PROBABILISTIC COMPLETION OF A KNOCKOUT TOURNAMENT[1]

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1. Summary. A knockout tournament is a procedure for selecting the best among 2^n players by, in the first round, splitting the 2^n players into 2^{n-1} pairs who play each other; the 2^{n-1} winners proceed to the next round and repeat the process; finally the one player left is declared the best. A method is given for estimating a complete ranking of the 2^n players given the results of the $(2^n - 1)$ matches in the tournament; the method is based on the assumption that all $(2^n)!$ orderings of the players are equally probable before the tournament begins.

2. Terminology and assumptions. A tournament T_n is a set A_n of 2^n players, together with the results of $(2^n - 1)$ matches between the players, played according to the above scheme. The rth *round* of the tournament consists of the results of matches between players who win at least $(r - 1)$ matches.

A *rank* of the players is a complete ordering of the players according to the ability tested by the game played; each match is a comparison between two players in which one is ranked above or below the other; the whole tournament is a set of $(2^n - 1)$ such comparisons, and may be regarded as a partial ordering, $<$, in which $a < b$ if and only if there is a sequence of players $a_1, \cdots, a_k$ such that $a = a_1$, $b = a_k$ and a_i beat a_{i-1} in the tournament.

A *completion* of a partial ordering is a rank such that every comparison which holds in the partial ordering also holds in the rank; the set of completions associated with a partial ordering identifies it, and it is convenient to make this identification in probabilistic work with partial orderings. Suppose S is the set of all ranks on A_n, $\mathcal{R}$ is the family of all subsets of S, and we are given a probability distribution on $\mathcal{R}$; in the finite case, we may specify the distribution by giving the probabilities $p(r)$ for all the individual ranks r; then for $R \in \mathcal{R}$,

$$P(R) = \textstyle\sum_{r \in R} p(r).$$

Now suppose that we know that some set of comparisons, R_1 say, holds and that we are interested in some other set R_2. If we denote by $T(R_1)$ the set of completions of R_1, i.e. the set of ranks consistent with all the comparisons in R_1, we may define $P(R_2 \mid R_1) = P[T(R_1) \cap T(R_2) \mid T(R_1)]$.

We will consider only the simple distribution in which all ranks have equal initial prior probability; given the tournament T_n, we will then have that all ranks consistent with the tournament are equally probable. This result is not of immediate practical value because the number of such ranks is very large; however we may now calculate such quantities as

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$$P_n(a < b) = P(a < b \mid T_n),$$

$$P_{n,a}(i \mid T_n) = P[a \text{ is in the } i\text{th position in the final order} \mid T_n],$$

$$E_{n,a}(i \mid T_n) = \text{average position of } a = \sum iP_{n,a}(i \mid T_n).$$

In particular we will define the estimated final ordering by $a < b$ if $E_{n,a}(i \mid T_n) > E_{n,b}(i \mid T_n)$.

Since $E_{n,a}(i \mid T_n) = \sum_b P(a < b \mid T_n) + 1$, by a trivial manipulation, this estimated final ordering is also according to the average win probability of each player.

3. Binary representation of a tournament. The number of players in a tournament T_n is 2^n, and the number of matches is $2^n - 1$, so it is natural to attempt to represent the results of the tournament in binary notation. Let the players greater than or equal to a (according to the tournament) be $a_1, a_2, \cdots, a_k$ and let the last round in which a_i appeared be the n_ith. Then the mapping f from A_n onto the integers $2^n, 2^n + 1, \cdots, 2^{n+1} - 1$ defined by $a \to \sum_{i=1}^k 2^{n_i-1}$ is a representation of the results of the tournament, such that each set of results generates a unique mapping, and each mapping generates a unique set of results. (The object space $2^n, 2^n + 1, \cdots, 2^{n+1} - 1$ is used, rather than say $1, 2, \cdots, 2^n$, because it simplifies some later calculations.) In binary notation we have $f(a) = \alpha_1\alpha_2 \cdots \alpha_{n+1}$ where $\alpha_j = 1$ if one of the a_i appeared for the last time in the jth round, and $\alpha_j = 0$ otherwise. For example in a tournament of $2^6 = 64$ persons, suppose a was beaten by a_2 in the 1st round, a_2 by a_3 in the 4th round and a_3 by a_4 in the 6th round; a_4 will finally appear (the winner) in the 7th round. Then $f(a) = 1001011 = 105$.

To check that there is a 1-1 correspondence between results of tournaments and mappings of A onto $2^n, 2^n + 1, \cdots, 2^{n+1} - 1$, let us first note that the mapping f defined above is onto; for if it is not, two elements in A must map into the same number and hence have the same "last-round" sequence $n_1, \cdots, n_k$. Suppose the corresponding players are $a, a_2, \cdots, a_k$ and $b, b_2, \cdots, b_k$; the winner of the tournament is the final element in both sequences so $a_k = b_k$ and $n_k = (n + 1)$; we then have that a_{k-1} and b_{k-1} appear for the last time at round n_{k-1}; since only the winner is ranked higher than a_{k-1} or b_{k-1} we must have that the winner beat a_{k-1} and b_{k-1} in round n_{k-1}; i.e. $a_{k-1} = b_{k-1}$. Continuing this argument, we find that $a = b$; thus two different elements cannot map into the same number.

Next we must show that every 1-1 mapping from A onto $2^n, \cdots$ corresponds to a set of tournament results; given the mapping f, let the rth round consist of elements a such that $2^{r-1} \mid f(a)$; i.e. is divisible by 2^{r-1}; a beats b in the rth round if $2^r \mid f(a)$, $2^{r-1} \mid f(b)$ and $f(a) = f(b) + 2^{r-1}$. This tournament generates a mapping f' identical to f; thus every mapping corresponds to a set of tournament results.

(More generally the partial ordering on A_n corresponding to the results of the tournament may be obtained from f by setting $a < b$ if for some r, $2^r \mid f(b)$ and $0 < f(a) - f(b) < 2^r$.)

4. Decomposition of a tournament. The main technique in the calculation of estimated completions is the use of recursion formulae relating large tournaments to smaller ones; these formulae require some method of splitting a tournament into smaller components. We have seen that T_n may be regarded as a set of completions, and also as the pair (A_n, f_n) where A_n is the set of 2^n players and f_n is a mapping of A_n onto $2^n, 2^n + 1, \cdots, 2^{n+1} - 1$.

THEOREM. *Let ω be the winner of the tournament $T_n = (A_n, f_n)$ and let C_ω be the set of all ranks of A_n in which ω is ranked first; let*

$$A_k = \{a \mid 2^k \leqq f_n(a) - 2^n \leqq 2^{k+1} - 1\},$$

$$f_k(a) = f_n(a) - 2^n \quad \text{for} \quad a \,\varepsilon\, A_k, \qquad T_k = (A_k, f_k).$$

Then,

$$A_n = \{\omega\} \cup A_0 \cup A_1 \cup \cdots \cup A_{n-1},$$

$$T_n = C_\omega \cap T_0 \cap T_1 \cap \cdots \cap T_{n-1}.$$

PROOF. Firstly let us show that

$$A_n = \{\omega\} \cup A_0 \cup A_1 \cup \cdots A_n;$$

$$\begin{aligned} A_n &= \{a \mid 2^n \leqq f(a) \leqq 2^{n+1} - 1\} \\ &= \{a \mid 0 \leqq f(a) - 2^n \leqq 2^n - 1\} \\ &= \{\omega\} \textstyle\bigcup_{k=0}^{n-1} \{a \mid 2^k \leqq f_n(a) - 2^n \leqq 2^{k+1} - 1\} \\ &= \{\omega\} \cup A_0 \cup A_1 \cup \cdots \cup A_{n-1}. \end{aligned}$$

Next we need to show $T_n = C_\omega \cap T_0 \cap T_1 \cap \cdots \cap T_{n-1}$; now we know in the tournament T_n that $a < b$ if, for some r, $2^r \mid f_n(b)$ and $0 < f_n(a) - f_n(b) < 2^r$; if $b \,\varepsilon\, A_k$, we have $2^k \leqq f_n(b) - 2^n \leqq 2^{k+1} - 1$; letting $f_n(b) = \alpha 2^r + 2^n$ for some integer α, we have $2^k \leqq \alpha 2^r \leqq 2^{k+1} - 1$, i.e. $2^{k-r} \leqq \alpha < 2^{k+1-r}$. Now $0 < f_n(a) - f_n(b) < 2^r$, so $2^k < f_n(a) - 2^n < (\alpha + 1) 2^r$, i.e. $2^k < f_n(a) - 2^n < 2^{k+1}$ since $(\alpha + 1) \leqq 2^{k+1-r}$. Thus if $a < b$ and $b \,\varepsilon\, A_k$, then $a \,\varepsilon\, A_k$; furthermore since $f_k(a) = f_n(a) - 2^n$, any relation which holds between a and b in T_k has an exactly corresponding relation holding between a and b in T_n. Finally the relations involving ω, namely $a < \omega$ whenever $a \neq \omega$, determine C_ω, the set of ranks in which ω is ranked first. We have now seen that any relation holding in T_n holds in exactly one of $C_\omega, T_0, \cdots, T_{n-1}$; thus the set of ranks consistent with T_n is $C_\omega \cap T_0 \cap T_1 \cap \cdots \cap T_{n-1}$, concluding the theorem.

The essential import of the theorem is that the set of players can be split into n disjoint sets of players, and the set consisting of the winner, in such a way that the relations holding in T_n are either relations within one of the n sets or relations with the winner. Furthermore the structure of the relations within each set is that of a tournament [of size $2^0, 2^1, \cdots, 2^{n-1}$ respectively]; the tournament of size 2^r is in fact the set of players less than or equal to the player beaten by the winner in the $(r + 1)$th round.

With this decomposition, problems concerning the whole tournament may frequently be reduced to problems involving smaller tournaments, and solved by repetition of the process.

5. Recursion formulae. We will develop formulae for the quantities

$$P_n(a < b) = P(a < b \mid T_n),$$

$$P_{n,a}(i \mid T_n) = P(a \text{ is in the } i\text{th position in the final rank} \mid T_n),$$

$$E_{n,a}(i \mid T_n) = \text{average position of } i.$$

We will need some elementary properties of the combination of two sets of objects. [See Wilks [2], p. 141 for similar calculations]; suppose we are given n_1 objects of type A and n_2 objects of type B; the $(n_1 + n_2)$ objects are ordered at random; then the probability that exactly i A's appear before the jth B is given by

$$u_{i,j,n_1,n_2} = \binom{i+j-1}{i}\binom{n_1-i+n_2-j}{n_i-i}\Big/\binom{n_1+n_2}{n_1};$$

the average number of i's appearing before the jth B is $jn_1/(n_2 + 1)$.

Let us now consider $P_{n,a}(i \mid T_n)$; if $a = \omega$ then

$$P_{n,a}(i \mid T_n) = 1 \qquad \text{if } i = 1,$$

$$P_{n,a}(i \mid T_n) = 0 \qquad \text{elsewhere; if } a \varepsilon A_k,$$

suppose that all elements in A_k are arranged in some order (compatible with the tournament) and all elements outside A_k have been arranged in some order. The only remaining uncertainty is in the intermeshing of these two orders, and this is a simple problem because the only relations which must hold between A_k and $A - A_k$ are $b < \omega$ for all $b \neq \omega$; thus every arrangement of the 2^k objects in A_k and the $2^n - 2^k - 1$ objects in $A - A_k - \{\omega\}$ is equally probable, conditional on the orders within A_k and $A - A_k$ being fixed.

If a is in position p in A_k, the probability that j objects of $A - A_k - \{\omega\}$ precede it is u_{j,p,n_1,n_2} where $n_1 = 2^n - 2^k - 1$, $n_2 = 2^k$; this probability is conditional on the orders within A_k and $A - A_k$ being fixed, but depends only on p, the position of a within A_k; the probability that a occupies the pth position in the order of A_k is $P_{k,a}(p \mid T_k)$. Noting finally that the position of a in A_n is $(j + p + 1)$, we have

$$P_{n,a}(i \mid T_n) = \textstyle\sum_{j+p=i-1} P_{k,a}(p \mid T_k)u_{j,p,n_1,n_2}. \tag{1}$$

Here is a formula which relates $P_{n,a}(i \mid T_n)$ for large tournaments to its values in smaller tournaments.

Next, consider $P_n(a < b)$; if a and b are in the same A_k, outside relations are irrelevant and we may set

$$P_n(a < b) = P_k(a < b). \tag{2}$$

If a and b are in different A_k, say A_{k_1} and A_{k_2}, their behaviour depends only on A_{k_1} and A_{k_2} and, setting $n_i = 2^{k_i}$

TABLE 1

Estimated final rank (and standard deviations) of players in tournaments T_n, $0 \leqq n \leqq 6$

$f_n(a)-2^n$	n = 0	1	2	3	4	5	6
0	1 (0)	1 (0)	1.0 (0.0)	1.0 (0.0)	1.0 (0.0)	1.0 (0.0)	1.0 (0)
1		2 (0)	3.0 (0.8)	5.0 (2.0)	9.0 (4.3)	17.0 (8.9)	33.0 (18)
2			2.3 (0.5)	3.7 (1.5)	6.3 (3.4)	11.7 (7.2)	22.3 (15)
3			3.7 (0.5)	6.3 (1.5)	11.7 (3.4)	22.3 (7.2)	43.7 (15)
4				2.6 (0.8)	4.2 (2.2)	7.4 (4.8)	13.8 (10)
5				5.8 (1.6)	10.6 (3.6)	20.2 (7.6)	39.4 (16)
6				4.7 (1.2)	8.5 (3.1)	15.9 (6.6)	30.9 (14)
7				6.9 (1.1)	12.7 (2.8)	24.5 (6.0)	47.9 (12)
8					2.8 (1.1)	4.6 (2.7)	8.1 (6)
9					9.9 (3.9)	18.8 (8.1)	36.6 (16)
10					7.5 (3.1)	14.0 (6.6)	27.1 (14)
11					12.3 (3.0)	23.5 (6.4)	46.0 (13)
12					5.6 (2.1)	10.2 (4.8)	19.5 (10)
13					11.3 (3.2)	21.6 (6.8)	42.2 (14)
14					9.4 (2.7)	17.8 (6.0)	34.7 (13)
15					13.2 (2.4)	25.4 (5.4)	49.8 (11)

$f_n(a)-2^n$	n = 5	n = 6	$f_n(a)-2^n$	n = 6	$f_n(a)-2^n$	n = 6
16	2.9 (1.2)	4.8 (3)	32	2.9 (1)	48	6.6 (3)
17	17.9 (8.4)	34.9 (17)	33	34.0 (18)	49	35.8 (17)
18	12.9 (6.8)	24.8 (14)	34	23.6 (14)	50	26.1 (14)
19	23.0 (6.8)	44.9 (14)	35	44.3 (14)	51	45.5 (13)
20	8.9 (4.6)	16.8 (10)	36	15.4 (10)	52	18.3 (10)
21	21.0 (7.2)	40.9 (15)	37	40.2 (15)	53	41.6 (14)
22	16.9 (6.2)	32.9 (13)	38	31.9 (13)	54	33.8 (13)
23	15.0 (5.6)	48.9 (12)	39	48.5 (12)	55	49.4 (11)
24	6.2 (2.7)	11.5 (6)	40	9.8 (6)	56	13.1 (6)
25	19.6 (7.6)	38.2 (16)	41	37.4 (16)	57	39.0 (15)
26	15.2 (6.3)	29.3 (13)	42	28.1 (13)	58	30.4 (13)
27	24.1 (6.1)	47.1 (13)	43	46.6 (13)	59	47.7 (12)
28	11.6 (4.5)	22.2 (10)	44	20.9 (10)	60	23.5 (9)
29	22.3 (6.4)	43.6 (13)	45	42.9 (14)	61	44.2 (13)
30	18.7 (5.7)	36.4 (12)	46	35.6 (12)	62	37.3 (12)
31	25.9 (5.0)	50.7 (11)	47	50.3 (11)	63	51.2 (10)

(3) $$P_n(a < b) = \sum_{k<i} P_{k_1,a}(i) P_{k_2,b}(j)\, u_{k,j}, n_1, n_2.$$

Finally consider $E_{n,a}(i \mid T_n)$; if $a = \omega$, $E_{n,a}(i \mid T_n) = 1$; if $a \,\varepsilon\, A_k$, consider the expected position of a given its position p within A_k; this is $p + p(2^n - 2^k - 1)/(2^k + 1) + 1$, where the first term is due to its position within A_k, the second term due to the players in $A - A_k - \{\omega\}$ we might expect to precede it, and the third term due to ω:

$$(4) \qquad E_{n,a}(i \mid T_n) = E(p + p(2^n - 2^k - 1)/(2^k + 1) \mid T_k),$$
$$E_{n,a}(i \mid T_n) = E_{k,a}(i \mid T_k)2^n/(2^k + 1) + 1.$$

The recursion formula for $E_{n,a}(i \mid T_n)$ is much simpler than those for the probabilities, which are very difficult to use for n large (about 7 or 8); it is possible to write the formula explicitly, (although it is easier to use (4) computationally); if a is less than or equal to players who appeared last in the $n_1 + 1, \cdots, n_k + 1$, rounds we have

$$\begin{aligned} E_{n,a}(i \mid T_n) = 1 &+ 2^n/(2^{n_{k-1}} + 1) \\ &+ 2^n \cdot 2^{n_{k-1}}/(2^{n_{k-1}} + 1)(2^{n_{k-2}} + 1) + \cdots \\ &+ 2^n \cdots 2^{n_2}/(2^{n_{k-1}} + 1) \cdots (2^{n_1} + 1). \end{aligned}$$

Finally let us state without proof a formula similar to (4) which will allow calculation of variances of final positions,

$$(5) \quad E_{n,a}(i(i - 1) \mid T_n) = E_{k,a}(i(i + 1) \mid T_k)2^n(2^n + 1)/(2^k + 1)(2^k + 2).$$

6. Estimated final rank. The estimated final rank for individuals in a tournament T_n, $0 \leqq n \leqq 6$, are given in Table 1; the individuals are identified by their record in the experiment which may be represented by the function $f_n(a) - 2^n$ taking values on $0, 1, \cdots, 2^n - 1$. The standard deviations of the final rank are also given; as a general rule, the standard deviations are of the same order of magnitude as the estimated final rank; thus those who do badly in the tournament have a higher standard deviation than those who do well; this reflects the nature of a knockout tournament, where we are principally interested in finding a winner, and so we can place those who win through to the last rounds much better than those knocked out in the early rounds.

A simple rule for deciding a final rank sets $a < b$ if a is knocked out in an earlier round than b, and if a and b are knocked out in the same round by a_2 and b_2 respectively, sets $a < b$ if $a_2 < b_2$. This rule is not obeyed by our estimated final rank; for example in the case of 16 people, compare the person who loses to the winner of the tournament on the first round, and the person who wins in the first round, but is then beaten by a person, who is beaten by a person, who is beaten by the winner. The first person has an expected rank of 9.0, the second person of 9.4; thus a person who lost in the first round is ranked above a person who won.

7. Applications and future developments. Probabilistic completion of partial orderings has application to the design and analysis of "paired comparison" experiments; suppose that we have a large number N of objects which must be ordered by some criterion; if each comparison is difficult, it is desirable to avoid making all $N(N - 1)/2$ comparisons; by accepting errors in the final rank, we may be able to design an experiment in which much fewer than $N(N - 1)/2$ comparisons are made. (The method is also applicable when all $N(N - 1)/2$ comparisons are attempted, but some comparisons can not be made, either because the objects are so close as to be indistinguishable, or so different as to be incomparable; in this case the final result is a partial ordering which may be analysed by probabilistic completion—for example it may be possible to decide between indistinguishability and incomparability. However analysis seems very difficult for large numbers of objects unless the partial order has some simple structure, a property which can be enforced in designing experiments, but not easily on general "paired comparison" data.) The experiment will in general be sequential, with each comparison selected on the basis of the results of previous ones; the result of the whole experiment will be a partial ordering on the objects, which may be probabilistically completed to give the estimated final rank, the probability that a is ranked above b, and other probability statements about the final rank. In general, we want the experiment to "define" the probability matrix $P(a < b)$ as much as possible, i.e. to make the matrix to consist of numbers near either 0 or 1; for example, we might try and make $S = \sum (P(a < b) - \frac{1}{2})^2$ as large as possible.

In a knockout-tournament, the main aim is to choose the best player, rather than to make comparisons between all players; and we have seen that the tournament does locate highly ranked players much better than lowly ranked ones. Considering the tournament as a sequential experiment, we see that at each stage "symmetrical" comparisons are selected between players a, b with $P(a < b) = \frac{1}{2}$, the probability being conditional on the results of previous experiments; for example in the first round, all players are equal in probability and the 2^{n-1} comparisons made are all symmetrical in the above sense; in the second round the winners of the first round are equal in probability, and the losers are equal in probability—we choose to make 2^{n-2} comparisons between the winners, and proceed similarly in later rounds, because the tournament is oriented towards finding winners.

The knockout tournament requires only $(2^n - 1)$ comparisons, which may be compared with the $2^{n-1}(2^n - 1)$ possible comparisons; it is interesting to note that a final estimated ordering which distinguishes all players may be made in fewer than $(2^n - 1)$ comparisons in some cases; for example with eight players 1 2 3 4 5 6 7 8, suppose we observe first $1 > 2, 3 > 4, 5 > 6, 7 > 8$, and then observe $1 > 3$; rather than compare (5, 7) as in a tournament, we compare (6, 8) and find $6 > 8$. The expected rankings are 1.8, 5.4, 4.2, 6.6, 2.4, 4.8, 3.6, 7.2; these are based on 6 comparisons against the 7 in a tournament; of course 5 would complain, as he had no chance of winning from the start.

It is possible to modify the tournament so that the information provided is less winner-oriented. Starting with the knockout tournament set up a relation R_p for the pth round by $(a, b) \varepsilon R_p$ if a played b in the previous rounds, and make the relation transitive so that $(a, b) \varepsilon R_p$, $(b, c) \varepsilon R_p$ implies $(a, c) \varepsilon R_p$, in the first round, 2^{n-1} pairs are chosen randomly from the 2^n players; in the second round, the 2^n players consist of 2^{n-1} winners and 2^{n-1} losers, and R_2 defines a correspondence between the two sets, since for each winner there is just one player (the person he beat) such that $(a, b) \varepsilon R_2$; the 2^{n-1} winners are split into 2^{n-2} pairs, producing new sets of 2^{n-2} winners and 2^{n-2} losers; using R_2 we make a corresponding division of the 2^{n-1} losers of the first round. The third round now consists of four sets of 2^{n-2} players, and R_3 defines a 1-1 correspondence between these four sets, which, after the winners of the third round have been decided, generates eight sets of the fourth round. Generally, the pth round of a knockout tournament may be regarded as 2^{p-1} sets of 2^{n-p+1} players, with a 1-1 correspondence between the sets defined by R_p ; all players within a particular set are equal in probability, and the 2^p sets of the $(p + 1)$th round are obtained by matching players who have won all their rounds so far. If we say a wins in the ith if $(a, b) \varepsilon R_i$ and b plays and wins in the pth round, and otherwise a loses on the ith round, the 2^{p-1} sets appearing in the pth round are sets of players with the same win-lose record. Of course, in a knockout tournament, only the set of 2^{n-p+1} winners need be considered in the pth round, but we plan to reduce the bias towards winners, by making comparisons in the pth round from one of the R_p-corresponding sets other than the winners $(p > 2)$. The pth round is played by those players who win on the $(p - 1)$th round, but have lost on all previous rounds ("win" and "lost" in the sense of R_p-correspondence). In this modified knockout scheme, all players except two play exactly twice; the total number of comparisons is of course $(2^n - 1)$.

For example, with sixteen players, the players in each round in a modified tournament are given below; the other R_p-corresponding sets are bracketed.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

[1 3 5 7 9 11 13 15] 2 4 6 8 10 12 14 16

[1 5 9 13] 3 7 11 15 [2 6 10 14][4 8 12 16]

[1 9] 5 13 [3 11][7 15][2 10][6 14][4 12][8 16]

Thus the third round is played by 3, 7, 11, 15 with 7 beating 3 and 15 beating 11. It seems very difficult to get a general recursion formula in the case of the modified knockout tournament but it will be interesting to compare the knockout tournament of size 8 with its modification. The knockout tournament would be

1 2 3 4 5 6 7 8

(1 3 5 7) 2 4 6 8

(1 5)(3 7)(2 6) 4 8

and the modified tournament would be

1 2 3 4 5 6 7 8

(1 3 5 7) 2 4 6 8

(1 5) 3 7 (2 6)(4 8).

Thus the only difference is that 3 plays 7 in the last round rather than 4 plays 8.

KNOCKOUT TOURNAMENT

Player.........	1	2	3	4	5	6	7	8
Exp. Rank	6.9	4.7	5.8	2.6	6.3	3.7	5.0	1.0
S.D. Rank	1.1	1.2	1.6	0.8	1.5	1.5	2.0	0.0

MODIFIED KNOCKOUT TOURNAMENT

Player.........	1	2	3	4	5	6	7	8
Exp. Rank	6.7	4.4	6.7	2.1	6.5	4.0	4.1	1.5
S.D. Rank	1.2	1.5	1.2	1.2	1.4	1.5	1.4	0.7

It is apparent that the S.D.'s are more uniform in the second case, and that the winning players 8 and 4 are placed much less accurately. Finally, let us consider a more general form of knockout tournament, with an arbitary number of players; the only restriction on the form of the tournament is that once a player loses, he plays no more. Examples of such tournaments are (1) the knockout tournament for 2^n players we have been considering (2) tournaments where the total number of players is not a power of 2, so that "byes" occur in some rounds (3) tournaments with star players who only enter in the later rounds. These general tournaments need have no unique winner, but it is convenient to introduce a player 0 who plays at the end of the tournament and beats the remaining players; in the terminology of graph theory [1], the results of the tournament may be represented by a "rooted tree". Recursion formulae for expected final ranks may be obtained for these general knockout tournaments; let $f(a)$ be the player who knocks out a, and let n_a be the number of players ranked less than or equal to a by the partial ordering; then

$$E(a) - E(f(a)) = E(f(a)) - E(f^2(a))n_{f(a)}/(n_a + 1),$$

where

$$E(0) = 0, \qquad E(f(0)) = -1.$$

This formula may be established by techniques similar to those used earlier.

It should be noted that throughout we have taken the attitude that the result of any particular match is not a random event; if player a is superior to player b he will always win the match. Probability enters only through our ignorance of the complete order of the players. We can afford this attitude in any generalised knockout tournament, because once a player is beaten he is removed from the contest: thus intransitivities, which might make the existence of an underlying order doubtful, are not allowed to appear.

REFERENCES

[1] BUSACKER, R. G. and SAATY, T. L. (1965). *Finite Graphs and Networks*. McGraw-Hill, New York.

[2] WILKS, S. S. (1963). *Mathematical Statistics*. Wiley, New York.