ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Central Regional meeting, Lafayette, Indiana, March 23–25, 1966. Additional abstracts appeared in the February issue.)

3. A two-sample estimate of the largest mean. KHURSHEED ALAM, Indiana University.

A two-sample procedure is considered for estimating the largest mean of $K \geqq 2$ normal populations with a common unknown variance. If the total sample size is fixed equal to n, say, we take m observations from each of the K populations in the first experiment and the remaining $n - mK$ observations in the second experiment from the population corresponding to the largest sample mean observed in the first experiment. The sample mean of this population is the estimate of the largest mean. The problem is to determine a proper choice of the value of m representing the distribution of the total sample size between the two experiments. With squared error divided by the common variance as the loss function a minimax value of m is given by $U_2 = .645$ and $U_3 = .68$, approximately where $U_K = m/(n - mK + m)$, the suffix of U representing the number of populations considered. Admissibility of the above estimate and the sampling rule is also discussed.

4. On the large sample properties of a generalized Wilcoxon-Mann-Whitney statistic (preliminary report). A. P. BASU, University of Minnesota.

Let there be two samples $X_1, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ $(N = m + n)$ from two populations with continuous cdf's $F(x)$ and $G(y)$. Let the first i ordered observations (out of the N combined observations) contain m_i x's and n_i y's $(m_i + n_i = i)$ where m_i and n_i are random numbers. Then to test the H_0: $F = G$ against the alternative that they are different, Sobel [Technical Report No. 69, University of Minnesota, (1965)] has proposed the statistic $V_r^{m,n} = \sum_{i=1}^{r} (nm_i - mn_i)$ based on the first r ordered observations only. In this paper the large sample properties of $V_r^{m,n}$ have been studied. The statistic is shown to be asymptotically normally distributed in the null and the non-null case, and is consistent. An expression for its efficacy is derived. Finally the test is compared with other tests proposed for life testing. K sample extensions of the problem are also considered.

5. Distribution-free independence tests. C. B. BELL and K. A. DOKSUM, Université de Paris. (By title)

Let G be the direct product with itself of the power group of the group of 1-1 strictly increasing continuous transformations of R_1 onto R_1; S, the direct product with itself of the permutation group of n elements; and $(x_1, y_1), \cdots, (x_n, y_n)$ be written $z = (x_1, \cdots, x_n; y_1, \cdots, y_n)$. The $(n!)^2$ images $S(z)$ for a.e. z in R_{2n} constitute the z-orbit; and v is a B-Pitman function if it assumes $(n!)^2$ distinct values on a.e. orbit. $T(v(z)) = \sum \epsilon \{v(z) - v(\gamma(z))\}$, for summation over S, is a B-Pitman statistic; and a statistic V is DF (SDF) if its distribution is invariant over H_0 (over the G-equivalence classes of $H_0 \cup H_1$). (A) (i) Each DF statistic has a discrete distribution with its probabilities multiples of $(n!)^{-2}$. (ii) There exists a DF statistic with any preassigned discrete distribution whose probabilities are multiples of $(n!)^{-2}$. (iii) V is DF (SDF) iff V is equivalent to a function of some B-Pitman statistic (non-sequential rank statistic). (B) For a "mildly regular" parametric alternative family $h(\theta, z)$ (i) the MP-DF statistic is given by Lehmann-Stein (1949), (ii) the locally MP-DF statistic is a B-Pitman statistic with $v = (\partial^r/\partial\theta^r)h \mid (\theta = 0)$ where

r is the smallest integer for which v is not invariant wrt S. (iii) for nonparametric Bhuchongkul (1963) alternatives, the normal scores test asymptotically maximizes minimum power.

6. Imbedded Markov chains in queueing systems $M/G/1$ and $GI/M/1$ with limited waiting room (preliminary report). U. N. Bhat, Michigan State University.

Let Q_n be the number of customers in the queueing system $M/G/1$ (Poisson arrivals, arbitrary service times and single server) with a limited waiting room capacity c, soon after the nth departure; $\{Q_n\}$ is a Markov chain satisfying the recurrence relation $Q_n = \min\{c, \max(X_n, Q_{n-1} + X_n - 1)\}$, where X_n is the number of arrivals during the nth service time. The basic process of this chain is $i + S_n - n$ for which we get $F_{ij}^{(n)} = \Pr\{i + S_r - r \leqq c\ (1 \leqq r < n)\ i + S_n - n = j\} = k_{n+j-i}^{(n)} - \sum_{m=1}^{n-1} (c+1-j)(n-m)^{-1} k_{n-m-c-1+j}^{(n-m)} k_{m+c+1-i}^{(m)}$, $(0 \leqq i, j \leqq c)$, where $S_n = \sum_1^n X_i$ and $\Pr\{S_n = j\} = k_j^{(n)}$. By combinational arguments the generating function ${}^c\phi_{ij}(z)$ of the transition probability ${}^cP_{ij}^{(n)} = \Pr\{Q_n = j, Q_r \leqq c\ (1 \leqq r < n) \mid Q_0 = i\}$ $i, j \leqq c$, can be derived as ${}^c\phi_{ij}(z) = f_{ij}(z) + f_{i0}(z)[f_{1j}(z) - f_{0j}(z)] \cdot [1 + f_{00}(z) - F_{10}(z)]^{-1}$, where $f_{ij}(z)$ is the probability generating function of $F_{ij}^{(n)}$. The general transition probabilities of Q_n follow from the fact that the state of overflow of the waiting room is a recurrent event. A similar study of the queueing system $GI/M/1$ (arbitrary inter-arrival times, exponential service times and single server) depends on the basic process $i + n - S_n$.

7. Some convergence theorems for linear combinations of independent random variables. Y. S. Chow, Purdue University.

Let a_n, a_{nk} be real numbers and x_n be independent random variables such that $E \exp[tx_n] \leqq \exp[t^2]$ for $-\infty < t < \infty$, $n = 1, 2, \cdots$. Theorem 1. *If* $\sum_{m=1}^{\infty} a_{nm}^2 = o(\log^{-1} n)$ *and* $T_n = \sum_{m=1}^{\infty} a_{nm} x_n$, then $T_n \to 0$ a.e. Theorem 2. *If* $\sum_{n=1}^{\infty} a_n^2 \log^{1+\delta} n < \infty$ *for some* $\delta > 0$ *and* $f(t) = \sum_{n=1}^{\infty} a_n x_n \cos nt$, *then there exists another stochastic process* $g(t)$ *such that* $f(t) = g(t)$ a.e. *for each* t *and* $g(t)$ *is* a.e. *sample continuous.* Theorem 3. *Let* $b_n = a_n - a_{n-1} \geqq 0$ *for* $n \geqq 2$, $b_1 = a_1 > 0$, $a_n \to \infty$ *and* $b_n = O(a_n/\log\log a_n)$. *Then* $\sum_1^n b_m x_m / a_n \to 0$ a.e. Corollary. *Let* y_n *be independent,* $Ey_n = 0$ *and* $Ey_n^2 = 1$. *If* $\sum_{m=1}^{n} a_{nm}^2 = 1$, *then* $n^{\frac{1}{2}} \sum_{m=1}^{n} a_{nm} y_n \to 0$ a.e. Theorem 1 extends a result of Hill, *Pacific J. Math.* **1** (1951). Theorems 2 and 3 are generalizations of results due to Salem and Zygmund, *Acta Math.* **91** (1954). The corollary was conjectured by Leon Gleser.

8. Non-parametric empirical Bayes procedures for selecting the best of k populations. John J. Deeley, Sandia Corporation.

Let $\pi_1, \pi_2, \cdots, \pi_k$ be k populations and corresponding to each population let X_i be an observable random variable with density $f(x \mid \omega_i)$ which is either continuous or discrete; ω_i being a parameter belonging to some set of real numbers Ω. We define the *best* population to be that population whose corresponding parameter is largest. We assume the existence of an unknown G_i, a distribution on Ω $(i = 1, 2, \cdots, k)$; and the availability of prior observations (x_{1i}, ω_{1i}), (x_{2i}, ω_{2i}), $\cdots$, (x_{ni}, ω_{ni}) on X_i, not ω_i, for each $i = 1, 2, \cdots, k$. If (i) $E[X_i \mid \omega_i] = \int_X x f(x \mid \omega_i)\, dx = \omega_i$, and (ii) $\int_X x^2 f(x \mid \omega_i)\, dx \leqq C_1 + C_2 \omega_i^r$ for some integer $r \geqq 2$, then procedures, t_n (based upon the n prior observations), are derived for selecting the best population. t_n is called empirical Bayes because its risk approaches that of the Bayes procedure; i.e. (iii) $\lim_{n\to\infty} R(t_n, G) = R(t_G, G)$ where t_G is the Bayes procedure with respect to G (t_G is unknown since G is unknown) under the linear loss function; and G is a member of the family $C = \{\prod_{i=1}^{k} G_i : G_i$ a distribution on Ω with $\int_\Omega \omega^r\, dG_i(\omega) < \infty\}$.

In this paper it is shown that the condition (iii) holds for the derived procedures for $G \varepsilon C$ by using a theorem of Robbins (*Ann. Math. Statist.* **35** (1964) 1–20) and conditions (i), (ii). The results are termed non-parametric in the sense that only assumptions about the mean of the random variable and about the relationship of the 2nd moment of f to the rth moment of G were made.

9. Estimation of the number of trials in the binomial case when several observations are available (preliminary report). DORIAN FELDMAN and MARTIN FOX, Michigan State University.

Let $X_1, \cdots, X_n$ be independent binomial random variables with known probabilities of success for each sequence of trials. Assume each sequence of trials is of the same but unknown length r. Estimates of r and their asymptotic properties are discussed. Motivated by the above for large r, let $X_1, \cdots, X_n$ be independent $N(\mu, \mu)$ random variables. It is shown that the minimum variance unbiased estimate of μ is $(Z/n)^{\frac{1}{2}} I_{n/2}(nZ)^{\frac{1}{2}} / I_{(n-2)/2}(nZ)^{\frac{1}{2}}$ where $Z = \sum_1^n X_i^2$ and I_ν is the modified Bessel function of first kind of order ν. The variance of this estimate is the Cramér-Rao lower bound, $2\mu^2/(4n\mu + 3n)$.

10. Maximum-likelihood estimation, from singly censored samples, of the location parameter of an extreme-value distribution with known scale parameter. H. LEON HARTER and ALBERT H. MOORE, Aerospace Laboratories, Wright-Patterson AFB and Air Force Institute of Technology, Wright-Patterson AFB.

If the random variable T has the two-parameter Weibull distribution with cumulative distribution function $F(t; \theta, K) = 1 - \exp[(-t/\theta)^K]$, where θ is the scale parameter and K is the shape parameter, then the random variable $X = \ln T$ has the (first) extreme-value distribution of smallest values with cumulative distribution function $F(x; u, b) = 1 - \exp\{-\exp[(x - u)/b]\}$, where $u = \ln \theta$ is the location parameter (mode) and $b = 1/K$ is the scale parameter. This well-known fact makes it possible to obtain the maximum-likelihood estimator $\hat{u}_{mn} \mid b$ of u, based on the first m order statistics of a sample of size n, when b is known, by a simple transformation of the corresponding estimator of θ when K is known. Use is made of the fact that $\hat{u}_{mn} \mid b = \ln \hat{\theta}_{mn} \mid K$, where $2m\ (\hat{\theta}_{mn} \mid K)^K/\theta^K$ has the chi-square distribution with $2m$ degrees of freedom, to set confidence bounds on u. The probability density function of $\hat{u}_{mm} \mid b$, which for given m is the same for any $n \geqq m$, is obtained by a simple transformation of that of $\hat{\theta}_{mn} \mid K$. Integration yields expressions, involving digamma and trigamma functions, for the expected value $E = E(\hat{u}_{mn} \mid b - u)$ and the variance $V(\hat{u}_{mn} \mid b)$. Values of $E/b(6DP)$ and of $V/b^2(6DP)$ are tabulated for $m = 1(1)100$. By subtracting the bias $E(\hat{u}_{mn} \mid b - u)$ from $\hat{u}_{mn} \mid b$, one obtains an unbiased estimator $\tilde{u}_{mn} \mid b$ which has the same variance as the maximum-likelihood estimator.

11. Fiducial theory and invariant prediction. R. B. HORA and R. J. BUEHLER, Purdue University and University of Minnesota. (Invited)

The prediction of future observations z given past observations y is considered where the joint distribution of (y, z) depends on an unknown parameter ω, and the spaces $\{(y, z)\}$ and $\{\omega\}$ are simultaneously transformed by a group $G = \{g\}$ essentially as in Fraser's fiducial theory. A function $\psi(z)$ is called "invariantly predictable" if $\psi(z_1) = \psi(z_2)$ implies $\psi(gz_1) = \psi(gz_2)$. For such functions, prediction limits based on the marginal fiducial distribution of ψ are shown to possess a frequency interpretation. Best predictors of ψ from an invariant decision-theoretic viewpoint are related to fiducial theory by means of the expectation identity described in an earlier abstract (*Ann. Math. Statist.* **35** (1964) 455).

12. Bayes risk of asymptotically Bayes sequential tests. GARY LORDEN, Northwestern University.

Schwarz (*Ann. Math. Statist.* **33** 222–236) considered the following procedure for sequential testing in a Bayesian setting: stop when the *a posteriori* risk is less than c, the cost per observation (and choose a decision with minimum *a posteriori* risk). Kiefer and Sacks (*Ann. Math. Statist.* **34** 705–750) proved under mild assumptions that the integrated risk of this procedure is asymptotically the same as the Bayes risk, as c approaches zero. The main result, under a slight modification of the assumptions of Kiefer and Sacks, is that this integrated risk exceeds the Bayes risk, which is $O(c|\log c|)$, by less than Mc. Computation of a sufficient M is possible in many common problems and is studied in detail for the general problem with finite parameter space. Moreover, the Schwarz procedure (which has smaller integrated risk of error than a Bayes procedure) has integrated ASN at most M larger than that of a Bayes procedure, for all c. $M = 11$ suffices in the problem of three normal means discussed on pages 710–713 of the paper by Kiever and Sacks. The key lemma states that Bayes procedures continue sampling whenever the *a posteriori* risk exceeds Mc.

13. Estimation of the population median. D. M. MAHAMUNULU and R. H. RODINE, State University of New York at Buffalo.

In this note we discuss the properties of the sample median as an estimator of the population median. We define the sample median, as usual, by: $\tilde{Y} = Y_m$ or $\tilde{Y} = 2^{-1}(Y_m + Y_{m+1})$ according as the sample size n is odd ($n = 2m - 1$) or even ($n = 2m$), $m = 1, 2, \cdots$. It is shown that $\tilde{Y}$ is a median unbiased estimator of the population median for odd sample sizes, and for even sample sizes $\tilde{Y}$ is median unbiased for symmetric populations, but it is not so in general. A class of pdf's for which the sample median is median unbiased and unbiased is given, and it is shown that for symmetric pdf's, $\tilde{Y}$ is not a minimum variance linear unbiased estimator of the population median. For symmetric populations, a class of unbiased and median unbiased estimators for the population median is given.

14. Point and interval estimation, from one order statistic, of the location parameter of an extreme-value distribution with known scale parameter. ALBERT H. MOORE and H. LEON HARTER, Air Force Institute of Technology, Wright-Patterson AFB and Aerospace Research Laboratories, Wright-Patterson AFB.

This paper derives a one-order statistic estimator $\tilde{u}_{mn} \mid b$, for the location parameter of the (first) extreme-value distribution of smallest values with cumulative distribution function $F(x; u, b) = 1 - \exp\{-\exp[(x - u)/b]\}$ using the minimum-variance unbiased one-order statistic estimator for the scale parameter of an exponential distribution. It is shown that exact confidence bounds, based on one-order statistic, can be derived for the location parameter of the extreme-value distribution, using exact confidence bounds for the scale parameter of an exponential distribution. The estimator for u is shown to be $b \ln C_{mn} + x_{mn}$ where x_{mn} is the mth order statistic from an ordered sample of size n from the extreme value distribution with scale parameter b and C_{mn} is the coefficient for a one-order statistic estimator of the scale parameter of an exponential distribution. Values of the factor C_{mn}, which has previously been tabulated for $n = 1(1)20$, are given for $n = 21(1)40$. The ratio of mean square error of this one-order statistic estimator, $\tilde{u}_{mn} \mid b$, to that of the maximum-likelihood m-order statistic estimator, $\hat{a}_{mn} \mid b$, is investigated by Monte Carlo methods. A section on the use of the table to estimate the location parameter of an extreme-value distribution with known scale parameter and a numerical example are included.

15. On selecting a subset containing the best of several discrete populations. K. NAGEL, Purdue University. (Introduced by S. Gupta.)

From k populations a random variable x_i, $i = 1, \cdots, k$, is observed, which can attain the integer values $0, \cdots, n$: $P(x_i = j) = a_j(t_i)$ when the distributions are assumed to be TP_2 with respect to t_i. The objective is to select a subset which contains the populations with the biggest parameter value t_i by the procedure R: Select Π_i if $x_i \geqq \max_j x_j - d$, $d \geqq 0$. A selection that contains the maximal Π_i is called a correct selection. For applications it is important to know the minimum of the probability of correct selection. Because of the TP_2-property the minimum is obtained for $t_1 = t_2 = \cdots = t_k = t$. In this case the probability of correct selection is of the form $P(CS \mid R) = \sum_{i=0}^{n} a_i(t)[\sum_{j=0}^{i+d} a_j(t)]^{k-1}$. In this paper the minimum of this expression is evaluated for arbitrary distributions, which gives an estimation for the minimum of any special discrete distribution, e.g. the binomial or power series distribution. It should be pointed out that the infimum of the probability of a correct selection for the case $a_i(t) = \binom{n}{i} t^i (1 - t)^{n-i}$ was evaluated in a paper by Gupta and Sobel (1960) [*Contributions to Probability and Statistics*, Chapter 20].

16. On certain structural properties of the logarithmic series distribution and the first type Stirling distribution. G. P. PATIL and J. K. WANI, Pennsylvania State University and McGill University.

The logarithmic series distribution and its n-fold convolution which we call the first type Stirling distribution have found applications in a variety of diverse fields. In this paper, we study properties of their probability and distribution functions, moments, cumulants and their inter-relations. We provide explicit and operational recurrence relations for moments and cumulants. Lastly, we obtain a few characterizations of these two distributions.

17. Compound distributions of infinitely divisible distributions with exponential family and with exponential type family. S. A. PATIL, Colorado State University.

The moment generating function of the compound distribution of infinitely divisible distribution with exponential family and the first moment of the distribution is obtained. The moment generating function of the exponential type family is expressed in terms of the mean function of the distribution. This is used to determine the compound distribution. It is shown that if an analytic infinitely divisible distribution belongs to a distribution of exponential type family then $dO_r(u) = A(u)e^{ru}\,d\mu(u)$, where $\log M(t) = tk_1(r) + \int_{-\infty}^{\infty} (e^{tu} - 1 - tu)\,dO_r(u)/u^2$. Here, $M(t)$ is the moment generating function of the infinitely divisible distribution and the above equation is its Kolmogorov representation. Further the mean function of the distribution $k_1(r)$ can be expressed as the Laplace transform of $A(u)/u$. This determines the moment generating function of the exponential type family, and from this moment generating function of the compound distribution can be determined. Some illustrative examples are discussed.

18. Estimation of the location of the cusp of a continuous density (preliminary report). B. L. S. PRAKASA RAO, Michigan State University.

Under the usual regularity conditions on the density, it is well known that the maximum likelihood estimator is consistent, asymptotically normal and asymptotically efficient.

Unfortunately, these conditions are not satisfied for distributions like double-exponential with location parameter θ. Daniels, in his paper in the Fourth Berkeley Symposium, has shown that there exist modified maximum likelihood estimators which are asymptotically efficient for the family of densities $f(x, \theta) = \text{Const.} \{-|x - \theta|^k\}$, where x and θ range over $(-\infty, \infty)$ and $\frac{1}{2} < k < 1$. We show in this paper that hyper-efficient estimators exist for θ when $0 < k < \frac{1}{2}$ and θ is restricted to a finite interval for a wider class of densities. We relate its asymptotic distribution to the distribution of the position of the maximum for a non-stationary Gaussian process. The estimation problem is reduced to that of a stochastic process and using theorems on convergence of distributions of stochastic processes in $C[0, 1]$, the asymptotic distribution is obtained. In fact, it can be shown that Bayes estimators for θ, for smooth prior densities, are also hyper-efficient and asymtotically the Bayes estimation of θ is equivalent to the estimation of the location parameter for a non-stationary Gaussian process.

19. The "secretary" problem. Herman Rubin, Michigan State University.

Let a_i, $i = 1, \cdots$, be a non-decreasing non-constant sequence of real numbers. Let individuals be selected at random from a set of n, with the agreement that, while any two can be compared, there is no other way of assessing the rank of an individual. Individuals are observed sequentially, and may be either accepted or rejected. When an individual is selected, the loss is a_i if the individual is ith in rank of the entire n. By looking at the randomization instead as the selection of the arrangement of the individuals, the position of the first, second, etc. being randomly selected in turn, the problem extends to $n = \infty$. We show (1) that the optimal procedure for n gives, for each $m < n$, a procedure whose risk is no greater; (2) the risk for $n = \infty$ is the limit of the risks for finite n; and (3) that the optimal procedure for $n = \infty$ can be obtained by solving the differential equation arising as the limit of the difference equations for the optimal solution for finite n.

20. Some inequalities among binomial and Poisson probabilities. S. M. Samuels and T. W. Anderson, Purdue University and Columbia University.

The binomial cumulative distribution function $B(k; n, \lambda/n) = \sum_{j=0}^{k} \binom{n}{j} (\lambda/n)^j \cdot (1 - \lambda/n)^{n-j}$, which is well known to converge to the Poisson cumulative distribution function $P(k; \lambda) = \sum_{j=0}^{k} e^{-\lambda}\lambda^j/j!$, is shown to be decreasing in n for $k \geqq \lambda$ and increasing in n for $k \leqq \lambda - \lambda/(n + 1)$. This, and other similar results, shows that using the Poisson as an approximation to the binomial in certain statistical problems is conservative.

21. Monotonicity of rank order likelihood ratios. K. M. Lal Saxena, University of Nebraska.

Suppose that $X_1, \cdots, X_m$; $Y_1, \cdots, Y_n$ are mutually independent random variables. The X_i's and the Y_i's have the continuous distribution functions $F(x, \theta)$ and $F(x, \eta)$ respectively. Define the vector $z = (z_1, \cdots, z_N)$ as follows: $z_i = O(1)$ if the ith smallest in the combined sample is an $X(Y)$ and $N = m + n$. Denote by $P_{\theta,\eta}(z)$ the probability of the rank order z. Define zRz' as follows: $z_i = z_i'$ for all $i = 1, \cdots, N$ except j and k $(j < k)$, $z_j = z_k' = 0$, $z_k = z_j' = 1$. If the interchanged components of z are at an end, then write zR^*z'. Whenever zRz', the monotonicity of the rank order likelihood ratio $P_{\theta,\eta}(z)/P_{\theta,\eta}(z')$, is exhibited in θ and η for (a) a Lehmann family, (b) a uniform family and (c) one observation from a population having monotone likelihood ratio any number of observations from another population. Whenever zR^*z', the monotonicity of the rank order likelihood ratio, is obtained for families satisfying a condition possessed by a normal and a logistic family.

With $m = n = 2$, the monotonicity of the rank order likelihood ratio is exhibited for a normal, a logistic and a double exponential family. Monotonicity of the rank order likelihood ratio is obtained in a neighborhood of $\theta = \eta$ when the populations sampled have monotone likelihood ratio. It is conjectured that the rank order likelihood ratio is monotone in both the arguments when the sampled populations have monotone likelihood ratio.

22. Some nonparametric Bayesian decision problems. K. M. LAL SAXENA, University of Nebraska. (By title)

Two sample nonparametric decision problems for single parameter families of distributions are considered, from the Bayesian viewpoint, when only the relative magnitudes of the observations are known, for the paired comparison data, the rank order data and the signed rank order data. The decision procedures are called nonparametric since they depend on nonparametric statistics. The likelihood functions for the three kinds of data are assumed to depend on the parameters of the sampled populations through H. A prior distribution is considered for the random variable H. The loss function for the Bayes estimation of H is the squared error. Some analytic properties of the posterior distribution of H and the orderings of the values of the Bayes estimate are obtained. Sufficient conditions are given for the risk of the Bayes estimate to go to zero as the sample sizes go to infinity. The Bayes two decision problem, the two decisions being $H > h$ and $H \leqq h$, with the $(0, 1)$ loss function, is considered for the paired comparison and the rank order data. Monotonic properties of the Bayes decision procedures are obtained. Sufficient conditions are given for the risk of the Bayes decision procedures to go to zero as the sample sizes go to infinity.

23. On the asymptotic theory of permutation tests. PRANAB KUMAR SEN, University of North Carolina, Chapel Hill. (Invited)

The object of the present expository paper is to present a systematic account of the various aspects of the asymptotic theory of permutation tests. This includes a review of the existing permutational central limit theorems by Wald-Wolfowitz-Noether-Hoeffding-Dwass-Motoo and Hajek, together with some recent developments on this line by the present author. The study of the asymptotic power properties of the permutation tests is also made, and some further generalizations of the results of Hoeffding and Chernoff and Savage are also considered here.

24. On the theory of rank-order tests and estimates for location in the multivariate one sample problem. PRANAB KUMAR SEN and MADAN L. PURI, University of North Carolina, Chapel Hill and Courant Institute of Mathematical Sciences, New York University.

In the multivariate one-sample location problem, the theory of permutation distribution under sign-invariant transformations is extended to a class of rank-order statistics, and is utilized in the formulation of a genuinely distribution free class of rank-order tests for location. Asymptotic properties of these permutation rank-order tests are studied and certain stochastic equivalence relations with a similar class of multivariate extensions of one-sample Chernoff-Savage-Hajek type tests are derived. The power properties of these tests are studied. Finally, Hodges-Lehmann technique (*Ann. Math. Statist.* **34** 598–611) of estimating shift parameters through rank-order tests is extended to the multivariate case which includes among other results, the results obtained by Bickel (*Ann. Math. Statist.* **35** 1079–1090 and *Ann. Math. Statist.* **36** 160–173) as special cases.

25. Use of truncated estimator of variance ratio in recovery of inter-block information. K. R. SHAH, Michigan State University and Cornell University.

A preliminary step in the recovery of inter-block information is to estimate the ratio of inter- to intra-block variances. Since the true value of this ratio exceeds unity, it is usually recommended that its estimate should be truncated from below at unity. However, some authors have used untruncated estimators of this ratio. In this paper it is shown that for a class of estimators of the variance ratio, truncation leads to smaller variance for the combined estimators of treatment differences. A table presented here serves to demonstrate that much of the gain due to recovery of inter-block information could be lost by using an untruncated estimator of variance ratio.

26. Computing the value of a stochastic sequence (preliminary report). DAVID O. SIEGMUND, Purdue University.

Let $(x_n, \mathfrak{F}_n)_{1 \leqq n}$ be a stochastic sequence such that $E|x_n| < \infty$ and let $\gamma_n = \text{ess}_{t \varepsilon C_n} \sup E(x_t \mid \mathfrak{F}_n)$, where C_n is the class of all stopping rules t such that $P(t \geqq n) = 1$ and Ex_t exists. It is of considerable interest in the theory of optimal stopping rules to have a constructive method for computing the sequence γ_n, $1 \leqq n$. Previous investigations have largely been directed at finding conditions under which considering the sequence $(x_n, \mathfrak{F}_n)_{1 \leqq n \leqq N}$, applying backwards induction, and then letting $N \rightarrow \infty$ yield the desired result [e.g., Chow, Y. S. and Robbins, H. E., *Zeit. Wahrscheinlichkeitstheorie* **2** (1963)]. This paper presents completely general methods for computing the sequence γ_n, $1 \leqq n$. The crucial point of the argument is proper application of a martingale convergence theorem much in the fashion of Snell [*Trans. Amer. Math. Soc.* (1952)]. Applications of the above results to the theory of optimal stopping rules are mentioned.

27. Selecting the t populations with the largest α-percentiles. MILTON SOBEL, University of Minnesota.

Given k populations, a nonparametric solution to this problem is developed by finding the required number of k-tuples (one component from each population) in order to have the probability of a correct selection $P\{CS\}$ at least P^* when the "distance" between any cdf $F_{[k-i+1]}$ with the ith largest α-percentile ($i \leqq t$) and any cdf with the jth largest ($j > t$) is at least d^*. The distance between $F_{[i]}$ and $F_{[j]}$ is the Min $[F_{[j]}(x) - F_{[i]}(x)]$ for $x_1 \leqq x \leqq x_2$, where $F_{[k-t+1]}(x_1) = \alpha - \epsilon^* < \alpha + \epsilon^* = F_{[k-t+1]}(x_2)$. Here ϵ^*, d^*, P^* are all preassigned and the cdf's are all assumed to be continuous. The same P^*-condition on the $P\{CS\}$ is considered with a second distance measure, namely that the minimum of the $\alpha - \epsilon^*$ percentiles of $F_{[j]}$ ($j \geqq k - t + 1$) is not less than the maximum of the $\alpha + \epsilon^*$ percentiles of $F_{[i]}$ ($i \leqq k - t$). For each of these two formulations, the effect of assuming that for all x, $F_{[j]}(x) \leqq F_{[k-t]}(x)$ and $F_{[i]} \geqq F_{[k-t+1]}(x)$ ($i \leqq k - t, j \geqq k - t + 1$) is studied. For each case a table of n-values is given for $k = 2, t = 1, \alpha = \frac{1}{2}$ and $\epsilon^* = d^*$. The asymptotic ($n \rightarrow \infty$) relative efficiency of these procedures for $\alpha = \frac{1}{2}$ is compared to other standard procedures under different alternatives.

28. On a new property of PBIB designs useful in an application of MANOVA in psychometrics. J. N. SRIVASTAVA and R. L. MAIK, University of Nebraska.

In a paper to be published in *Psychometrika* (1966), Srivastava has considered the problem of testing the hypothesis that the $(p \times p)$ population dispersion matrix (under the usual normal set-up) is a linear function of known matrices Σ_i, $i = 0, 1, \cdots, l$. A complete solution (based on the likelihood ratio test) given there, can be used under conditions slightly more general than the following: (1) There exists an orthogonal matrix P, such that $D_j = P'\Sigma_j P$ is diagonal, for all j. (2) $\Sigma_0 = I, \Sigma_1, \cdots, \Sigma_l$ generate a linear associative algebra. (3) The matrix $\Lambda(\Lambda'\Lambda)^{-1}\Lambda'$ has all elements non-negative, where the jth column of $\Lambda(p \times \overline{l+1})$ has in order, the diagonal elements of D_j. An important special case arises when, for example, in a psychological experiment, the battery of tests has the structure of a PB association scheme. Then the Σ's are the association matrices of the scheme, and are known to satisfy (1) and (2). In this paper, it is shown that they also satisfy (3), so that the above likelihood ratio test can be used.

29. Bayes risk consistency of classification procedures using density estimation. J. VAN RYZIN, Argonne National Laboratory.

Let X be a random variable on $(\mathfrak{X}, \mathfrak{F}, \mu)$ with density $f_i(x)$ under $P_i \ll \mu$; $i = 0, 1$. With ξ as the *a priori* probability that X is distributed as P_1 and with unit cost of misclassifying X, let $\phi(\xi)$ denote the risk of the Bayes procedure which classifies $X = x$ as P_1 if $\xi f_1(x) > (1 - \xi)f_0(x)$ and as P_0 otherwise. Estimating the densities $f_i(x)$ based on random samples $S_i = \{X_{i1}, \cdots, X_{in_i}\}$ from P_i, $i = 0, 1$, classification procedures are given. With $R_{n_0,n_1}(\xi)$ denoting the risk of such a procedure, conditions under which $\alpha(n_0, n_1)\{R_{n_0,n_1}(\xi) - \phi(\xi)\} \to 0$ as both $n_0, n_1 \to \infty$ are examined. Many cases are considered using the results of Parzen [*Ann. Math. Statist.* **33** 1065–1076], Cacoullos [Tech. Report No. 40, Dept. of Statistics, Univ. of Minn.], and Čencov [*Soviet Math.* **3** 1559–1562]. In all cases the above convergence is satisfied with $\alpha(n_0, n_1) \equiv 1$ under very general conditions, while under certain more restrictive (but still very general) conditions higher order results with $\alpha(n_0, n_1) \uparrow \infty$ as $n_0, n_1 \to \infty$ are given in all the above cases for suitable sequences $\{\alpha(n_0, n_1)\}$.

30. Sequential sampling for confidence. K. T. WALLENIUS, Yale University.

A lot of size N is to be sequentially inspected for defective items. Sampling terminates when the inspector can state, with confidence $1 - \alpha$, that the uninspected portion of the lot contains fewer than D^* defective items. (If inspection has been rectifying, the above quality statement would pertain to the entire lot.) Consider an inspection plan characterized by the stopping points $(0, n_0), \cdots, (d, n_d), \cdots, (N - D^* + 1, n_{N-D^*+1})$ where $n_d \leqq n_{d+1}$; the rule being to continue inspection until a stopping point is reached. If $P_D(d)$ denotes the probability of stopping at (d, n_d) when the lot initially has D defectives, then the above requirement is equivalent to the system of inequalities $\sum_{i=0}^{d} P_{D^*+d}(i) \leqq \alpha$ for $d = 0, 1, \cdots N - D^*$. Some obvious relationships and the following fundamental LEMMA. $P_k(i) = P_i(i)h(N, n_i, k; i)/h(N, n_i, i; i)$ *where* $h(N, n, D; x)$ *is a hypergeometric probability, are the tools necessary to recursively calculate the stopping boundary. The boundary obtained is optimal in the sense that, given* $n_0, n_1, \cdots, n_{d-1}$, *we require* n_d *to be the smallest stopping number consistent with the confidence condition.* Formulas for expected sample size and operating characteristics are given and results are compared with the asymptotic solution given by Wurtele [*J. Roy. Statist. Soc. Ser. B* **17** (1955) 124–127].

31. The cumulants of s, $1/s$, **and** t. JOHN S. WHITE, General Motors Research Laboratories.

Let $x_1, \cdots, x_n$ be NID(0, 1). It is well known that the sample standard deviation s is distributed as χ with $m = n - 1$ degrees of freedom. The raw moments of s are $E(s^k) = (2/m)^{k/2}\,\Gamma(m/2 + k/2)/\Gamma(m/2)$. The central moments and cumulants of s and $1/s$ may be computed from the raw moments in the usual way. Unfortunately, the round-off errors involved may be so large as to make the computed results useless. In this paper, expansions of $E(s^k)$ are obtained from Stirling's formula and are used to obtain expansions for the cumulants of s, $1/s$, and $t = n^{\frac{1}{2}}\bar{x}/s$. Expansions for the mixed cumulants of $1/s$ and t are also given. These results may be used to obtain the cumulants of the non-central t, defined as $t(\epsilon, m) = (n^{\frac{1}{2}}\bar{x} + \epsilon)/s$. The distribution function and percentile points of the non-central t may then be found by substituting these cumulants in the appropriate Edgeworth or Cornish-Fisher series.

32. Efficiency curves for a class of layer tests of bivariate independence (preliminary report). George G. Woodworth, University of Minnesota.

Let $\mathbf{X}_1, \cdots, \mathbf{X}_n$, $\mathbf{X}_i = (X_i, Y_i)$, be a sample from a continuous bivariate population, ordered according to the size of the X-component; i.e., $X_1 \leqq X_2 \leqq \cdots \leqq X_n$. Let l_j be the rank of Y_j among $Y_1, \cdots, Y_j$. Let $\{c_n(u, v)\}$, $n \geqq 1$, be a sequence of functions on the unit square which converge to a function $c(u, v)$ in $2 + \delta$-mean for some $\delta > 0$. We define test statistics of the form: $T_n = n^{-\frac{1}{2}}\sum_{j=1}^{n} c_n(l_j/j + 1, j/n + 1)$, and we consider tests of bivariate independence of the form: $T_{nc} \geqq k_n$, k_n being a constant. In this paper we derive efficiency curves of the type proposed by Klotz. [*Ann. Math. Statist.* **33** (1964) 1099–1114]. Let $\{H_\theta ; \theta \geqq 0\}$ be a family of alternative distributions, where H_θ has independent marginals when $\theta = 0$. Using Klotz's notation, our results in parametric form are: $e_c(\theta) = -\int \ln(\int \exp (h\cdot c(u, v))\, du)\, dv + h\cdot x$, where h is the solution of $\int\int c(H_\theta(x, y)/F_\theta(x), F_\theta(x))\, dH_\theta(x, y) = x = \int\{\int c(u, v) \exp (h\cdot c(u, v))\, du/\int \exp (h\cdot c(u, v))\, du\}\, dv$, where F_θ is the marginal of x. Curves have been computed for various choices of c, one of which gives a test equivalent to Kendall's tau, and for various families $\{H_\theta\}$, among them the bivariate normal.

33. A sequential estimation of the mean of a log-normal distribution, which guarantees a prescribed closeness. Shelemyahu Zacks, Kansas State University.

Let $X_1, X_2, \cdots$ be a sequence of independent random variables, identically distributed like $\exp\{N(\mu, \sigma^2)\}$, $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$. The expected value of X is $\xi = \exp\{\mu + \sigma^2/2\}$ and its variance is $\tau^2 = \xi^2(e^{\sigma^2} - 1)$. An estimator $\phi(X_1, X_2, \cdots)$ of ξ is said to have a closeness level γ and closeness rate δ if $P[|\phi(X_1, X_2, \cdots) - \xi| \leqq \delta\xi] \geqq \gamma$ for all ξ. The problem is to find a sampling procedure and an estimator which will satisfy prescribed closeness level γ and closeness rate δ. If σ^2 is known, a fixed sample of size $n \geqq \chi_\gamma^2[1]\, \sigma^2/\log^2(1 + \delta)$, and the estimator $\hat{\xi}(\bar{Y}_n, \sigma^2) = \exp\{\bar{Y}_n + \sigma^2/2\}$, where $Y_i = \log X_i$ $(i = 1, \cdots, n)$ and $\bar{Y}_n = \sum_{i=1}^{n} Y_i/n$, have the required property. If σ^2 is unknown one cannot attain the required property by a fixed sample procedure. The following sequential estimation procedure is proposed: Given $X_1, \cdots, X_k$ $(k = 1, 2, \cdots)$ compute $v_k = \sum_{i=1}^{n} (Y_i - \bar{Y}_k)^2/k$. Stop after taking K observations, where K = smallest integer $k \geqq F_\gamma[1, k]v_k(1 + v_k/2)/\log^2(1 + \delta)$. Estimate ξ by the estimator $\hat{\xi}(\bar{Y}_K, v_K) = \exp\{\bar{Y}_K + \frac{1}{2}v_K\}$. The reason for using this procedure is that, for a given $K = n$, $\exp\{\bar{Y}_n + \frac{1}{2}v_n\}$ is the maximum likelihood estimator of ξ whose asymptotic distribution, as $n \to \infty$, is $N(\xi, (\xi^2\sigma^2/n)(1 + \sigma^2/2))$. It is proven that this sequential procedure has the asymptotic properties: (1) $\lim_{\delta\to 0} [\log^2(1 + \delta)K/\chi_\gamma^2[1]\sigma^2(1 + \sigma^2/2)] = 1$ a.s., for all $0 < \sigma^2 < \infty$, (2) $\lim_{\delta\to 0} P[|\hat{\xi}(\bar{Y}_K, v_K) - \xi| \leqq \delta\xi] \geqq \dot{\gamma}$, for all $0 < \xi < \infty$ and (3) $\lim_{\delta\to 0} [\log^2(1 + \delta)E\{K\}/\chi_\gamma^2[1]\sigma^2(1 + \sigma^2/2)] = 1$, for all $0 < \sigma^2 < \infty$.

(Abstracts of papers to be presented at the Eastern Regional meeting, Upton, Long Island, New York, April 27–29, 1966. Additional papers will appear in the June issue.)

1. Subminimax estimation of the mean of a normal random variable. A. J. BARANCHIK, Columbia University.

For a sequence of independent normal (known variance) mean estimation problems an easy to compute empirical Bayes sequence of estimators is obtained. The sequence is asymptotically optimal versus normal priors, and also possesses the Bayes minimax property, i.e. for any prior measure, the Bayes risk of any estimator in the sequence is no more than the minimax value of the problem.

2. Estimation associated with linear discriminants. SEYMOUR GEISSER, State University of New York at Buffalo.

Suppose we have two p-variate normal populations where Π_1 is $N(\mu_1, \Sigma)$ and Π_2 is $N(\mu_2, \Sigma)$ and sample estimates $\bar{x}_1$ of μ_1, $\bar{x}_2$ of μ_2 based on n_1 and n_2 observations respectively. Further we assume we have an independent estimate S of Σ based on ν degrees of freedom. Let the population and sample (or index) linear discriminants be $U = [z - \frac{1}{2}(\mu_1 + \mu_2)]'\Sigma^{-1}(\mu_1 - \mu_2)$, $V = [z - \frac{1}{2}(\bar{x}_1 + \bar{x}_2)]'S^{-1}(\bar{x}_1 - \bar{x}_2)$, respectively, where the new observation z has prior probability q_1 of being from Π_1 and q_2 from Π_2. For $r = q_2/q_1$, $U > \log r$ assigns z to Π_1, $U < \log r$ assigns z to Π_2 and $V > \log r$ assigns z to Π_1, $V < \log r$ assigns z to Π_2. The following problems are attacked from the Bayesian point of view: (1) The posterior distribution of U and the estimator of U for fixed z. (2) Estimation of ϵ_1 and ϵ_2, the "true" errors of misclassification, i.e., obtained from U. (3) Estimation of β_1 and β_2, the "index" errors of misclassification, i.e., obtained from V where V is considered to be used on all future observations z.

3. Wilcoxon's signed rank test as a large sample competitor of Wilcoxon's rank sum test. MYLES HOLLANDER, The Florida State University.

Let $X_1, X_2, \cdots, X_n$ be iid according to F_1 and $Y_1, Y_2, \cdots, Y_n$ be iid according to F_2 with F_1, F_2 continuous. In this equal sample size case we may randomly pair the X's with the Y's and base a test of $H_0 : F_1 = F_2 = F$ (unknown) on Wilcoxon's signed rank statistic (W). We consider this procedure as a competitor of the more commonly used test based on Wilcoxon's rank sum statistic (U). Assuming $0 < \int F_1\, dF_2 < 1$, it is shown that the asymptotic distribution of (W, U), with each component suitably normed, is bivariate normal. Furthermore, the null correlation $r_0^n(F)$ between W, U depends on F (except for $n = 1$ and $n = 2$), as does $r^*(F) = \lim r_0^n(F)$. Specifically, $r_0^n(F) = [n^2(24\mu(F) - 6) + n(23 - 72\mu(F)) + (48\mu(F) - 14)]/\{(2n+1)[n(n+1)/2]^{\frac{1}{2}}\}$ and $r^*(F) = [24\mu(F) - 6]/2^{\frac{1}{2}}$ where $\mu(F) = P(X_1 < X_2 ; X_1 < X_3 + X_4 - X_5)$ when X_1, X_2, X_3, X_4, X_5 are idd according to F. The parameter $\mu(F)$ has also appeared in a different correlation obtained by the author (*Ann. Math. Statist.* (Abstract) **36** 1083). For F normal, rectangular and exponential, the values of $r^*(F)$ are, respectively, .976, .990, and .943. For the translation alternatives $F_2(x) = F(x - \theta)$ we determine the Bahadur efficiency $B_\theta(W, U)$ and find $\lim_{\theta\to\infty} B_\theta(W, U) = .5$. The corresponding Pitman efficiency expression is $2[\int g^2/\int f^2]^2$ where g is the density of $X_1 - X_2$ when X_1, X_2 are iid according to F. For F normal, rectangular, and exponential, the Pitman efficiency values are, respectively, 1, .889, and .5.

4. Least squares estimation of the components of a symmetric matrix. HAROLD J. LARSON, U. S. Naval Postgraduate School, Monterey.

In the simultaneous equation model $Y = XB + E$, where Y is an $s \times t$ matrix of observable random variables, X is an $s \times t$ matrix of known constants, B is a $t \times t$ matrix of unknown parameters and E is an $s \times t$ matrix of unobservable random variables, $s > t$, standard statistical techniques are available for computing the least squares estimate of B. Certain physical problems require that B be symmetric, which the general solution does not necessarily satisfy. The form of the symmetric matrix B which minimizes tr $E'E$ is derived and a simple numerical example is presented.

5. On a class of multivariate multisample rank-order tests. Madan L. Puri and Pranab K. Sen, Courant Institute of Mathematical Sciences, New York University and University of North Carolina.

A class of rank-order tests for the multivariate several sample location and scale problems is proposed and studied. The principle of rank-permutation tests by Chatterjee and Sen is utilized to make these tests strictly distribution-free. The asymptotic properties of rank-permutation tests are studied with the aid of Wald-Wolfowitz-Noether-Hajek theorems (Hajek: *Ann. Math. Statist.* **32** 506–523), and certain stochastic equivalence relations of these tests with the multivariate extensions of the tests discussed by Puri (*Ann. Math. Statist.* **35** 102–121, and *Ann. Math. Statist.* **36** 1084) are derived.

6. Minimax property of the maximum likelihood estimators for normal multivariate regression. Stanley L. Sclove, Columbia University.

Consider estimating the regression function in the regression of $Y(q \times 1)$ on $X(p \times 1)$, where $(Y, X)'$ has a multivariate normal distribution. Take as the loss the expected square of the Euclidean distance between Y and its predicted value, in the metric of the residual covariance matrix, when a prediction is made on the basis of $n \geqq p + 3$ previous observations on $(Y, X)'$ and a new random observation on X. The risk of the mle is computed. Application of Kiefer's general invariance theorem and the Hunt-Stein method, similar to that made in the univariate case $(q = 1)$ by Stein, proves the mle is minimax.

(Abstracts of papers not connected with any meeting of the Institute.)

1. Asymptotic theory of Bayes solutions (preliminary report). Peter Bickel and Joseph Yahav, University of California, Berkeley, and Tel Aviv University.

Under suitable regularity conditions we establish asymptotic normality of Bayes estimates and convergence of normalized Bayes posterior risks for convex loss functions which need not be bounded. In particular, quadratic loss is included in our class. Under more stringent conditions this extends work of Le Cam. Similarly, we show the nth root of the Bayes posterior risk in testing converges under suitable conditions. The above results generalize theorems proved and stated in Bickel and Yahav "Asymptotically Pointwise Optimal Sequential Decision Procedures" (to appear in *Proc. Fifth Berkeley Symp. Math. Statist. Prob.*).

2. Asymptotically optimal sequential Bayes estimates (preliminary report). Peter Bickel and Joseph Yahav, University of California, Berkeley, and Tel Aviv University.

Let Y_n be the Bayes posterior stopping risk for estimation with quadratic loss. We have Theorem (a). *Under the regularity conditions guaranteeing that* $nY_n \rightarrow_{\text{a.s.}} I(\theta)$ *where* $I(\theta)$ *is the Fisher information number, the rule* $t(c)$: *Stop as soon as* $Y_n/n \leqq c$ *is asymptotically point-*

wise optimal. (See Bickel and Yahav "A.P.O. Sequential Decision Procedures".) THEOREM (b). *If* $\sup_n E(nY_n) < \infty$, $t(c)$ *is asymptotically optimal in the sense that if* $s(c)$ *is any other sequence of stopping rules,* $\lim\sup_{c\to 0} \{[E(Y_{t(c)}) + cE(t(c))]/[E(Y_{s(c)}) + cE(s(c))]\} \leqq 1$.

These results have been extended to the case $l(\theta, d) = |\theta - d|^r$, $r > 0$. Under further regularity conditions it may be shown that these rules are asymptotically minimax in the sense of Wald.

3. Some optimum properties of ranking procedures. MORRIS EATON, Stanford University.

Let $X = (X_1, \cdots, X_k)$ be a vector of real observations with density $p(x, \theta)\, d\mu(x)$, $x = (x_1, \cdots, x_k)$ and $\theta = \theta_1, \cdots, \theta_k)$. For the problem of deciding, on the basis of X, which θ_i is largest, we let our action space be $\mathfrak{A} = \{1, 2, \cdots, k\}$ and $L_i(\theta)$ be the loss for taking action i when θ is the parameter point. If $Z = (Z_1, \cdots, Z_k)$ and σ is a permutation of $\{1, 2, \cdots, k\}$ define Z_σ by $Z_\sigma = (Z_{\sigma 1}, Z_{\sigma 2}, \cdots, Z_{\sigma k})$. Assume that $p(x, \theta)\, d\mu(x) = p(x_\sigma, \theta_\sigma)\, d\mu(x_\sigma)$ and $L_i(\theta) = L_{\sigma i}(\theta_\sigma)$ for all σ, $i = 1, \cdots, k$. Also assume that $0 \leqq L_i(\theta) \leqq L_j(\theta)$ if $\theta_i \geqq \theta_j$, $i \neq j$. Let φ_0 be the decision function which assigns probability 1 to action i when $X_i > X_j$ for all $j \neq i$ (assume the obvious randomization on boundary points). The main result is the following: If $p(x, \theta)$ has property M (see the definition below) then: (i) φ_0 is Bayes for every prior distribution $F(\theta)$ which is symmetric in its arguments, (ii) φ_0 is minimax, and (iii) φ_0 is admissible. The density $p(x, \theta)$ is said to have property M if for each i, j $(i \neq j)$, $x_i \geqq x_j$ and $\theta_i \geqq \theta_j$ implies $p(x, \theta) \geqq p(x, \theta_{(i,j)})$ where (i, j) is the permutation which interchanges i and j. For the obvious decision procedure, the above result is extended to the following case: on the basis of X, decide on the k_1 largest θ_i, the k_2 next largest θ_i, $\cdots$, k_s smallest θ_i, where $1 \leqq k_i < k$ and $\sum_1^s k_i = k$ [see Bechhofer (*Ann. Math. Statist.* **25** 16–39)].

4. Characterization of distributions by the indentical distribution of linear forms. MORRIS EATON, Stanford University.

Let $X_0, X_1, \cdots, X_n$ be independent identically distributed random p-dimensional row vectors and let B_i, $i = 1, \cdots, n$, be fixed $p \times p$ real symmetric non-singular matrices. Suppose that the distribution of X_0 is the same as the distribution of $\sum_{i=1}^n X_i B_i + b$ where b is a fixed p-dimensional row vector. It is shown that if all the eigenvalues of each B_i are in the open interval $(-1, 1)$, then the distribution of X_0 is infinitely divisible. If it is further assumed that the eigenvalues of $\sum_1^n B_i^2$ are all greater than or equal to 1, it is shown that the distribution of X_0 must be multivariate normal (possibly degenerate). Laha and Lukacs (*Pacific J. Math.* **15** 207–214) consider the univariate version of the above problem. In addition to the multivariate results above, some univariate results concerning symmetric stable laws are presented. More specifically, let $Y_0, Y_1, \cdots, Y_n$ be independent identically distributed symmetric real valued random variables and let $2 \leqq m < n$ for m and n integers. If $(\log n)/\log m$ is irrational and if the distribution of Y_0 is the same as the distribution of $(\sum_1^m Y_i)/m^{1/\alpha}$ and $(\sum_1^n Y_i)/n^{1/\alpha}$ $(0 < \alpha \leqq 2)$, then Y_0 has a symmetric stable distribution of order α.

5. On tests of the equality of two covariance matrices. N. GIRI, Indian Institute of Technology, Kanpur.

Let X, Y be independently, normally distributed p-dimensional column vectors with unknown means ξ, η and unknown covariance matrices Σ_1, Σ_2 respectively. Let (S_1, S_2) be minimal sufficient for (Σ_1, Σ_2), and $\theta_1, \cdots, \theta_p$ be the characteristic roots of $\Sigma_1\Sigma_2^{-1}$. In this paper it will be shown that for testing $H_0: \theta_1 = \cdots = \theta_p = 1$ against the alternative $H_1: \theta_i \geqq 1$ for all i and $\sum_1^p \theta_i > p$, the test which rejects H_0 if $\operatorname{tr} S_2(S_1 + S_2)^{-1}$ is less than some constant, depending on the size of the test; is uniformly most powerful invariant if $(\theta_i - 1) \to 0^+$ for all i; and for the dual problem of testing H_0 against $H_2 : \theta_i \leqq 1$ for all i and $\sum_1^p \theta_i < p$, the test which rejects H_0 if $\text{tr}S_2(S_1 + S_2)^{-1}$ is greater than some constant, depending on the size of the test; is uniformly most powerful invariant if $(\theta_i - 1) \to 0^-$ for all i.

6. A new approach to sampling from finite populations I—sufficiency and linear estimation. V. P. GODAMBE, The Johns Hopkins University.

Barnard [*J. Roy. Statist. Soc.* (1963)] introduced the concept of linear sufficiency, in connection with Gauss-Markoff set-up of estimation. In the present article, the concept of linear sufficiency is *redefined*, to suit the problem of estimation in survey-sampling where as demonstrated by the present author [*J. Roy. Statist. Soc.* (1955); *Inter. Statist. Inst. Rev.* 1965)] the Gauss-Markoff set-up is fundamentally inadequate. According to this redefined concept, a certain estimate is shown to be uniquely (up to a constant multiplier), linearly sufficient for the population total, in the *entire* class (defined by the author [*J. Roy. Statist. Soc.* (1955)]) of linear estimators. The significance of this result would be clear on the background of the author's (1955) previous result demonstrating the non-existence of a uniformly least variance estimator, in the *entire* class of unbiased linear estimators for the population total.

7. A new approach to sampling from finite populations II—distribution-free sufficiency. V. P. GODAMBE, The Johns Hopkins University.

The idea that in some situations the prior knowledge about the unknown parameters could be formulated as a class of prior distributions, which could be used, not necessarily through Bayes *posteriori* probability, for subsequent inference, is already present in the author's [*J. Roy. Statist. Soc.* (1955)] earlier paper. In the present paper, the concept of distribution-free linear sufficiency or in short linear sufficiency, originally due to Barnard [*J. Roy. Statist. Soc.* (1963)], but redefined by the present author (Part I), is *extended* by defining distribution-free sufficiency, *removing* the restriction of *linearity*. This extension again is based on the assumption that in some situations prior knowledge could be formulated as a class of prior distributions. A certain linear estimator of the population total which in Part I was shown to satisfy the redefined criteria of linear sufficiency uniquely in the class of all linear estimators, is now shown to satisfy this extended criteria of distribution-free sufficiency in the entire class of estimators. Further the general relationship between the linear sufficiency of the Part I and the distribution-free sufficiency introduced here, is investigated. Broadly the result is, if we restrict to linear estimators only, the distribution-free sufficiency is identical with the linear sufficiency. Finally some remarks are offered by way of comparison between the result obtained by the author [*J. Roy. Statist. Soc.* (1955)] previously and the result here, about the utilization of the prior information.

8. Bayes and empirical Bayes estimation in sampling finite populations. V. P. GODAMBE, The Johns Hopkins University.

The problem of estimation, in sampling finite population has been studied exhaustively by Godambe (1955) and Godambe and Joshi (1965). A conclusion of great practical significance is this: "For almost all of the commonly known sampling designs, an unbiased minimum variance (UMV) estimator for the population total does not exist." In this paper after deriving Bayes estimators, it is shown that the ratio-type estimators for the population total which include the estimator based on sample mean, are, "empirical Bayes" wrt squared error as the loss-function. A concept of "global admissibility" is introduced and Bayes and empirical Bayes estimators are shown to be globally admissible.

9. Asymptotic efficiency of the Kolmogorov-Smirnov test. Jerome Klotz, University of Wisconsin.

Asymptotic efficiency for the two sample Kolmogorov-Smirnov test is obtained by comparing the exponential rate of convergence to zero of the type I error (α) while keeping the type II error (β) fixed ($0 < \beta < 1$). In particular, for the one sided test with equal samples of size n it is shown, using the closed form expression for the null distribution, that $\lim_{n\to\infty} -(1/n) \log \alpha_n = (1 - \rho) \log (1 - \rho) + (1 + \rho) \log (1 + \rho)$ where $\rho = \sup (F - G)$. For a location parameter family of alternatives $(F(x), G(x) = F(x - \Delta))$ with a symmetric density, the expression reduces to $2F(\Delta/2) \log F(\Delta/2) + 2F(-\Delta/2) \log F(-\Delta/2)$ which is the same expression obtained for the Mood and Brown median test. As a consequence, the limiting efficiency relative to the Mood and Brown test is one for such alternatives. The efficiency relative to the t test for normal location alternatives gives $2/\pi$ as Δ goes to zero.

10. Exact moments of order statistics from the Pareto distribution. Henrick John Malik, Western Reserve University.

Let $X_{1,N} < X_{2,N} < \cdots < X_{N,N}$ denote a set of order statistics in a random sample of N independent and identically distributed random variables $X_1, X_2, \cdots, X_N$ from a distribution having a pdf $va^v x^{-v-1}$ (for $a > 0$, $v > 0$, $x \geqq a$) and zero elsewhere. The characteristic function of the kth order statistic is obtained and moments about the origin of the kth order statistic are expressed in terms of gamma function. An exact expression for the covariance of any two order statistics $X_{i,N} < X_{j,N}$ is obtained. Various recurrence relations between the expected values of order statistics are obtained. Using the results, the expected values of the kth order statistic are tabulated for sample sizes up to and including 12 and for the index parameter $v = 2.5(0.5)7$, and the expected values of cross-products of all pairs of order statistics are obtained and tabulated for sample of sizes up to and including 12 and for the index parameter $v = 2.5(0.5)7$. The expressions for the variance and covariance of the kth order statistic are also obtained in closed form and tabulated.

11. A probability distribution for the number of completed pregnancies. K. B. Pathak, Banaras Hindu University. (Introduced by S. N. Singh)

A probability distribution for the variation in number of births per couple during period $(0, T)$ has been obtained on the basis of certain assumptions similar to those of Singh (*J. Amer. Statist. Assoc.* **58** 721–27). Let X denote the number of conceptions during period $(0, T)$, then $F(x, T) = 1 - \alpha + [\alpha(1 - \beta)/(h - 1)] \sum_{d=s}^{h'} F_1(x - 1, T - d) + [\alpha(1 - \beta)/h'] \cdot \sum_{d=1}^{s-1} F_1(x, T - d) + \alpha\beta F(x, T)$ where $F_1(x, T) = \sum_{r=0}^{x} [\binom{T-rh'}{r} + \sum_{l=1}^{h'} \binom{T-l-h'(r-1)}{r-1} q^{h-l}] \cdot p^r q^{T-rh}$. $h' = h - 1$, where h is the duration of rest period; $h - s$ is the length of the gestation period; α is the proportion of fecund couples and β is the proportion of the fecund couples not exposed to the risk of conception in the first unit of the observational period. The problem of finding out the consistent estimates of the parameters has been discussed.

12. A note on inflated power series distribution. K. B. Pathak, Banaras Hindu University. (Introduced by S. N. Singh.)

Inflated Power Series Distribution (IPSD) is a mixture of a Simple Power Series Distribution (SPSD) and some degenerate distributions. It is especially useful where the SPSD describes the observed data except for the cells which are inflated. The IPSD's of different orders have been derived. The maximum likelihood estimates of the parameters and their variances have been obtained. The order of the IPSD is determined on the basis of the number of degenerate distributions in the mixture.

13. On the first conceptive delay. K. B. PATHAK, Banaras Hindu University. (Introduced by S. N. Singh.)

On the basis of certain assumptions, we have formulated two probability distributions given below to describe the first conceptive delay after the marriage: (1) Inflated Geometric Distribution whose pdf is given by $P(X = 1) = 1 - \alpha + \alpha p$, $P(X = x) = \alpha p q^{x-1}$ for $x > 1$, where X denotes the month in which the couple conceives. (2) Inflated Geometric Type I Distribution whose pdf is given by $P(X = 1) = 1 - \alpha + \alpha a/(a + b)$, $P(X = x) = \alpha B(a + 1, b + x - 1)/B(a, b)$ for $x > 1$ where $(1 - \alpha)$ is the proportion of the couples who conceive prior to their marriage but report the first month of the marriage for their conception. The models have been applied to examples which appeared in Singh (*Sankhyā Ser. B* **26** 95–102) and Potter and Parker (Population Studies **17** 99–116).

14. Estimation for unimodal densities and for distributions with monotone failure rate (preliminary report). B. L. S. PRAKASA RAO, Michigan State University.

Recently Marshall and Proschan (*Ann. Math. Statist.* **36** 69–77) have derived the maximum likelihood estimates for distributions with monotone failure rate and they have shown these estimators are consistent. We obtain the asymptotic distribution of estimators using the results of Chernoff in his paper on the estimation of mode. The estimation problem is reduced at first to that of a stochastic process and the asymptotic distribution is obtained by means of theorems on convergence of distributions of stochastic processes. Similar results are obtained for distributions with unimodal densities.

15. Some distribution free multiple decision procedures for certain problems in analysis of variance. MADAN L. PURI and PREM S. PURI, Courant Institute of Mathematical Sciences, New York University and University of California, Berkeley.

This paper is concerned with single-sample multiple-decision procedures based on the ranks of the observations for selecting from c continuous populations (a) the "best t" populations *without* regard to order, (b) the "best t" populations *with* regard to order, and (c) a subset which contains all populations "as good or better than a standard one." The "bestness" of a population is characterized by its location parameter. Large-sample methods are provided for computing the sample sizes necessary to guarantee a preassigned probability of a correct grouping (or ranking) under specified conditions on location parameters. It is shown that the asymptotic efficiency of these procedures relative to the normal theory procedures (see, for example, Bechhofer, *Ann. Math. Statist.* (1954) 273–289 and Gupta-Sobel, *Ann. Math. Statist.* (1958) 235–244) is the same as that of the associated tests in one-way analysis of variance model I problems. If the ratio of the sample equals this efficiency, the two procedures being compared are shown to have the same asymptotic performance characteristics. Finally, in the case of problem (c) two alternative rank-score procedures are proposed which are asymptotically equi-efficient. The results are obtained by following the methods due to Lehmann (Math. Annalen (1963) 268–275).

16. On two methods of bias reduction in the estimation of ratios. J. N. K. RAO and J. T. WEBSTER, Texas A&M University and Southern Methodist University.

Suppose r denotes the customary ratio estimator of the form $r = \bar{y}/\bar{x}$ based on n observations whose bias is $cn^{-1} + O(n^{-2})$ where c is a constant. Let the sample be divided at random into g groups, each of size p, where $n = pg$. Let r_j' denote the ratio estimator calculated from the sample after omitting the jth group. Then the estimator $r_Q = gr - [(g-1)/g] \sum r_j'$ has a bias of order n^{-2} at most. J. N. K. Rao [*Biometrika* **52** (1965)] has shown that, if the regression of y on x is linear and x is normally distributed, *both* the bias and variance of r_Q to $O(n^{-3})$ are decreasing functions of g so that $g = n$ would be the optimum choice. In this paper, we show that the *exact* bias and variance of r_Q are decreasing functions of g and r_Q has a smaller variance than r for $g > 2$ when the regression of y on x is linear and x has a Γ-distribution. We also derive the exact bias and variance of the modified ratio estimator r_M [Tin, *J. Amer. Statist. Assoc.* **60** (1965) 294–307] assuming the above model.

17. Distributions of some "sequential type" statistics of Rayleigh class random vectors. HAROLD SACKROWITZ, Columbia University.

Let $X_1, \cdots, X_n$ be p-dimensional vectors of Rayleigh class with mean zero and positive definite covariance matrix M. Let $Z_k = g_k(r_1, \cdots, r_{k+1})X_{k+1}$, $k = 1, \cdots, n-1$ where r_j is the Euclidean norm of X_j and the g_k satisfy some mild regularity conditions. We are interested in obtaining, when possible, the joint density function of $Z_1, \cdots, Z_{n-1}$ in closed form. A typical result being that if the statistic $T(X_1, \cdots, X_n) = (Z_1, \cdots, Z_{n-1})$ is invariant under multiplication of the X vectors by a positive scalar, then the density can be given in terms of the hypergeometric function. Similar results are obtained, for other cases, in terms of Bessel functions.

18. Asymptotically optimal tests for Markov chains (preliminary report). R. C. SRIVASTAVA, Ohio State University.

We consider tests of simple and composite hypothesis about the transition probability matrix of certain types of Markov chains. Following Hoeffding [*Ann. Math. Statist.* (1965)], we consider tests whose size α_n tends to zero as n tends to infinity. The main result is "for any test of size α_n which is sufficiently different from a likelihood ratio test, there exists a likelihood ratio test of size less than or equal to α_n which is more powerful then the given test at most points in the set of alternatives as n tends to infinity and α_n tends to zero at a suitable rate." In particular we compare the chi-square test and the likelihood ratio test for a simple hypothesis and extend our results for a composite hypothesis. Finally we compare our results with those of Anderson and Goodman [*Ann. Math. Statist.* (1957)] who considered the stochastic comparison of the likelihood ratio test and the chi-square test for Markov chains.

19. On the asymptotic properties of Ruben's estimator of the fundamental interaction parameter in the emigration-immigration process (preliminary report). R. C. SRIVASTAVA, Ohio State University.

Recently Ruben (*Ann. Math. Statist.* **34** 1963) has considered the problem of estimating the interaction parameter in the emigration-immigration process. The emigration-immigration process $\mathbf{n}(t) = (n_1(t), \cdots, n_m(t))$ is a vector-valued stochastic process and for any fixed t, the distribution of $\mathbf{n}(t)$ is $\prod_{r=1}^{m} e^{-\nu_r}(\nu_r)^{n_r(t)}/(n_r(t))!$. Let $\mathbf{n}(\tau), \cdots, \mathbf{n}(\kappa\tau)$ be k observations on the process $\mathbf{n}(t)$ at times $\tau, \cdots, \kappa\tau$ respectively. The estimator of the interaction parameter is constructed from the consecutive differences $\mathbf{n}(i\tau) - \mathbf{n}((i-1)\tau)$ and formula for the large sample variance is given and its relative efficiency is investigated. In this paper we prove that Ruben's estimator is consistent and asymptotically normally distributed and obtain an expression for its efficiency.