.000069×10^6	1.5

in a 3^4 factorial experiment. The error variance σ^2 was therefore 1. The ordered shape-scaled gammas $\{S_i\}$, shown in Table A, are twice the ordered mean squares, and the shape parameters $\{\eta_i\}$ of the $\{S_i\}$ are half the degrees of freedom $\{\nu_i\}$ of the ordered mean squares.

The second set of data is from an example discussed in Bennett and Franklin [(1954), pp. 589–592]. The ordered shape-scaled gammas $\{S_i\}$ and the corresponding $\{\eta_i\}$, for this example, are shown in Table B.

To illustrate the application of the results of this section to the data in Table A, suppose M were taken to be 7 with $K = 15$. Then, $\sum_{j=1}^{7} \eta_j = 20$, $\eta_7 S_7 = 3.3729$ and $\sum_{j=1}^{7} \eta_j S_j/\eta_7 S_7 = 6.2622$. The solution, $\hat{\zeta}$, of Equation (9) turns out to be 1.7313, yielding an estimate for λ of $\hat{\lambda} = \hat{\zeta}/\eta_7 S_7 = 0.5133$. [Note: The true value $\lambda = 1/2\sigma^2 = 0.5$.]

Next, suppose one were to pool the "higher order interaction" sum of squares, i.e. the four sums of squares each with 8 degrees of freedom and the single sum of squares with 16 degrees of freedom, and also pool the remaining sums of squares thereby obtaining two pooled mean squares one with 48 degrees of freedom and the other with 32 degrees of freedom. The two shape-scaled gammas corresponding to these mean squares are $S_1 = 1.77396$ with $\eta_1 = 24$ and $S_2 = 1.94714$ with $\eta_2 = 16$. Using $K = 2$ and $M = 1$, in this version with pooled mean squares, the estimate of λ may be obtained from the solution of Equation (13) as provided in Table II of the Appendix. Corresponding to $\eta_1 = 24$ and $\eta_2 = 16$, from Table II of the Appendix (interpolating between 17.789, for $\eta_1 = 20$, $\eta_2 = 16$, and 22.630, for $\eta_1 = 25$, $\eta_2 = 16$), the value of $\hat{\zeta} = 21.662$. The estimate of λ is $\hat{\lambda} = \hat{\zeta}/\eta_1 S_1 = 0.5088$.

Using the data from Table B above, again for illustrating the procedures described earlier in this section, suppose $M = 3$ with $K = 9$. Then, $\sum_{j=1}^{3} \eta_j = 10.5$, $\eta_3 S_3 = 0.532692 \times 10^6$, and $\sum_{j=1}^{3} \eta_j S_j/\eta_3 S_3 = 2.0459$. The solution, $\hat{\zeta}$, of Equation (9) is 2.9913, leading to an estimate of $\hat{\lambda} = \hat{\zeta}/\eta_3 S_3 = 5.6154 \times 10^{-6}$. This corresponds to an estimate of error variance of $\hat{\sigma}^2 = 1/2\hat{\lambda} = 0.890 \times 10^5$. Bennett and Franklin (1954) in their analysis employ a "residual" mean square with 21 degrees of freedom, to obtain their error variance estimate of $.884 \times 10^5$.

As another possibility, one might consider pooling into two groups, a main-effects group and a group of the remainder. In such a procedure, the shape-scaled gammas turn out to be 3.136242×10^6 with shape parameter 7.5 and $.168126 \times 10^6$, with shape parameter 24 respectively. Using Table II, one then finds the estimate $\hat{\lambda} = 5.467 \times 10^{-6}$, leading to an error variance estimate of $.915 \times 10^5$.

4. Estimation from largest shape-scaled gammas. The attention of the present section is directed toward the estimation of λ based on $S_{M+1}, \cdots, S_K$, the $K - M$ largest shape-scaled gammas, having shape parameters $\eta_{M+1}, \cdots, \eta_K$, respectively, under the random sampling constraint that the observations from the remaining M populations are all less than S_{M+1}.

Thus, given that the observed $S_{M+1}, \cdots, S_K$, came from populations having shape parameters $\eta_{M+1}, \cdots, \eta_K$, respectively, while the remaining unobserved $S_1, \cdots, S_M$, might come from any permutation of the populations with shape parameters $\eta_1, \cdots, \eta_M$, the likelihood of λ is proportional to,

$$\lambda^{\eta} \exp \{-\lambda \textstyle\sum_{j=M+1}^{K} \eta_j S_j\} \prod_1^M \int_0^{S_{M+1}} t^{\eta_j - 1} e^{-\lambda \eta_j t}\, dt, \tag{15}$$

where $\eta = \sum_1^K \eta_j$.

The likelihood equation reduces to,

$$1/\lambda = (1/\eta)[\textstyle\sum_{M+1}^{K} \eta_j S_j + \sum_1^M \eta_j E(U_j \mid U_j < S_{M+1})], \tag{16}$$

where U_j is a shape-scaled gamma random variable with density as in Equation (1), having parameters λ and η_j, and where $E(U_j \mid U_j < S_{M+1})$ denotes the conditional expectation of U_j under the constraint that U_j is less than the observed S_{M+1}.

When $M = 0$ in this formulation, one gets the usual pooled estimate as in Equation (8). When $M > 0$ the form of the estimate is one in which the unobserved $S_1, \cdots, S_M$ are replaced in the pooling by their conditional expectations.

Computationally, it is convenient to rewrite Equation (16) as,

$$\zeta \textstyle\sum_{M+1}^{K} \eta_j S_j / \eta_{M+1} S_{M+1} - \sum_{M+1}^{K} \eta_j = \sum_1^M G(\eta_j, \eta_j \zeta / \eta_{M+1}), \tag{17}$$

where

$$\zeta = \lambda \eta_{M+1} S_{M+1}, \qquad G(a, b) = e^{-b}/D(a, b),$$

and

$$D(a, b) = \int_0^1 t^{a-1} e^{-bt}\, dt, \qquad a > 0, b > 0.$$

Note that

$$H(a, b) = e^{-b}/[\Gamma(a)/b^a - D(a, b)]. \tag{18}$$

Tables of the function $G(a, b)$ are given in Table III in the Appendix, tabulated in terms of a and a/b on the grid :

$$a = 1(1)10(2)20, \qquad a = 5(5)50(10)100,$$

$$a/b = .5(.1)2(.2)4(.3)5.5(.5)7(1)10(2)20(5)40(10)60, 75, 80, 100.$$

To obtain values of the $G(a, b)$ function the computational procedure for $D(a, b)$ was that described in Wilk, Gnanadesikan and Huyett (1962).

The solution of Equation (17) for ζ, given tables of $G(a, b)$, may be carried out as described for the comparable situation in Section 3.

Given the root $\hat{\zeta}$, the maximum likelihood estimate of λ is then given by $\hat{\lambda} = \hat{\zeta}/\eta_{M+1}S_{M+1}$.

The root $\hat{\zeta}$ may be shown to lie between the following bounds :

$$B_l, B_l\{1 + \textstyle\sum_1^M G[\eta_j, (\eta_j/\eta_{M+1})B_l]/\sum_{M+1}^K \eta_j\}, \tag{19}$$

where $B_l = \sum_{M+1}^K \eta_j/[\sum_{M+1}^K \eta_j S_j/\eta_{M+1}S_{M+1}]$.

For the special case where $K = 2$ and $K - M = 1$, the estimating Equation (17) reduces to

$$\zeta - \eta_2 = G[\eta_1, (\eta_1/\eta_2)\zeta]. \tag{20}$$

In this case, the root $\hat{\zeta}$ of the equation may be tabulated as a function of η_1 and η_2 alone, since the equation does not involve the observations. These roots are tabulated in Table IV in the Appendix for a range of values

$$\eta_1, \eta_2 = .5(.5)5(1)10(2)20(5)50.$$

Another special case when $K - M = 1$ but $K > 2$ gives the estimating equation as

$$\zeta = \eta_K + \textstyle\sum_1^{K-1} G[\eta_j, (\eta_j/\eta_K)\zeta]. \tag{21}$$

This equation is analogous to Equation (14) in the previous section. As in that case, $\hat{\zeta}$ may be interpreted as an adjusted shape parameter. However, in this case, $\hat{\zeta}$ will always exceed η_K, whereas in Equation (14) $\hat{\zeta}$ was always less than η_1. These results are quite intuitive. Thus, for this case of estimation based on the single largest value, the effect of the adjustment of the shape parameter is to associate with the largest order statistic an inflated shape parameter so that the largest observation would become a "representative" value from the "adjusted" distribution. The bounds given in Equation (19) for the root $\hat{\zeta}$ become, in this special case of $K - M = 1$, $\eta_K < \hat{\zeta} < \eta_K + \sum_1^{K-1} G(\eta_j, \eta_j)$.

Applying the methods of this section to the data in Table A, taking $K - M = 8$ with $K = 15$, one obtains $\sum_{j=8}^{15} \eta_j = 20$, $\eta_8 S_8 = 1.8091$ and $\sum_{j=8}^{15} \eta_j S_j/\eta_8 S_8 = 29.0793$. Solving Equation (17) yields the solution $\hat{\zeta} = 0.9543$. The estimate for λ is then $\hat{\lambda} = \hat{\zeta}/\eta_8 S_8 = 0.5275$, the true value of λ being 0.5.

Pooling the four sums of squares with 8 degrees of freedom with the sum of squares with 16 degrees of freedom, and pooling the remaining sums of squares, one obtains the two shape-scaled gammas of $S_1 = 1.77396$ with $\eta_1 = 24$, and $S_2 = 1.94714$ with $\eta_2 = 16$. For $K - M = 1$ and $K = 2$, the estimate of λ obtained from solving Equation (20) can be obtained from Table IV of the Appendix. Interpolating in Table IV the value of $\hat{\zeta}$ is 18.121 and the estimate of λ is $\hat{\lambda} = 0.5817$.

From the data in Table B, taking $K - M = 6$ with $K = 9$, it follows that

$$\textstyle\sum_{j=4}^9 \eta_j = 21, \qquad \sum_{j=4}^9 \eta_j S_j/\eta_4 S_4 = 36.46699/1.85635 = 19.64446.$$

Solving Equation (17) then leads to the estimate $\hat{\lambda} = 1.1295 \times 10^{-6}$, corresponding to an error variance estimate of 4.427×10^5. As one would expect, this estimate is biased upwards considerably by the inclusion of possibly non-null sums of squares and hence is much larger than the estimate obtained in Section 3.

5. Estimation from intermediate shape-scaled gammas. There exist situations in the analysis of experiments in which the smallest mean squares may not be responsive to all suspected sources of variation (e.g., with apparent but not "real" replication) and in which the largest mean squares are believed to reflect systematic effects. Then, one may wish to base the estimation of the error variance on a subset of the intermediate valued mean squares.

In the present formulation, this corresponds to a desire to estimate λ, the unknown common scale parameter, from the intermediate L observed shape-scaled gammas $S_{M+1}, \cdots, S_{M+L}$, having associated shape parameters $\eta_{M+1}, \cdots, \eta_{M+L}$, under the restraint that there are M shape-scaled gammas ordered according to some permutation of the populations with parameters $\eta_1, \cdots, \eta_M$, smaller than S_{M+1} and $K - M - L$ shape-scaled gammas with some permutation of shape parameters $\eta_{M+L+1}, \cdots, \eta_K$, larger than S_{M+L}.

The likelihood function for this situation is proportional to

$$(22) \quad \lambda^{\eta} \exp\{-\lambda \textstyle\sum_{M+1}^{M+L} \eta_j S_j\} \prod_1^M \int_0^{S_{M+1}} t^{\eta_j - 1} e^{-\lambda \eta_j t}\, dt \prod_{M+L+1}^{K} \int_{S_{M+L}}^{\infty} t^{\eta_j-1} e^{-\lambda\eta_j t} dt,$$

where $\eta = \sum_1^K \eta_j$.

The likelihood equation may be reduced to

$$(23) \quad 1/\lambda = (1/\eta)[\textstyle\sum_1^M \eta_j E(U_j \mid U_j < S_{M+1}) + \sum_{M+1}^{M+L} \eta_j S_j + \sum_{M+L+1}^{K} \eta_j E(U_j \mid U_j > S_{M+L})].$$

(The notation of Equation (23) has been defined in the two preceding sections.)

Note that this case includes, mathematically as special cases, those considered in Sections 3 and 4 and, as before, when $M = 0$ and $L = K$ one obtains the usual pooled estimate.

An alternate form of the estimating equation is

$$(24)\ \lambda \textstyle\sum_{M+1}^{M+L} \eta_j S_j - \sum_{M+1}^{M+L} \eta_j = \sum_1^M G(\eta_j, \lambda S_{M+1}\eta_j) - \sum_{M+L+1}^{K} H(\eta_j, \lambda S_{M+L}\eta_j).$$

This equation can be solved iteratively for λ using Tables I and III in the Appendix or by means of available, operational computer programs.

For the special case when $L = 1$, i.e., only a single intermediate shape-scaled gamma, S_{M+1}, is observed, the estimating equation simplifies to

$$(25) \quad \zeta = \eta_{M+1} + \textstyle\sum_1^M G[\eta_j, (\eta_j/\eta_{M+1})\zeta] - \sum_{M+2}^{K} H[\eta_j, (\eta_j/\eta_{M+1})\zeta],$$

where $\zeta = \lambda \eta_{M+1} S_{M+1}$.

Applying the methods of this section to the data of Table A, taking $L = 1$, with $M = 7$ and $K - M - L = 7$, the solution of Equation (25) gives $\hat{\lambda} = .4883$, in reasonable concordance with the results of Sections 3 and 4 for this Monte Carlo null example.

For the example of Table B, using $L = 1$, $M = 3$, and $K - M - L = 5$, the solution of Equation (25) is $\hat{\lambda} = 4.991 \times 10^{-6}$. This corresponds to an estimate of error variance of 1.002×10^5. This is in good agreement with the estimate of Section 3, since the possibly non-null largest sums of squares are excluded from *directly* affecting the estimate. It so happens that the present estimate is based directly on the "residual mean square," with a value of 0.884×10^5, which was used by Bennett and Franklin (1954) as their estimate of error variance. Of course, in the present order statistics formulation, this mean square is employed with recognition of the fact that it is the fourth ordered mean square in the analysis of variance involving 9 mean squares. Even though the largest may be non-null their inclusion in the basis for specification of K, but exclusion from the estimate directly, has only a small biasing effect.

6. Some properties of the estimation procedures. It is difficult to obtain analytical results, for small samples, on statistical properties of the estimators developed in the paper. However, some empirical insight may be obtained concerning some features of the methods and such results for the two examples of Table A and Table B are given in this section.

Table A1 gives the results of applying the methods of Section 3 to the data of Table A using a sequence of M values of 1(1)15; i.e., the estimate is based on the M smallest shape-scaled gammas.

It will be seen that the estimates of λ for varying M are reasonably concordant. Also given in Table A1 is the estimated asymptotic variance of the estimates (EAV), namely,

$$\begin{aligned} \text{EAV} &= -[d^2 \log \mathcal{L}/d\lambda^2]_{\hat{\lambda}}^{-1} \\ &= \hat{\lambda}^2\{ \textstyle\sum_1^M \eta_j \\ &\quad - \textstyle\sum_{M+1}^K H(\eta_j, \hat{\lambda}\eta_j S_M)[1 + \hat{\lambda}\eta_j S_M - \eta_j - H(\eta_j, \hat{\lambda}\eta_j S_M)]\}^{-1}. \end{aligned} \tag{26}$$

TABLE A1

Some properties of Section 3 estimation procedures applied to the data of Table A (true value of λ = .5)

M	$\hat{\lambda}$	EAV	
1	.5081	.1845	
2	.6255	.0882	
3	.7499	.0916	
4	.8247	.0586	
5	.5365	.0139	
6	.4933	.0106	
7	.5133	.0105	
8	.5125	.0100	
9	.5345	.0098	
10	.5136	.0084	
11	.5110	.0077	
12	.5175	.0076	
13	.5268	.0074	
14	.5298	.0072	
15	.5425	.0073	(Theoretical Value is .0063)

For the special case of $M = K$, EAV $= \hat{\lambda}^2/\sum_1^K \eta_j$. The values of EAV generally decrease with increasing M, as might be expected. It will be noted that the estimates of λ are all within $\pm 2(\text{EAV})^{\frac{1}{2}}$ of the true value of $\lambda = .5$ and of each other.

Similar results for the data of Table B are given in Table B1. Clearly the

TABLE B1

Some properties of Section 3 estimation procedures applied to the data of Table B (Bennett and Franklin)

M	$\hat{\lambda} \times 10^6$	EAV $\times 10^{12}$
1	4.5861	3.3294
2	5.2757	3.2097
3	5.6154	2.1898
4	5.1540	1.1366
5	4.6151	.8045
6	4.0871	.5992
7	3.7872	.4920
8	1.3705	.0637
9	1.1431	.0415

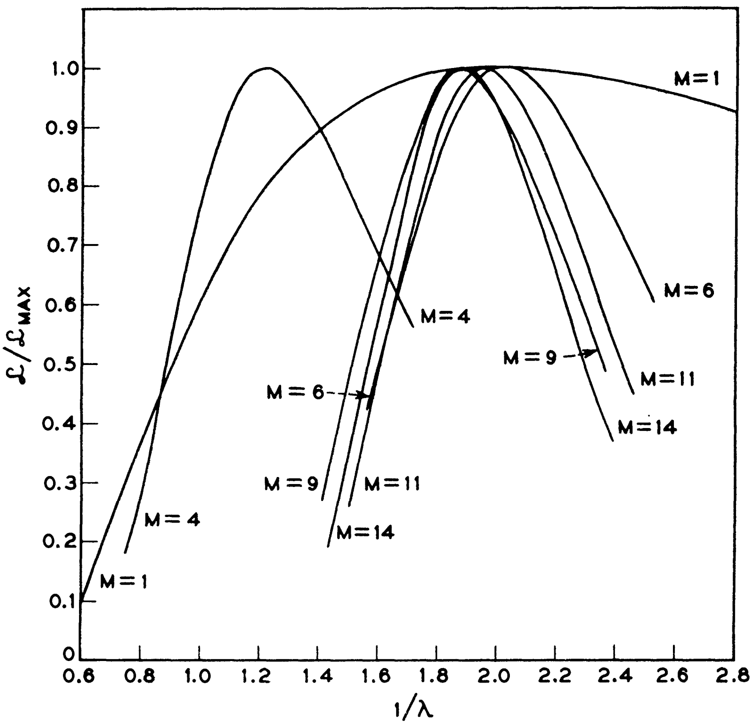

FIG. A1. Ratio of likelihoods for estimation from smallest shape-scaled gammas. (See Tables A and A1.)

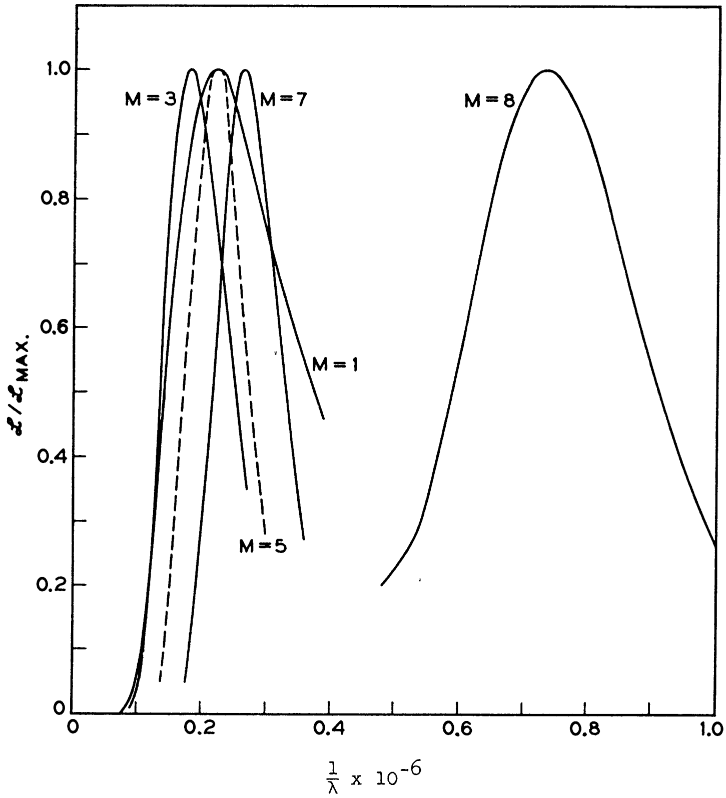

FIG. B1. Ratio of likelihoods for estimation from smallest shaped-scaled gammas. (See Tables B and B1.)

TABLE A2

Some properties of Section 4 estimation procedures applied to the data of Table A (λ = .5)

$K - M$	$\hat{\lambda}$	EAV
14	.5430	.0074
13	.5454	.0074
12	.5452	.0074
11	.5253	.0070
10	.5183	.0071
9	.5255	.0073
8	.5275	.0075
7	.5297	.0077
6	.5196	.0080
5	.5134	.0083
4	.5349	.0103
3	.5429	.0120
2	.5679	.0209
1	.7081	.0613

TABLE B2

Some properties of Section 4 estimation procedures applied to the data of Table B *(Bennett and Franklin)*

$K - M$	$\hat{\lambda} \times 10^6$	EAV $\times 10^{12}$
8	1.1459	.0417
7	1.1441	.0416
6	1.1295	.0406
5	1.0738	.0368
4	0.9678	.0303
3	0.8759	.0253
2	0.2907	.0054
1	0.2358	.0067

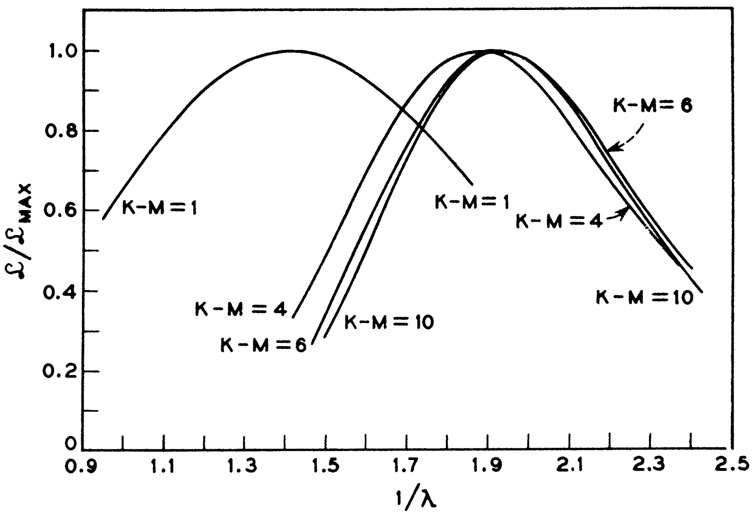

FIG. A2. Ratio of likelihoods for estimation from largest shape-scaled gammas. (See Tables A and A2.)

sequence of estimates $\hat{\lambda}$ are not all measures of the same statistical entity. Indeed as the larger values are included directly in the estimate (i.e., M increases) the value of $\hat{\lambda}$ drops sharply. The estimates are not all within $\pm 2(\text{EAV})^{\frac{1}{2}}$ of one another. This analysis would suggest that the two largest values include non-null variability. However, the conditioning conception employed for these sequences of results would not be appropriate for an internal comparisons approach (see Wilk and Gnanadesikan (1964b), (1964c)).

Further indication of statistical properties may be obtained from the likelihood plots shown in Figure A1 (for Table A example) and Figure B1 (for Table B example). These figures show the ratio of the log likelihood to the maximum log likelihood as a function of $1/\lambda$, for various values of M.

For the null example shown in Figure A1 the likelihood curve becomes

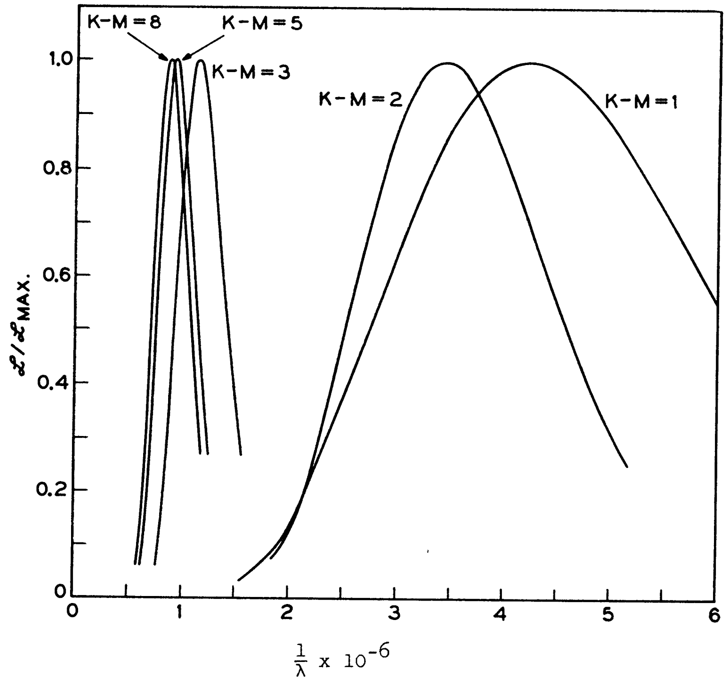

FIG. B2. Ratio of likelihoods for estimation from largest shape-scaled gammas. (See Tables B and B2.)

TABLE A3

Some properties of Section 5 estimation procedures with $L = 1$ applied to the data of Table A ($\lambda = .5$)

M	$K - M - L$	$\hat{\lambda}$	EAV
0	14	.5081	.1845
1	13	.6315	.0894
2	12	.7878	.0981
3	11	.8486	.0615
4	10	.5042	.0128
5	9	.4551	.0097
6	8	.4859	.0104
7	7	.4883	.0101
8	6	.5160	.0103
9	5	.4808	.0089
10	4	.4717	.0083
11	3	.4986	.0100
12	2	.5175	.0114
13	1	.5352	.0177
14	0	.7081	.0613

TABLE B3

Some properties of Section 5 estimation procedures with $L = 1$ applied to the data of Table B (Bennett and Franklin)

M	$K - M - L$	$\lambda \times 10^6$	EAV $\times 10^{12}$
0	8	4.5861	3.3294
1	7	5.5734	3.5065
2	6	5.7858	2.3517
3	5	4.9911	1.1365
4	4	3.8961	0.7157
5	3	2.7482	0.4022
6	2	2.1916	0.2849
7	1	.2991	0.0066
8	0	.2358	0.0067

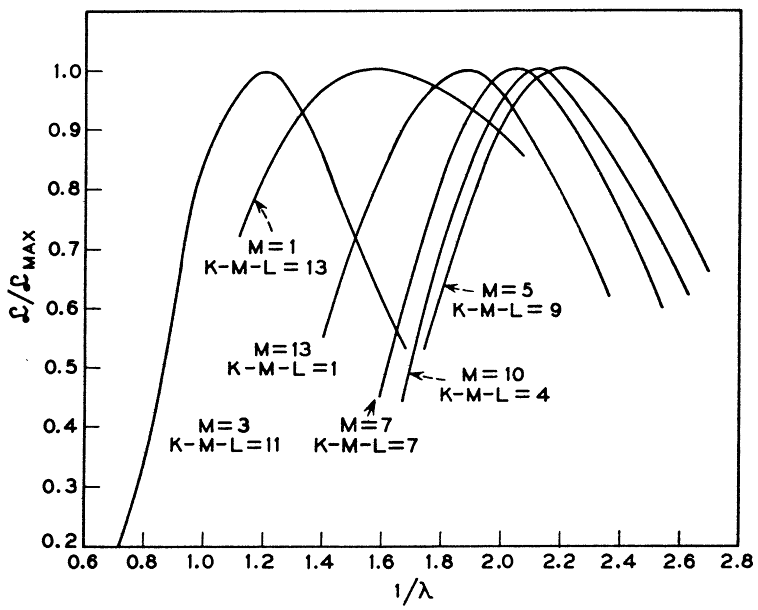

FIG. A3. Ratio of likelihoods for estimation from a single intermediate shape-scaled gamma. (See Tables A and A3.)

"sharper" as M increases, while the position of the maximum does not shift appreciably.

For the Bennett and Franklin (1954) example depicted in Figure B1, the likelihood curve again becomes sharper with increasing M, but the location of the maximum shifts abruptly at $M = 8$.

Similar results for the procedure of Section 4 are given in Table A2 and Figure A2, for the example of Table A, and in Table B2 and Figure B2, for the example of Table B. The EAV's shown in Tables A2 and B2 are calculated from the

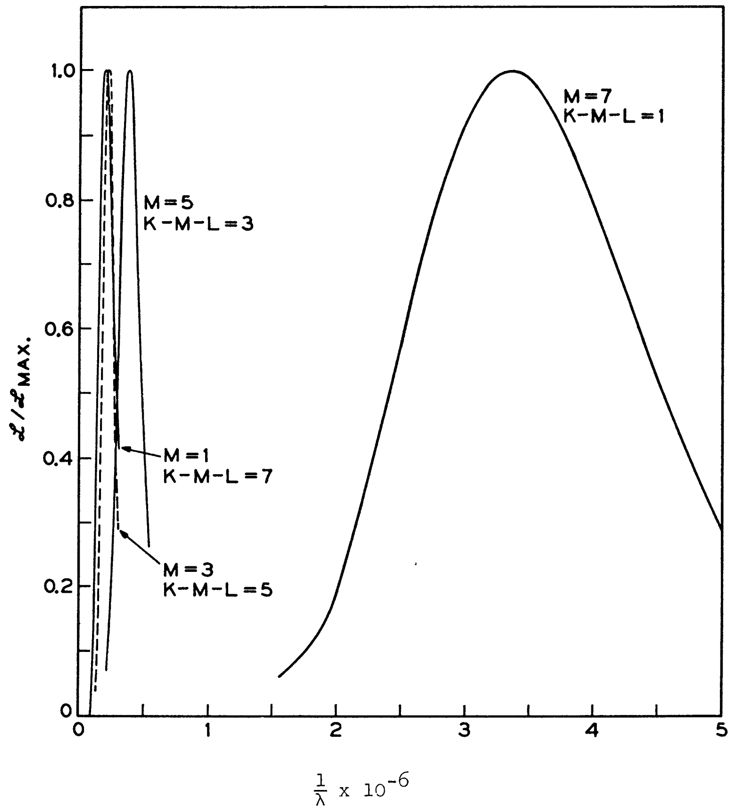

FIG. B3. Ratio of likelihoods for estimation from a single intermediate shape-scaled gamma. (See Tables B and B3.)

expression

$$(27)\quad \text{EAV} = -[d^2 \log \mathcal{L}/d\lambda^2]_{\hat{\lambda}}^{-1} = \hat{\lambda}^2\{\textstyle\sum_{M+1}^{K} \eta_j + \sum_1^M G(\eta_j, \hat{\lambda}\eta_j S_{M+1}) \cdot [1 + \hat{\lambda}\eta_j S_{M+1} - \eta_j + G(\eta_j, \hat{\lambda}\eta_j S_{M+1})]\}^{-1}.$$

For the procedures of Section 5, results in which each of the ordered shape-scaled gammas are used individually, i.e., $L = 1$, are given in Tables A3 and B3 and Figures A3 and B3, for the two examples. The formula for evaluating the EAV's shown in Tables A3 and B3 is

$$(28)\quad \begin{aligned} \text{EAV} = -[d^2 \log \mathcal{L}/d\lambda^2]_{\hat{\lambda}}^{-1} = \hat{\lambda}^2\{&\textstyle\sum_{M+1}^{M+L} \eta_j + \sum_1^M G(\eta_j, \hat{\lambda}\eta_j S_{M+1}) \\ &\cdot [1 + \hat{\lambda}\eta_j S_{M+1} - \eta_j + G(\eta_j, \hat{\lambda}\eta_j S_{M+1})] \\ &- \textstyle\sum_{M+L+1}^{K} H(\eta_j, \hat{\lambda}\eta_j S_{M+L}) \\ &\cdot [1 + \hat{\lambda}\eta_j S_{M+L} - \eta_j - H(\eta_j, \hat{\lambda}\eta_j S_{M+L})]\}^{-1}. \end{aligned}$$

7. Discussion. The new conceptual and theoretical considerations in the present paper derive from the formulation of an estimation problem in terms of order statistics from a set of observations, each of which may come from a different distribution. This formulation raises questions (that do not arise in the equal components case) concerning what is the relevant statistical background for the inferences of interest.

In particular, in the equal components case, the fact that the observations all come from the same distribution implies that marginal joint distributions of the order statistics remain the same whether or not any conditioning is made in the association of the ordered observations with populations. This of course is not true in the case of unequal components. The precise nature of the conditioning to be applied should depend on the specifics of the particular problem. In the view of the authors, it may be profitable to employ various schemes of conditioning for obtaining different insights into the same set of data.

Specifically, the statistical conditioning which is appropriate should be influenced by the objects and purposes of the analysis as well as by the actual information available.

Thus, if one wished to evolve a complete set of nominal estimates of error variance, for comparative purposes, from each mean square in turn in an analysis of variance, then the appropriate view may be that of "complete conditioning." That is, the statistical sampling is envisaged as being restricted so that the order relationship of the mean squares, in respect of the populations with which they are associated, is the same as for the actual set under study. Procedures for generating such a sequence of estimates or, for generalized probability plotting for purposes of internal comparisons, are considered in Wilk and Gnanadesikan (1964b), (1964c).

In Sections 3, 4 and 5 of the present paper, the conception employed has been that of "group conditioning." Thus, for example, in the methods of Section 3, the statistical sampling is envisaged as being constrained by the order relationships of the observed smallest M mean squares while the unobserved $(K-M)$ larger mean squares may be sampled from any permutation of the corresponding populations.

In connection with the estimation of a supposedly common scale parameter from a set of ordered observations on shape-scaled gamma random variables, one may usefully distinguish two possible biasing effects. Consider two approaches: First, to base the estimate on all the observations. Second, to base it on a "conservative" subset of the ordered observations, but treating these as order statistics from a sample whose size is that of the complete set of observations. In the event that the total sample is not homogeneous in regard to the scale parameter, and thus, say, some of the observations may have derived from populations whose scale parameters are smaller (or larger) than the scale parameter of estimation interest, then either of the above approaches will tend to bias the estimate. The first procedure will have a downward (upward) bias due to the direct numerical inclusion of observations which are "too large" (or too small). The second procedure will have a bias from the indirect effect of inappropriately large specification of the total sample size of which one has a subset of order statistics. For example, suppose one treats the bottom M order statistics as having come from a sample of size K when in fact it actually arose from a sample of size $K' < K$. Then, it is intuitively clear that the resulting estimate of the scale parameter λ will be smaller using K than it would be using K' and hence will tend to be biased downward.

From experience, as well as heuristic considerations, it appears reasonable that the biasing effect in the second approach will tend to be much smaller than that in the first. This is exemplified by the results in Table B1. Note that when M is "small," i.e., $\leqq 7$, the value of $\hat{\lambda} \times 10^6$ remains reasonably the same, but the inclusion of the 8th and 9th ordered mean squares causes a sharp drop in the estimate. For comparison, if one bases the estimate on the first four mean squares, but using, in sequence, the values $K = 9, 8, 7$, then the resulting estimates, $\hat{\lambda} \times 10^6$, are 5.154, 5.399, and 5.674.

APPENDIX

TABLE I

Values of $H(a, b) = e^{-b} \Big/ \int_1^{\infty} t^{a-1} e^{-bt}\, dt$

(Enter table with value of a and derived value of a/b. Linear interpolation is accurate to at least 1%.)

a/b\a	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0	16.0	18.0	20.0
0.10	10.000	19.048	28.067	37.077	46.083	55.088	64.091	73.093	82.095	91.097	109.099	127.101	145.102	163.104	181.104
0.11	9.091	17.234	25.346	33.448	41.546	49.642	57.737	65.830	73.924	82.016	98.201	114.385	130.568	146.751	162.934
0.12	8.333	15.723	23.080	30.426	37.767	45.106	52.443	59.780	67.116	74.451	89.121	103.790	118.459	133.127	147.795
0.13	7.692	14.446	21.163	27.870	34.571	41.269	47.966	54.662	61.357	68.052	81.440	94.827	108.213	121.599	134.985
0.14	7.143	13.351	19.522	25.680	31.832	37.982	44.130	50.277	56.423	62.568	74.857	87.146	99.434	111.721	124.008
0.15	6.667	12.403	18.100	23.783	29.460	35.134	40.807	46.477	52.147	57.817	69.154	80.491	91.827	103.162	114.497
0.16	6.250	11.574	16.856	22.124	27.386	32.644	37.900	43.155	48.408	53.661	64.166	74.669	85.172	95.674	106.176
0.17	5.882	10.843	15.760	20.662	25.557	30.448	35.337	40.224	45.111	49.996	59.766	69.535	79.302	89.069	98.836
0.18	5.556	10.194	14.786	19.362	23.932	28.497	33.060	37.621	42.181	46.740	55.857	64.972	74.086	83.200	92.313
0.19	5.263	9.613	13.915	18.201	22.479	26.752	31.023	35.292	39.560	43.827	52.360	60.890	69.420	77.949	86.478
0.20	5.000	9.091	13.132	17.156	21.172	25.183	29.192	33.198	37.203	41.208	49.214	57.219	65.223	73.226	81.228
0.22	4.545	8.190	11.782	15.354	18.918	22.476	26.031	29.584	33.136	36.686	43.785	50.881	57.977	65.071	72.165
0.24	4.167	7.440	10.658	13.855	17.043	20.224	23.402	26.577	29.751	32.923	39.265	45.605	51.944	58.281	64.618
0.26	3.846	6.807	9.710	12.590	15.459	18.322	21.180	24.036	26.891	29.743	35.446	41.146	46.844	52.541	58.237
0.28	3.571	6.266	8.898	11.507	14.104	16.694	19.280	21.863	24.443	27.022	32.177	37.329	42.478	47.627	52.774
0.30	3.333	5.797	8.197	10.571	12.933	15.287	17.636	19.983	22.326	24.668	29.348	34.025	38.699	43.372	48.044
0.32	3.125	5.388	7.584	9.754	11.911	14.059	16.202	18.341	20.478	22.612	26.878	31.139	35.398	39.655	43.911
0.34	2.941	5.028	7.046	9.036	11.011	12.978	14.939	16.896	18.850	20.802	24.702	28.597	32.490	36.380	40.269
0.36	2.778	4.708	6.568	8.399	10.214	12.020	13.819	15.615	17.407	19.197	22.772	26.342	29.910	33.474	37.038
0.38	2.632	4.423	6.142	7.830	9.503	11.165	12.821	14.472	16.120	17.765	21.050	24.330	27.606	30.879	34.151
0.40	2.500	4.167	5.759	7.321	8.865	10.398	11.925	13.447	14.964	16.480	19.504	22.522	25.537	28.549	31.559
0.43	2.326	3.828	5.255	6.648	8.023	9.387	10.743	12.093	13.439	14.783	17.462	20.135	22.804	25.469	28.132
0.46	2.174	3.535	4.818	6.066	7.295	8.512	9.720	10.923	12.120	13.315	15.696	18.069	20.438	22.803	25.165
0.49	2.041	3.278	4.436	5.559	6.660	7.749	8.829	9.902	10.970	12.034	14.154	16.266	18.372	20.474	22.573
0.52	1.923	3.053	4.101	5.113	6.103	7.079	8.046	9.005	9.959	10.909	12.800	14.681	16.556	18.426	20.292
0.55	1.818	2.852	3.804	4.718	5.609	6.486	7.353	8.212	9.065	9.914	11.601	13.278	14.948	16.612	18.273
0.58	1.724	2.673	3.539	4.366	5.171	5.959	6.737	7.507	8.271	9.030	10.536	12.031	13.518	14.999	16.475
0.61	1.639	2.512	3.302	4.052	4.778	5.488	6.187	6.877	7.561	8.239	9.584	10.916	12.239	13.556	14.868
0.64	1.563	2.367	3.089	3.769	4.425	5.065	5.693	6.312	6.924	7.530	8.729	9.915	11.092	12.261	13.424
0.67	1.493	2.236	2.895	3.514	4.107	4.684	5.248	5.802	6.350	6.891	7.960	9.015	10.059	11.095	12.125
0.70	1.429	2.116	2.720	3.282	3.819	4.339	4.845	5.342	5.831	6.314	7.265	8.201	9.126	10.041	10.959
0.75	1.333	1.939	2.462	2.942	3.396	3.832	4.255	4.668	5.072	5.469	6.249	7.011	7.762	8.502	9.234
0.80	1.250	1.786	2.238	2.648	3.032	3.398	3.749	4.090	4.423	4.748	5.382	5.998	6.600	7.191	7.774
0.85	1.176	1.651	2.043	2.394	2.717	3.022	3.314	3.594	3.865	4.129	4.639	5.130	5.607	6.072	6.527
0.90	1.111	1.533	1.873	2.171	2.443	2.696	2.936	3.164	3.383	3.595	4.000	4.386	4.756	5.114	5.462
0.95	1.053	1.427	1.722	1.976	2.203	2.412	2.607	2.791	2.966	3.133	3.449	3.746	4.027	4.295	4.553
1.00	1.000	1.333	1.588	1.803	1.992	2.162	2.319	2.465	2.602	2.732	2.973	3.195	3.401	3.595	3.778
1.10	0.909	1.173	1.362	1.513	1.639	1.749	1.845	1.931	2.009	2.081	2.206	2.314	2.408	2.490	2.563
1.20	0.833	1.042	1.179	1.281	1.360	1.424	1.477	1.520	1.556	1.586	1.632	1.664	1.685	1.697	1.703
1.30	0.769	0.932	1.029	1.093	1.137	1.167	1.187	1.200	1.207	1.210	1.204	1.189	1.166	1.138	1.108
1.40	0.714	0.840	0.905	0.939	0.956	0.962	0.959	0.951	0.939	0.923	0.886	0.843	0.798	0.752	0.707
1.50	0.667	0.762	0.800	0.812	0.809	0.796	0.778	0.756	0.731	0.705	0.650	0.595	0.542	0.491	0.443
1.60	0.625	0.694	0.711	0.705	0.687	0.662	0.633	0.603	0.571	0.539	0.477	0.418	0.365	0.317	0.274
1.70	0.588	0.636	0.636	0.616	0.587	0.553	0.518	0.482	0.447	0.413	0.349	0.293	0.245	0.203	0.168
1.80	0.556	0.585	0.571	0.540	0.503	0.464	0.425	0.386	0.350	0.316	0.256	0.205	0.163	0.129	0.102
1.90	0.526	0.540	0.515	0.476	0.434	0.391	0.350	0.311	0.275	0.243	0.188	0.144	0.109	0.082	0.062
2.00	0.500	0.500	0.466	0.421	0.375	0.330	0.289	0.251	0.217	0.187	0.138	0.101	0.073	0.052	0.037
2.20	0.455	0.433	0.385	0.333	0.283	0.238	0.199	0.165	0.136	0.112	0.075	0.050	0.033	0.021	0.014
2.40	0.417	0.379	0.322	0.266	0.217	0.174	0.139	0.110	0.087	0.068	0.041	0.025	0.015	0.009	0.005
2.60	0.385	0.334	0.272	0.216	0.168	0.129	0.098	0.075	0.056	0.042	0.023	0.013	0.007	0.004	0.002
2.80	0.357	0.298	0.232	0.176	0.132	0.097	0.071	0.051	0.037	0.026	0.013	0.007	0.003	0.002	0.001
3.00	0.333	0.267	0.200	0.146	0.104	0.073	0.051	0.035	0.024	0.017	0.008	0.004	0.002	0.001	0.000
3.50	0.286	0.208	0.142	0.093	0.060	0.038	0.024	0.015	0.009	0.006	0.002	0.001	0.000	0.000	0.000
4.00	0.250	0.167	0.104	0.063	0.037	0.021	0.012	0.007	0.004	0.002	0.001	0.000	0.000	0.000	0.000

TABLE I—*Continued*

a/b\a	10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0	100.0
0.50	11.643	21.780	31.840	41.874	51.896	61.912	71.924	81.933	91.940	101.945
0.51	11.269	21.021	30.692	40.337	49.968	59.592	69.213	78.830	88.446	98.060
0.52	10.909	20.292	29.590	38.860	48.116	57.365	66.609	75.850	85.089	94.327
0.53	10.564	19.592	28.532	37.441	46.336	55.223	64.106	72.985	81.861	90.736
0.54	10.233	18.919	27.514	36.077	44.624	53.164	61.697	70.228	78.756	87.283
0.55	9.914	18.273	26.535	34.764	42.977	51.181	59.380	67.575	75.767	83.957
0.56	9.608	17.650	25.593	33.501	41.391	49.272	57.148	65.018	72.887	80.753
0.57	9.313	17.052	24.686	32.284	39.864	47.433	54.996	62.555	70.111	77.665
0.58	9.030	16.475	23.813	31.112	38.391	45.660	52.922	60.180	67.435	74.686
0.59	8.756	15.919	22.971	29.981	36.971	43.950	50.922	57.888	64.851	71.812
0.60	8.493	15.384	22.159	28.891	35.601	42.300	48.990	55.676	62.358	69.037
0.62	7.994	14.369	20.620	26.823	33.002	39.168	45.325	51.476	57.623	63.767
0.64	7.530	13.424	19.185	24.895	30.577	36.245	41.902	47.553	53.199	58.841
0.66	7.097	12.543	17.847	23.094	28.311	33.512	38.701	43.883	49.059	54.231
0.68	6.692	11.720	16.596	21.410	26.192	30.954	35.704	40.445	45.180	49.910
0.70	6.314	10.950	15.425	19.834	24.206	28.558	32.894	37.221	41.541	45.855
0.72	5.960	10.230	14.330	18.358	22.346	26.311	30.259	34.196	38.125	42.048
0.74	5.628	9.555	13.304	16.974	20.602	24.203	27.786	31.356	34.917	38.471
0.76	5.316	8.923	12.341	15.677	18.965	22.224	25.463	28.688	31.902	35.108
0.78	5.023	8.330	11.439	14.460	17.430	20.367	23.283	26.182	29.069	31.946
0.80	4.748	7.774	10.593	13.319	15.989	18.625	21.235	23.828	26.407	28.975
0.82	4.489	7.251	9.800	12.248	14.638	16.990	19.314	21.619	23.908	26.184
0.84	4.245	6.761	9.056	11.245	13.372	15.458	17.513	19.547	21.563	23.566
0.86	4.016	6.300	8.358	10.305	12.186	14.023	15.827	17.606	19.367	21.112
0.88	3.799	5.868	7.704	9.426	11.077	12.681	14.250	15.792	17.314	18.818
0.90	3.595	5.462	7.092	8.603	10.041	11.428	12.778	14.100	15.398	16.679
0.92	3.402	5.081	6.519	7.835	9.075	10.261	11.409	12.526	13.618	14.690
0.94	3.220	4.723	5.983	7.118	8.175	9.177	10.138	11.066	11.968	12.849
0.96	3.048	4.388	5.483	6.451	7.340	8.172	8.962	9.718	10.447	11.154
0.98	2.886	4.073	5.016	5.831	6.567	7.245	7.879	8.480	9.052	9.601
1.00	2.732	3.778	4.581	5.257	5.853	6.391	6.887	7.348	7.781	8.190
1.05	2.384	3.120	3.622	4.005	4.312	4.566	4.781	4.965	5.124	5.262
1.10	2.081	2.563	2.831	2.993	3.092	3.148	3.174	3.179	3.166	3.141
1.15	1.816	2.095	2.185	2.191	2.153	2.087	2.007	1.917	1.822	1.726
1.20	1.586	1.703	1.666	1.571	1.454	1.329	1.205	1.085	0.973	0.869
1.25	1.385	1.377	1.254	1.103	0.952	0.812	0.687	0.577	0.482	0.401
1.30	1.210	1.108	0.932	0.759	0.606	0.477	0.372	0.289	0.222	0.170
1.35	1.057	0.887	0.685	0.512	0.374	0.270	0.193	0.137	0.096	0.067
1.40	0.923	0.707	0.498	0.339	0.226	0.148	0.096	0.062	0.039	0.025
1.45	0.807	0.561	0.359	0.221	0.133	0.079	0.046	0.027	0.015	0.009
1.50	0.705	0.443	0.256	0.142	0.077	0.041	0.021	0.011	0.006	0.003
1.55	0.616	0.349	0.181	0.090	0.044	0.021	0.010	0.005	0.002	0.001
1.60	0.539	0.274	0.127	0.057	0.024	0.010	0.004	0.002	0.001	0.000
1.65	0.471	0.215	0.089	0.035	0.013	0.005	0.002	0.001	0.000	0.000
1.70	0.413	0.168	0.062	0.022	0.007	0.002	0.001	0.000	0.000	0.000
1.75	0.361	0.131	0.043	0.013	0.004	0.001	0.000	0.000	0.000	0.000
1.80	0.316	0.102	0.030	0.008	0.002	0.001	0.000	0.000	0.000	0.000
1.85	0.277	0.080	0.020	0.005	0.001	0.000	0.000	0.000	0.000	0.000
1.90	0.243	0.062	0.014	0.003	0.001	0.000	0.000	0.000	0.000	0.000
1.95	0.213	0.048	0.010	0.002	0.000	0.000	0.000	0.000	0.000	0.000
2.00	0.187	0.037	0.007	0.001	0.000	0.000	0.000	0.000	0.000	0.000

TABLE II

Roots for estimation from the smaller of two shape-scaled gammas

ETA(1) / ETA(2)	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	6.0	7.0	8.0	9.0	10.0
0.5	0.161	0.167	0.176	0.185	0.193	0.201	0.208	0.214	0.220	0.225	0.234	0.243	0.250	0.256	0.262
1.0	0.525	0.500	0.497	0.500	0.506	0.512	0.519	0.526	0.532	0.538	0.550	0.561	0.571	0.580	0.588
1.5	0.955	0.900	0.880	0.873	0.873	0.875	0.878	0.883	0.888	0.893	0.904	0.915	0.925	0.935	0.944
2.0	1.412	1.333	1.298	1.281	1.272	1.269	1.268	1.270	1.272	1.276	1.284	1.293	1.303	1.312	1.321
2.5	1.883	1.786	1.737	1.710	1.694	1.684	1.680	1.677	1.677	1.678	1.683	1.689	1.697	1.705	1.713
3.0	2.363	2.250	2.190	2.154	2.130	2.116	2.106	2.100	2.096	2.095	2.095	2.099	2.104	2.110	2.118
3.5	2.847	2.722	2.652	2.608	2.579	2.558	2.544	2.534	2.527	2.523	2.519	2.519	2.522	2.527	2.532
4.0	3.335	3.200	3.122	3.071	3.035	3.010	2.991	2.977	2.968	2.960	2.952	2.948	2.948	2.950	2.954
4.5	3.826	3.682	3.596	3.538	3.498	3.468	3.445	3.428	3.415	3.405	3.392	3.385	3.382	3.382	3.384
5.0	4.317	4.167	4.074	4.012	3.966	3.932	3.905	3.885	3.869	3.856	3.839	3.828	3.822	3.820	3.819
6.0	5.305	5.143	5.040	4.968	4.913	4.872	4.838	4.812	4.790	4.773	4.747	4.730	4.718	4.710	4.705
7.0	6.296	6.125	6.014	5.934	5.872	5.824	5.785	5.753	5.726	5.705	5.671	5.647	5.629	5.617	5.608
8.0	7.290	7.111	6.994	6.907	6.839	6.786	6.742	6.705	6.674	6.648	6.607	6.576	6.553	6.536	6.523
9.0	8.284	8.100	7.977	7.884	7.812	7.754	7.705	7.664	7.629	7.600	7.551	7.515	7.488	7.466	7.449
10.0	9.279	9.091	8.963	8.867	8.790	8.726	8.674	8.630	8.591	8.558	8.504	8.463	8.430	8.404	8.383
12.0	11.273	11.076	10.942	10.838	10.754	10.684	10.625	10.574	10.530	10.490	10.426	10.375	10.334	10.300	10.272
14.0	13.268	13.066	12.926	12.817	12.726	12.651	12.588	12.531	12.482	12.439	12.364	12.306	12.256	12.215	12.181
16.0	15.265	15.060	14.913	14.800	14.706	14.626	14.558	14.497	14.442	1[illegible].396	14.315	14.247	14.192	14.146	14.104
18.0	17.261	17.052	16.905	16.786	16.689	16.606	16.533	16.470	16.412	16.362	16.274	16.199	16.138	16.085	16.041
20.0	19.259	19.047	18.895	18.776	18.673	18.588	18.512	18.446	18.385	18.331	18.239	18.160	18.094	18.036	17.987
25.0	24.254	24.037	23.882	23.753	23.650	23.555	23.476	23.402	23.335	23.277	23.174	23.082	23.006	22.939	22.878
30.0	29.251	29.031	28.870	28.742	28.632	28.533	28.449	28.372	28.303	28.237	28.123	28.028	27.940	27.867	27.797
35.0	34.250	34.028	33.866	33.729	33.618	33.520	33.430	33.349	33.276	33.208	33.088	32.986	32.892	32.810	32.738
40.0	39.250	39.026	38.860	38.723	38.606	38.503	38.416	38.333	38.254	38.186	38.059	37.952	37.854	37.766	37.688
45.0	44.245	44.019	43.855	43.717	43.602	43.498	43.404	43.316	43.239	43.168	43.036	42.926	42.822	42.734	42.652
50.0	49.246	49.020	48.849	48.715	48.593	48.489	48.392	48.306	48.227	48.154	48.019	47.903	47.800	47.702	47.617

ETA(1) / ETA(2)	12.0	14.0	16.0	18.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0
0.5	0.272	0.281	0.288	0.295	0.300	0.312	0.322	0.330	0.337	0.343	0.348
1.0	0.603	0.616	0.627	0.637	0.646	0.666	0.681	0.695	0.706	0.716	0.724
1.5	0.961	0.976	0.990	1.002	1.013	1.038	1.058	1.075	1.089	1.102	1.114
2.0	1.338	1.354	1.369	1.382	1.395	1.422	1.446	1.465	1.483	1.498	1.512
2.5	1.730	1.746	1.761	1.775	1.788	1.817	1.843	1.865	1.884	1.901	1.917
3.0	2.133	2.148	2.162	2.177	2.190	2.221	2.247	2.271	2.292	2.311	2.328
3.5	2.544	2.559	2.572	2.586	2.599	2.630	2.658	2.683	2.705	2.725	2.744
4.0	2.965	2.977	2.990	3.002	3.015	3.046	3.074	3.100	3.123	3.144	3.163
4.5	3.391	3.401	3.413	3.425	3.437	3.467	3.495	3.521	3.544	3.566	3.586
5.0	3.824	3.831	3.841	3.852	3.864	3.893	3.920	3.946	3.970	3.993	4.013
6.0	4.703	4.706	4.712	4.720	4.730	4.755	4.782	4.807	4.831	4.853	4.875
7.0	5.599	5.596	5.598	5.603	5.610	5.632	5.655	5.679	5.702	5.726	5.747
8.0	6.507	6.500	6.497	6.499	6.502	6.518	6.540	6.562	6.584	6.606	6.627
9.0	7.426	7.413	7.406	7.403	7.404	7.415	7.433	7.453	7.473	7.495	7.516
10.0	8.354	8.334	8.323	8.317	8.316	8.321	8.334	8.351	8.371	8.391	8.410
12.0	10.230	10.200	10.180	10.167	10.158	10.151	10.156	10.167	10.183	10.199	10.217
14.0	12.128	12.089	12.059	12.039	12.025	12.005	12.001	12.005	12.015	12.029	12.042
16.0	14.040	13.993	13.956	13.929	13.909	13.876	13.862	13.860	13.864	13.874	13.886
18.0	15.966	15.911	15.868	15.835	15.806	15.762	15.740	15.729	15.729	15.734	15.740
20.0	17.904	17.841	17.789	17.750	17.716	17.660	17.628	17.611	17.604	17.604	17.609
25.0	22.777	22.697	22.630	22.578	22.533	22.450	22.395	22.359	22.337	22.325	22.319
30.0	27.684	27.589	27.512	27.446	27.387	27.281	27.208	27.156	27.120	27.094	27.076
35.0	32.610	32.503	32.417	32.340	32.272	32.144	32.054	31.986	31.935	31.896	31.870
40.0	37.551	37.439	37.336	37.253	37.180	37.034	36.921	36.838	36.780	36.731	36.692
45.0	42.503	42.383	42.273	42.185	42.102	41.938	41.811	41.718	41.641	41.581	41.537
50.0	47.464	47.336	47.220	47.122	47.037	46.854	46.719	46.609	46.524	46.457	46.402

TABLE III

Values of $G(a, b) = e^{-b} \Big/ \int_0^1 t^{a-1}e^{-bt}\,dt$

(Enter table with value of a and derived value of a/b. Linear interpolation is accurate to at least 1%.)

a/b\a	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0	16.0	18.0	20.0
0.5	0.313	0.323	0.285	0.239	0.195	0.156	0.123	0.097	0.075	0.058	0.035	0.020	0.012	0.007	0.004
0.6	0.388	0.469	0.481	0.466	0.438	0.406	0.371	0.336	0.303	0.272	0.216	0.170	0.133	0.103	0.079
0.7	0.450	0.602	0.676	0.714	0.729	0.732	0.725	0.712	0.695	0.675	0.629	0.581	0.532	0.484	0.439
0.8	0.502	0.720	0.858	0.955	1.027	1.081	1.123	1.155	1.180	1.198	1.220	1.227	1.225	1.214	1.198
0.9	0.545	0.822	1.021	1.178	1.309	1.422	1.521	1.609	1.688	1.759	1.884	1.990	2.082	2.161	2.230
1.0	0.582	0.911	1.165	1.379	1.568	1.739	1.895	2.041	2.178	2.308	2.549	2.771	2.977	3.171	3.354
1.1	0.613	0.989	1.293	1.559	1.801	2.026	2.238	2.440	2.633	2.820	3.176	3.514	3.838	4.151	4.454
1.2	0.641	1.057	1.406	1.719	2.010	2.284	2.547	2.801	3.047	3.287	3.751	4.199	4.635	5.061	5.479
1.3	0.664	1.117	1.506	1.861	2.196	2.516	2.825	3.126	3.420	3.708	4.271	4.820	5.358	5.888	6.411
1.4	0.685	1.170	1.595	1.988	2.363	2.723	3.074	3.417	3.755	4.087	4.739	5.379	6.010	6.634	7.252
1.5	0.703	1.217	1.674	2.102	2.512	2.910	3.298	3.680	4.056	4.427	5.160	5.882	6.596	7.304	8.007
1.6	0.720	1.260	1.746	2.204	2.646	3.077	3.499	3.916	4.327	4.734	5.539	6.334	7.123	7.906	8.685
1.7	0.735	1.298	1.810	2.297	2.768	3.228	3.681	4.129	4.571	5.011	5.881	6.743	7.598	8.449	9.297
1.8	0.748	1.332	1.868	2.380	2.878	3.366	3.846	4.322	4.793	5.261	6.190	7.112	8.028	8.940	9.849
1.9	0.760	1.364	1.921	2.456	2.978	3.490	3.996	4.497	4.995	5.489	6.472	7.447	8.418	9.385	10.349
2.0	0.771	1.392	1.970	2.526	3.069	3.604	4.133	4.658	5.179	5.697	6.728	7.752	8.773	9.790	10.804
2.2	0.790	1.442	2.055	2.648	3.237	3.804	4.374	4.939	5.501	6.062	7.177	8.287	9.394	10.498	11.60[illegible]
2.4	0.806	1.485	2.127	2.752	3.366	3.974	4.578	5.178	5.775	6.371	7.557	8.740	9.919	11.096	12.272
2.6	0.820	1.522	2.189	2.841	3.484	4.120	4.753	5.382	6.010	6.636	7.883	9.127	10.368	11.608	12.846
2.8	0.832	1.553	2.243	2.919	3.585	4.247	4.905	5.560	6.213	6.865	8.165	9.462	10.757	12.050	13.342
3.0	0.843	1.581	2.291	2.986	3.674	4.358	5.038	5.715	6.391	7.065	8.411	9.754	11.095	12.435	13.774
3.2	0.852	1.606	2.332	3.046	3.753	4.455	5.155	5.852	6.548	7.242	8.628	10.011	11.393	12.774	14.153
3.4	0.860	1.628	2.370	3.100	3.823	4.542	5.259	5.973	6.687	7.399	8.820	10.240	11.657	13.074	14.490
3.6	0.868	1.647	2.403	3.147	3.886	4.620	5.352	6.082	6.811	7.539	8.992	10.443	11.893	13.342	14.790
3.8	0.874	1.665	2.433	3.190	3.942	4.690	5.436	6.180	6.923	7.664	9.146	10.626	12.104	13.582	15.059
4.0	0.880	1.681	2.460	3.229	3.993	4.753	5.511	6.268	7.023	7.778	9.285	10.791	12.295	13.799	15.302
4.3	0.888	1.702	2.496	3.281	4.061	4.837	5.612	6.385	7.158	7.929	9.471	11.010	12.549	14.087	15.625
4.6	0.895	1.721	2.528	3.326	4.120	4.911	5.700	6.488	7.275	8.061	9.632	11.202	12.771	14.339	15.906
4.9	0.901	1.737	2.556	3.366	4.172	4.976	5.778	6.578	7.378	8.177	9.775	11.370	12.965	14.560	16.154
5.2	0.907	1.752	2.580	3.401	4.219	5.033	5.847	6.659	7.470	8.281	9.901	11.520	13.138	14.756	16.373
5.5	0.912	1.765	2.603	3.433	4.26[illegible]	5.085	5.908	6.730	7.552	8.373	10.013	11.653	13.292	14.930	16.569
6.0	0.919	1.784	2.635	3.479	4.320	5.159	5.997	6.834	7.670	8.506	10.177	11.846	13.515	15.184	16.852
6.5	0.925	1.800	2.662	3.518	4.371	5.223	6.073	6.922	7.771	8.619	10.315	12.010	13.704	15.398	17.092
7.0	0.930	1.814	2.686	3.552	4.416	5.277	6.138	6.998	7.858	8.717	10.434	12.151	13.867	15.582	17.298
8.0	0.939	1.837	2.724	3.607	4.487	5.366	6.244	7.122	7.999	8.875	10.628	12.380	14.131	15.88[illegible]	17.633
9.0	0.945	1.855	2.754	3.650	4.543	5.436	6.327	7.218	8.109	8.999	10.779	12.558	14.337	16.116	17.895
10.0	0.951	1.869	2.778	3.684	4.589	5.491	6.394	7.296	8.197	9.098	10.909	12.702	14.503	16.303	18.104
12.0	0.959	1.890	2.815	3.736	4.656	5.575	6.494	7.412	8.330	9.247	11.082	12.917	14.751	16.585	18.419
14.0	0.965	1.906	2.841	3.774	4.705	5.636	6.566	7.495	8.425	9.354	11.213	13.071	14.929	16.786	18.644
16.0	0.969	1.918	2.861	3.80[illegible]	4.742	5.681	6.620	7.558	8.496	9.435	11.311	13.186	15.062	16.938	18.813
18.0	0.972	1.927	2.876	3.824	4.77[illegible]	5.716	6.662	7.607	8.552	9.497	11.387	13.277	15.166	17.055	18.944
20.0	0.975	1.934	2.888	3.841	4.793	5.744	6.695	7.646	8.597	9.547	11.448	13.349	15.249	17.149	19.050
25.0	0.980	1.947	2.911	3.873	4.834	5.795	6.756	7.717	8.677	9.637	11.558	13.479	15.399	17.319	19.239
30.0	0.983	1.956	2.925	3.894	4.862	5.829	6.797	7.764	8.731	9.698	11.632	13.565	15.499	17.432	19.366
35.0	0.986	1.962	2.936	3.909	4.881	5.854	6.825	7.797	8.769	9.741	11.684	13.627	15.570	17.513	19.456
40.0	0.988	1.967	2.944	3.920	4.896	5.872	6.847	7.823	8.798	9.773	11.724	13.674	15.624	17.574	19.524
50.0	0.990	1.973	2.955	3.936	4.917	5.897	6.878	7.858	8.838	9.818	11.779	13.739	15.699	17.659	19.619
60.0	0.992	1.978	2.963	3.947	4.931	5.914	6.898	7.882	8.865	9.849	11.816	13.782	15.749	17.716	19.683
75.0	0.993	1.982	2.970	3.957	4.945	5.932	6.918	7.905	8.892	9.879	11.852	13.826	15.799	17.773	19.746
80.0	0.994	1.983	2.972	3.960	4.948	5.936	6.924	7.911	8.899	9.886	11.862	13.837	15.812	17.787	19.762
100.0	0.995	1.987	2.978	3.968	4.958	5.949	6.939	7.929	8.919	9.909	11.889	13.869	15.849	17.830	19.810

TABLE III—*Continued*

a/b\a	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0	60.0	70.0	80.0	90.0	100.0
0.5	0.195	0.058	0.015	0.004	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.6	0.438	0.272	0.150	0.079	0.040	0.020	0.010	0.005	0.002	0.001	0.000	0.000	0.000	0.000	0.000
0.7	0.729	0.675	0.556	0.439	0.338	0.256	0.192	0.143	0.106	0.078	0.041	0.022	0.011	0.006	0.003
0.8	1.027	1.198	1.227	1.198	1.142	1.072	0.996	0.919	0.844	0.771	0.637	0.521	0.423	0.342	0.274
0.9	1.309	1.759	2.038	2.230	2.370	2.473	2.550	2.607	2.647	2.675	2.702	2.700	2.677	2.640	2.591
1.0	1.568	2.308	2.876	3.354	3.775	4.156	4.507	4.833	5.139	5.428	5.967	6.462	6.923	7.356	7.766
1.1	1.801	2.820	3.678	4.454	5.179	5.867	6.529	7.169	7.792	8.402	9.586	10.736	11.858	12.957	14.039
1.2	2.010	3.287	4.419	5.479	6.495	7.481	8.446	9.393	10.328	11.252	13.074	14.871	16.649	18.412	20.163
1.3	2.196	3.708	5.090	6.411	7.695	8.955	10.197	11.427	12.646	13.857	16.261	18.647	21.019	23.382	25.737
1.4	2.363	4.087	5.696	7.252	8.777	10.282	11.774	13.255	14.728	16.196	19.117	22.025	24.924	27.816	30.702
1.5	2.512	4.427	6.240	8.007	9.748	11.472	13.186	14.891	16.590	18.285	21.663	25.032	28.394	31.751	35.103
1.6	2.646	4.734	6.730	8.685	10.619	12.539	14.450	16.355	18.254	20.150	23.934	27.710	31.480	35.247	39.011
1.7	2.768	5.011	7.171	9.297	11.403	13.498	15.586	17.667	19.745	21.820	25.964	30.101	34.234	38.365	42.493
1.8	2.878	5.261	7.570	9.849	12.110	14.362	16.608	18.849	21.086	23.321	27.786	32.246	36.703	41.157	45.610
1.9	2.978	5.489	7.933	10.349	12.751	15.144	17.532	19.916	22.297	24.676	29.429	34.178	38.925	43.670	48.413
2.0	3.069	5.697	8.263	10.804	13.333	15.854	18.370	20.883	23.394	25.903	30.916	35.927	40.935	45.941	50.946
2.2	3.230	6.062	8.841	11.600	14.349	17.092	19.832	22.569	25.304	28.037	33.502	38.964	44.424	49.883	55.342
2.4	3.366	6.371	9.330	12.272	15.206	18.135	21.061	23.985	26.908	29.830	35.670	41.509	47.347	53.184	59.020
2.6	3.484	6.636	9.748	12.846	15.937	19.024	22.109	25.191	28.273	31.354	37.514	43.672	49.829	55.986	62.142
2.8	3.585	6.865	10.109	13.342	16.568	19.790	23.011	26.230	29.448	32.666	39.099	45.532	51.963	58.394	64.824
3.0	3.674	7.065	10.425	13.774	17.117	20.457	23.796	27.134	30.470	33.806	40.477	47.147	53.816	60.484	67.152
3.2	3.753	7.242	10.703	14.153	17.600	21.043	24.485	27.926	31.367	34.807	41.685	48.563	55.440	62.316	69.193
3.4	3.823	7.399	10.949	14.490	18.027	21.562	25.095	28.628	32.160	35.691	42.753	49.814	56.874	63.935	70.995
3.6	3.886	7.539	11.168	14.790	18.408	22.024	25.638	29.252	32.865	36.478	43.703	50.927	58.151	65.375	72.598
3.8	3.942	7.664	11.365	15.059	18.749	22.438	26.125	29.812	33.498	37.184	44.555	51.925	59.295	66.664	74.033
4.0	3.993	7.778	11.543	15.302	19.057	22.811	26.564	30.316	34.068	37.820	45.322	52.823	60.324	67.825	75.326
4.3	4.061	7.929	11.780	15.625	19.467	23.307	27.147	30.986	34.825	38.663	46.340	54.015	61.691	69.366	77.041
4.6	4.120	8.061	11.986	15.906	19.823	23.739	27.655	31.569	35.484	39.398	47.226	55.053	62.880	70.707	78.533
4.9	4.172	8.177	12.168	16.154	20.137	24.119	28.101	32.082	36.063	40.043	48.004	55.964	63.924	71.884	79.844
5.2	4.219	8.281	12.329	16.373	20.415	24.456	28.496	32.536	36.575	40.614	48.693	56.771	64.848	72.926	81.003
5.5	4.260	8.373	12.473	16.569	20.663	24.756	28.848	32.940	37.032	41.124	49.307	57.490	65.672	73.854	82.037
6.0	4.320	8.506	12.681	16.852	21.021	25.190	29.358	33.525	37.693	41.860	50.195	58.529	66.863	75.196	83.530
6.5	4.371	8.619	12.857	17.092	21.325	25.557	29.789	34.021	38.253	42.484	50.946	59.409	67.871	76.332	84.794
7.0	4.416	8.717	13.009	17.298	21.586	25.873	30.160	34.446	38.733	43.019	51.591	60.163	68.735	77.307	85.878
8.0	4.487	8.875	13.256	17.633	22.010	26.386	30.762	35.138	39.513	43.889	52.639	61.390	70.140	78.891	87.641
9.0	4.543	8.999	13.448	17.895	22.341	26.786	31.231	35.676	40.121	44.566	53.455	62.345	71.234	80.123	89.012
10.0	4.589	9.098	13.602	18.104	22.605	27.106	31.607	36.108	40.608	45.108	54.109	63.109	72.109	81.109	90.110
12.0	4.656	9.247	13.834	18.419	23.003	27.587	32.171	36.755	41.338	45.922	55.089	64.256	73.423	82.590	91.756
14.0	4.705	9.354	14.000	18.644	23.288	27.931	32.574	37.217	41.861	46.504	55.790	65.076	74.361	83.647	92.933
16.0	4.742	9.435	14.124	18.813	23.501	28.189	32.877	37.565	42.252	46.940	56.315	65.691	75.066	84.441	93.816
18.0	4.770	9.497	14.221	18.944	23.667	28.390	33.112	37.835	42.557	47.280	56.724	66.169	75.614	85.058	94.503
20.0	4.793	9.547	14.299	19.050	23.800	28.551	33.301	38.051	42.801	47.551	57.052	66.552	76.052	85.552	95.052
25.0	4.834	9.637	14.439	19.239	24.040	28.840	33.640	38.441	43.241	48.041	57.641	67.241	76.841	86.441	96.041
30.0	4.862	9.698	14.532	19.366	24.200	29.033	33.867	38.700	43.534	48.367	58.034	67.701	77.367	87.034	96.701
35.0	4.881	9.741	14.599	19.456	24.314	29.171	34.029	38.886	43.743	48.600	58.315	68.029	77.743	87.458	97.172
40.0	4.896	9.773	14.649	19.524	24.400	29.275	34.150	39.025	43.900	48.775	58.525	68.275	78.025	87.775	97.525
50.0	4.917	9.818	14.719	19.619	24.520	29.420	34.320	39.220	44.120	49.020	58.820	68.620	78.420	88.220	98.020
60.0	4.931	9.849	14.766	19.683	24.600	29.516	34.433	39.350	44.267	49.183	59.017	68.850	78.683	88.517	98.350
75.0	4.945	9.879	14.813	19.746	24.680	29.613	34.546	39.480	44.413	49.347	59.213	69.080	78.947	88.813	98.680
80.0	4.948	9.886	14.824	19.762	24.700	29.637	34.575	39.512	44.450	49.387	59.262	69.137	79.012	88.888	98.763
100.0	4.958	9.909	14.859	19.810	24.760	29.710	34.660	39.610	44.560	49.510	59.410	69.310	79.210	89.110	99.010

TABLE IV

Roots for estimation from the larger of two shape-scaled gammas

ETA(1) / ETA(2)	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	6.0	7.0	8.0	9.0	10.0
0.5	0.789	1.316	1.827	2.334	2.837	3.340	3.843	4.344	4.845	5.345	6.347	7.348	8.350	9.350	10.350
1.0	0.870	1.445	1.981	2.502	3.015	3.525	4.033	4.538	5.043	5.546	6.552	7.555	8.559	9.561	10.563
1.5	0.896	1.507	2.065	2.601	3.126	3.644	4.157	4.667	5.176	5.684	6.696	7.703	8.711	9.715	10.719
2.0	0.903	1.538	2.115	2.663	3.199	3.725	4.247	4.763	5.275	5.788	6.805	7.818	8.829	9.836	10.843
2.5	0.901	1.553	2.144	2.706	3.250	3.785	4.312	4.834	5.351	5.867	6.891	7.909	8.924	9.935	10.945
3.0	0.897	1.562	2.164	2.735	3.288	3.829	4.362	4.889	5.411	5.930	6.961	7.985	9.003	10.018	11.030
3.5	0.891	1.564	2.175	2.755	3.316	3.863	4.400	4.933	5.459	5.981	7.019	8.047	9.070	10.089	11.105
4.0	0.883	1.564	2.182	2.768	3.335	3.888	4.431	4.968	5.498	6.024	7.068	8.100	9.128	10.150	11.169
4.5	0.877	1.562	2.186	2.778	3.350	3.908	4.455	4.996	5.530	6.059	7.108	8.145	9.177	10.203	11.225
5.0	0.870	1.558	2.187	2.785	3.361	3.923	4.475	5.019	5.556	6.088	7.143	8.186	9.220	10.249	11.277
6.0	0.857	1.549	2.185	2.790	3.374	3.943	4.503	5.052	5.596	6.133	7.198	8.249	9.251	10.327	11.357
7.0	0.845	1.538	2.180	2.789	3.380	3.956	4.519	5.074	5.624	6.167	7.239	8.298	9.347	10.389	11.424
8.0	0.834	1.527	2.173	2.786	3.382	3.962	4.530	5.091	5.643	6.191	7.271	8.337	9.392	10.438	11.478
9.0	0.824	1.517	2.164	2.782	3.380	3.964	4.537	5.100	5.657	6.208	7.294	8.366	9.427	10.479	11.524
10.0	0.815	1.508	2.155	2.776	3.376	3.963	4.540	5.106	5.666	6.220	7.312	8.390	9.455	10.512	11.561
12.0	0.799	1.490	2.138	2.761	3.366	3.958	4.539	5.110	5.674	6.233	7.336	8.423	9.497	10.563	11.620
14.0	0.786	1.473	2.120	2.746	3.353	3.948	4.532	5.107	5.676	6.239	7.348	8.443	9.524	10.597	11.661
16.0	0.774	1.457	2.105	2.730	3.339	3.935	4.523	5.101	5.672	6.238	7.354	8.454	9.542	10.620	11.690
18.0	0.764	1.444	2.091	2.716	3.324	3.923	4.512	5.092	5.665	6.233	7.354	8.459	9.552	10.636	11.710
20.0	0.755	1.432	2.076	2.702	3.311	3.910	4.500	5.082	5.657	6.226	7.352	8.462	9.558	10.645	11.724
25.0	0.737	1.405	2.047	2.668	3.279	3.878	4.471	5.054	5.631	6.205	7.337	8.454	9.559	10.654	11.741
30.0	0.722	1.383	2.020	2.641	3.249	3.849	4.440	5.025	5.606	6.180	7.316	8.437	9.548	10.648	11.741
35.0	0.710	1.365	1.998	2.617	3.223	3.822	4.414	4.998	5.579	6.154	7.292	8.418	9.532	10.637	11.733
40.0	0.700	1.350	1.980	2.595	3.200	3.798	4.389	4.974	5.554	6.130	7.269	8.396	9.513	10.620	11.720
45.0	0.691	1.337	1.962	2.575	3.179	3.776	4.365	4.950	5.529	6.106	7.246	8.374	9.492	10.601	11.704
50.0	0.684	1.325	1.947	2.559	3.161	3.756	4.345	4.928	5.508	6.083	7.224	8.352	9.472	10.583	11.686

ETA(1) / ETA(2)	12.0	14.0	16.0	18.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0
0.5	12.351	14.351	16.352	18.352	20.352	25.352	30.352	35.354	40.354	45.354	50.354
1.0	12.567	14.569	16.570	18.571	20.572	25.575	30.576	35.577	40.577	45.578	50.578
1.5	12.725	14.730	16.734	18.737	20.739	25.743	30.746	35.748	40.749	45.751	50.752
2.0	12.854	14.861	16.868	18.871	20.875	25.882	30.887	35.891	40.893	45.894	50.896
2.5	12.959	14.971	16.979	18.986	20.991	26.001	31.008	36.014	41.018	46.020	51.023
3.0	13.051	15.065	17.076	19.085	21.093	26.107	31.116	36.123	41.127	46.132	51.135
3.5	13.128	15.147	17.161	19.173	21.182	26.200	31.212	36.221	41.227	46.232	51.237
4.0	13.198	15.220	17.237	19.251	21.263	26.284	31.298	36.310	41.318	46.323	51.329
4.5	13.258	15.285	17.306	19.321	21.336	26.360	31.378	36.391	41.401	46.409	51.415
5.0	13.313	15.344	17.367	19.385	21.402	26.431	31.452	36.468	41.478	46.488	51.495
6.0	13.407	15.444	17.473	19.498	21.519	26.556	31.583	36.604	41.619	46.631	51.641
7.0	13.482	15.528	17.563	19.593	21.619	26.665	31.698	36.724	41.743	46.758	51.770
8.0	13.546	15.598	17.640	19.676	21.703	26.759	31.799	36.831	41.853	46.873	51.887
9.0	13.599	15.658	17.706	19.746	21.780	26.844	31.891	36.926	41.953	46.976	51.994
10.0	13.644	15.710	17.763	19.808	21.847	26.920	31.973	37.013	42.045	47.071	52.091
12.0	13.716	15.793	17.858	19.911	21.957	27.048	32.114	37.165	42.205	47.237	52.265
14.0	13.769	15.857	17.930	19.993	22.046	27.153	32.232	37.292	42.341	47.380	52.414
16.0	13.808	15.905	17.988	20.059	22.119	27.241	32.331	37.402	42.459	47.505	52.545
18.0	13.837	15.944	18.034	20.111	22.179	27.314	32.416	37.496	42.561	47.616	52.661
20.0	13.860	15.973	18.071	20.154	22.228	27.377	32.490	37.580	42.652	47.713	52.764
25.0	13.892	16.021	18.134	20.231	22.318	27.495	32.633	37.746	42.836	47.912	52.976
30.0	13.905	16.045	18.169	20.279	22.376	27.578	32.738	37.868	42.976	48.066	53.144
35.0	13.905	16.056	18.188	20.307	22.412	27.636	32.814	37.960	43.083	48.188	53.277
40.0	13.899	16.056	18.196	20.321	22.435	27.675	32.870	38.032	43.167	48.284	53.385
45.0	13.889	16.051	18.197	20.328	22.447	27.703	32.911	38.086	43.234	48.362	53.473
50.0	13.875	16.042	18.193	20.329	22.452	27.721	32.941	38.126	43.287	48.425	53.546

REFERENCES

BENNETT, C. A. and FRANKLIN, N. L. (1954). *Statistical Analysis in Chemistry and the Chemical Industry*. Wiley, New York.

BLÖM, GUNNAR (1958). *Statistical Estimates and Transformed Beta Variables*. Wiley, New York.

FISHER, R. A. (1956). *Statistical Methods and Scientific Inference*. Hafner, New York.

SARHAN, A. E. and GREENBERG, B. (1963). *Contributions to Order Statistics*. Wiley, New York.

WILK, M. B. and GNANADESIKAN, R. (1961). Graphical analysis of multiresponse experimental data using ordered distances. *Proc. Nat. Acad. Sci. U.S.A.* **47** 1209–1212.

WILK, M. B. and GNANADESIKAN, R. (1964a). Graphical methods for internal comparisons in multiresponse experiments. *Ann. Math. Statist.* **35** 613–631.

WILK, M. B. and GNANADESIKAN, R. (1964b). A probability plotting procedure for internal comparisons in a general analysis of variance. Unpublished manuscript. (Invited paper given at Meetings of Royal Statistical Society, Cardiff, Wales, Sept. 1964.)

WILK, M. B. and GNANADESIKAN, R. (1964c). Internal comparison methods in the analysis of variance. Unpublished manuscript. (Invited paper given at Meetings of American Statistical Association, Chicago, Dec. 1964.)

WILK, M. B., GNANADESIKAN, R., and FREENY, ANNE E. (1963). Estimation of error variance from smallest ordered contrasts. *J. Amer. Statist. Assoc.* **58** 152–160.

WILK, M. B., GNANADESIKAN, R. and HUYETT, MARILYN J. (1962). Estimation of parameters of the gamma distribution using order statistics. *Biometrika* **49** 525–545.

WILK, M. B., GNANADESIKAN, R. and HUYETT, MARILYN J. (1963). Separate maximum likelihood estimation of scale or shape parameters of the gamma distribution using order statistics. *Biometrika* **50** 217–221.