ON CERTAIN DISTRIBUTION PROBLEMS BASED ON POSITIVE DEFINITE QUADRATIC FUNCTIONS IN NORMAL VECTORS

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1. Introduction and summary. Let $X:p \times n$ be a matrix of random real variates such that the column vectors of X are independently and identically distributed as multivariate normals with zero mean vectors. Then a positive definite quadratic function in normal vectors is defined as XLX' where L is a symmetric positive definite (p.d.) matrix with real elements. In the analysis of variance, such functions appear. In the previous study, Khatri [14], [16], has established the necessary and sufficient conditions for the independence and the Wishartness of such functions. In this paper, we study the distribution of a positive definite quadratic function and the distribution of $Y'(XLX')^{-1}Y$ where $Y:p \times m$ is independently distributed of X and its columns are independently and identically distributed as multivariate normals with zero mean vectors. Moreover, we study the distribution of the characteristic (ch.) roots of $(YY')(XLX')^{-1}$ and the similar related problems. When $p = 1$, the distribution of a p.d. quadratic function in normal variates (central or noncentral) has been studied by a number of people (see references).

In the study of the above and related topics in multivariate distribution theory, we are using zonal polynomials. A. T. James [10], [11], [12], [13], and Constantine [1], [2], have used them successfully and have given the final results in a very compact form, using hypergeometric functions ${}_pF_q(S)$ in matrix arguments. These functions are defined by

$$(1)\quad {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; Z) = \sum_{k=0}^{\infty} \sum_{\kappa} [(a_1)_\kappa \cdots (a_p)_\kappa/(b_1)_\kappa \cdots (b_q)_\kappa][C_\kappa(Z)/k!]$$

where $C_\kappa(Z)$ is a symmetric homogeneous polynomial of degree k in the latent roots of Z, called zonal polynomials (for more detail study of zonal polynomials, see the references of A. T. James and Constantine), $\kappa = (k_1, \cdots, k_p)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_p \geqq 0$, $k_1 + k_2 + \cdots + k_p = k$; $a_1, \cdots, a_p, b_1, \cdots, b_q$ are real or complex constants, none of the b_j is an integer or half integer $\leqq \frac{1}{2}(m-1)$ (otherwise some of the denominators in (1) will vanish),

$$(2)\quad (a)_\kappa = \prod_{j=1}^{m} (a - \tfrac{1}{2}(j-1))_{k_j} = \Gamma_m(a, \kappa)/\Gamma_m(a),$$
$$(x)_n = x(x+1) \cdots (x+n-1),\ (x)_0 = 1$$

and

$$(3)\quad \Gamma_m(a) = \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^{m} \Gamma(a - \tfrac{1}{2}(j-1))$$
$$\text{and}\quad \Gamma_m(a, \kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^{m} \Gamma(a + k_j - \tfrac{1}{2}(j-1)).$$

Received 21 June 1965.

In (1), Z is a complex symmetric $m \times m$ matrix, and it is assumed that $p \leqq q + 1$, otherwise the series may converge for $Z = 0$. For $p = q + 1$, the series converge for $\|Z\| < 1$, where $\|Z\|$ denote the maximum of the absolute value of ch. roots of Z. For $p \leqq q$, the series converge for all Z. Similarly we define

$$(2b)\quad {}_pF_q^{(m)}(a_1, a_2, \cdots, a_p; b_1, \cdots, b_q; S, R) = \sum_{k=0}^{\infty} \sum_{\kappa} [(a_1)_\kappa \cdots (a_p)_\kappa/(b_1)_\kappa \cdots (b_q)_\kappa][C_\kappa(S)C_\kappa(R)/C_\kappa(I_m)k!].$$

The Section 2 gives some results on integration with the help of zonal polynomials, the Section 3 derives the distributions based on p.d. quadratic functions, the Section 4 gives the moments of certain statistics arising in the study of multivariate distributions, and the Section 5 gives the results for complex multivariate Gaussian variates.

2. Some results on integration. We shall write $X > X_0$ for $X - X_0$ to be p.d., $O(m)$ for orthogonal group of $m \times m$ orthogonal matrices and $C_\kappa(X)$ for a zonal polynomial of degree k. $R(Z)$ means the real part of Z. We shall denote

$$(4)\quad \Gamma_m(t, -\kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^{m} \Gamma(t - k_j - \tfrac{1}{2}m + \tfrac{1}{2}j) \text{ and } R(t) > \tfrac{1}{2}(m-1) + k_1 .$$

LEMMA 1. *Let $S:m \times m$ and $T:m \times m$ be symmetric matrices. Then*

$$(5)\quad \int_{O(m)} C_\kappa(SHTH')\, dH = C_\kappa(S)C_\kappa(T)/C_\kappa(I_m).$$

(See A. T. James [10].)

LEMMA 2. *Let $Z:m \times m$ be a complex symmetric matrix whose real part is* p.d. *and let $T:m \times m$ be an arbitrary complex symmetric matrix. Then*

$$(6)\quad \int_{S>0} \exp(-\mathrm{tr}ZS)|S|^{t-\frac{1}{2}(m+1)} C_\kappa(TS)\, dS = \Gamma_m(t, \kappa)|Z|^{-t} C_\kappa(TZ^{-1})$$

where $\Gamma_m(t, \kappa)$ is defined in (3) *and $R(t) > \frac{1}{2}(m-1)$.* (See Constantine [1].)

LEMMA 3. *If R is any* p.d. *$m \times m$ matrix, then*

$$(7)\quad \int_0^I |S|^{t-\frac{1}{2}m-\frac{1}{2}} |I - S|^{u-\frac{1}{2}m-\frac{1}{2}} C_\kappa(RS)\, dS = \Gamma_m(t, \kappa)\Gamma_m(u)C_\kappa(R)/\Gamma_m(t + u, \kappa).$$

(See Constantine [1].)

LEMMA 4. *Let Z be a complex symmetric matrix such that $R(Z) > 0$, and let T be an arbitrary complex symmetric matrix. Then.*

$$(8)\quad \int_{S>0} \exp(-\mathrm{tr}ZS)|S|^{t-\frac{1}{2}(m+1)} C_\kappa(TS^{-1})\, dS = \Gamma_m(t, -\kappa)|Z|^{-t} C_\kappa(ZT),$$

where $R(t) > \frac{1}{2}(m-1) + k_1$ and $\Gamma_m(t, -\kappa)$ is defined by (4).

PROOF. First, we shall prove the result for the special case $Z = I_m$, the $m \times m$ identity matrix. Put

$$(9)\quad f(T) = \int_{S>0} \exp(-\mathrm{tr}S)|S|^{t-\frac{1}{2}(m+1)} C_\kappa(TS^{-1})\, dS.$$

Then $f(T)$ is clearly a symmetric function of T (in fact, a homogeneous symmetric polynomial). Hence, making the transformation $T \to H'TH$ and integrating H over $O(m)$ with the help of (5), we have

$$(10)\quad f(T) = [f(I)/C_\kappa(I)]C_\kappa(T).$$

To find $f(I)/C_\kappa(I)$, let us assume that T is diagonal. Then Constantine [1] has showed that

$$C_\kappa(T) = d_{\kappa,\kappa} t_1^{k_1} t_2^{k_2} \cdots t_m^{k_m} + \text{"lower terms"}$$

and

$$C_\kappa(TS^{-1}) = d_{\kappa,\kappa} \prod_{j=1}^m t_j^{k_j} |(S^{-1})_j|^{k_j - k_{j+1}} + \cdots$$

with $k_{m+1} = 0$ and $(S^{-1})_j = (s^{uv})$, $u, v, = 1, 2, \cdots, j$. Then using these results in (10) and comparing the coefficients of $t_1^{k_1} \cdots t_m^{k_m}$ from both sides, we get

$$(11) \quad f(I)/C_\kappa(I) = \int_{S>0} \exp(-\operatorname{tr} S) |S|^{t - \frac{1}{2}(m+1)} \prod_{j=1}^m |(S^{-1})_j|^{k_j - k_{j+1}} \, dS.$$

Let $S = VV'$ where V is an upper triangular matrix. The Jacobian of the transformation is $2^m \prod_{j=1}^m v_{jj}^j$, and $|(S^{-1})_j| = \prod_{a=1}^j v_{aa}^{-2}$. Then using these in (11), we get

$$(12) \quad \begin{aligned} f(I)/C_\kappa(I) &= \int \cdots \int \exp\left(-\sum_{a=1}^m \sum_{j=a}^m v_{aj}^2\right) \prod_{j=1}^m (v_{jj}^2)^{t - k_j - \frac{1}{2}(m-j) - 1} \\ &\quad \cdot \prod_{j=1}^m dv_{jj}^2 \prod_{a<j} dv_{aj} \\ &= \pi^{\frac{1}{4}(m-1)m} \prod_{j=1}^m \Gamma(t - k_j - \tfrac{1}{2}m + \tfrac{1}{2}j) = \Gamma_m(t, -\kappa), \end{aligned}$$

the range of integration being $0 \leqq v_{aa} \leqq \infty$, $-\infty \leqq v_{aj} (a < j) \leqq \infty$. For the general case, substitute $Z^{\frac{1}{2}} S Z^{\frac{1}{2}}$ for S in $f(T)$ with the Jacobian of the transformation $|Z|^{\frac{1}{2}(m+1)}$

LEMMA 5. *If R is any arbitrary symmetric complex $m \times m$ matrix, then*

$$(13) \quad \int_{S>0} |S|^{t - \frac{1}{2}(m+1)} |I + S|^{-t-u} C_\kappa(RS) \, dS = \Gamma_m(t, \kappa) \Gamma_m(u, -\kappa) C_\kappa(R) / \Gamma_m(t+u)$$

and

$$(14) \quad \int_{S>0} |S|^{t - \frac{1}{2}(m+1)} |I + S|^{-t-u} C_\kappa(RS^{-1}) \, dS = \Gamma_m(t, -\kappa) \Gamma_m(u, \kappa) C_\kappa(R) / \Gamma_m(t+u).$$

PROOF. By (6), we have for any p.d. matrix Z

$$(15) \quad \int_{S>0} \exp(-\operatorname{tr} ZS) |S|^{t - \frac{1}{2}(m+1)} C_\kappa(RS) |Z|^t \, dS = \Gamma_m(t, \kappa) C_\kappa(RZ^{-1}).$$

Multiplying both the sides of (15) by $\exp(-\operatorname{tr} Z) |Z|^{u - \frac{1}{2}(m+1)}$ and integrating over $Z > 0$, we get

$$(16) \quad \begin{aligned} \int_{S>0} \Gamma_m(t+u) |S|^{t - \frac{1}{2}(m+1)} |I + S|^{-t-u} C_\kappa(RS) \, dS \\ = \Gamma_m(t, \kappa) \int_{Z>0} \exp(-\operatorname{tr} Z) |Z|^{u - \frac{1}{2}m - \frac{1}{2}} C_\kappa(RZ^{-1}) \, dZ \end{aligned}$$

and now the use of (8) on the right side of (16) gives (13). If we transform S to S^{-1} in (13), then t and u will be interchanged and we shall get (14).

LEMMA 6. *Let R be a* p.d. *matrix. Then for $t \geqq m/2 + \frac{1}{2}k_1$,*

$$(17) \quad \int_0^I |S|^{t - \frac{1}{2}(m+1)} |I - S|^{u - \frac{1}{2}(m+1)} C_\kappa(RS^{-1}) \, dS = \Gamma_m(t, -\kappa) \Gamma_m(u) C_\kappa(R) / \Gamma_m(t+u, -\kappa).$$

PROOF. The left hand side of (17) is a symmetric function $F(R)$ of R, so that, as in the proof of Lemma 4,

$$F(R) = [F(I)/C_\kappa(I)]C_\kappa(R). \tag{18}$$

For obtaining $F(I)/C_\kappa(I)$, we note that

$$\int_{S>0} \exp(-\mathrm{tr}ZS)|S|^{t-\frac{1}{2}(m+1)}C_\kappa(Z^{-1}S^{-1})\, dS\, |Z|^t = \Gamma_m(t, -\kappa)C_\kappa(I).$$

Multiplying the expression by $|Z|^{u-\frac{1}{2}(m+1)} \exp(-\mathrm{tr}Z)$, integrating over $Z > 0$ and transforming $(S^{-1} + I)^{-1}$ to S, we get

$$\int_0^I |S|^{t-\frac{1}{2}(m+1)}|I - S|^{u-\frac{1}{2}(m+1)}C_\kappa(S^{-1})\, dS\Gamma_m(t + u, -\kappa) = \Gamma_m(u)\Gamma_m(t, -\kappa)C_\kappa(I),$$

which gives the expression for $F(I)/C_\kappa(I)$. The use of this result in (18) proves (17).

LEMMA 7. *Let T be any arbitrary complex symmetric matrix. Then*

$$\int_{S>0} \exp(-\mathrm{tr}S)|S|^{t-\frac{1}{2}(m+1)}(\mathrm{tr}S)^jC_\kappa(TS)\, dS = \Gamma_m(t, \kappa)\Gamma(mt + j + k)C_\kappa(T)/\Gamma(mt + k) \tag{19}$$

while

$$\int_{S>0} \exp(-\mathrm{tr}S)|S|^{t-\frac{1}{2}(m+1)}(\mathrm{tr}S)^jC_\kappa(TS^{-1})\, dS = \Gamma_m(t, -\kappa)\Gamma(mt + j - k)C_\kappa(T)/\Gamma(mt - k). \tag{20}$$

PROOF. We shall only prove the result (19). We have by (6) for $q < 1$,

$$\int_{S>0} \exp(-\mathrm{tr}S(1 - q))|S|^{t-\frac{1}{2}(m+1)}C_\kappa(TS)\, dS = (1 - q)^{-tm-k}\Gamma_m(t, \kappa)C_\kappa(T). \tag{21}$$

Equating the coefficient of $q^j/j!$ from both the sides of (21), we get (19). (20) can be proved in the same way.

LEMMA 8. *Let B be symmetric matrix of order $n \times n$ and let A be a* p.d. *symmetric $p \times p$ matrix with $n \geqq p$. Let $X:p \times n$ be a real matrix. Then*

$$\int_{XX'=S} \exp(\mathrm{tr}AXBX')\, dX = (\pi)^{\frac{1}{2}pn}\{\Gamma_p(\tfrac{1}{2}n)\}^{-1}|S|^{\frac{1}{2}(n-p-1)}\, {}_0F_0^{(n)}(AS, B). \tag{22}$$

PROOF. Since $\mathrm{tr}(AXBX')$ can be written as the function of a symmetric matrix, it is easy to see from the results of A. T. James [12], [13], and Constantine [1] that

$$\exp(\mathrm{tr}AXBX') = {}_0F_0(X'AXB). \tag{23}$$

Now let

$$g(B) = \int_{XX'=S} {}_0F_0(X'AXB)\, dX. \tag{24}$$

Since $g(B)$ is a homogeneous symmetric function in B, by making the transformation $B \to HBH'$ and integrating H over $O(n)$ with the help of (5), we get

$$g(B) = \int_{XX'=S} {}_0F_0^{(n)}(AXX', B)\, dX = \pi^{\frac{1}{2}pn}|S|^{\frac{1}{2}(n-p-1)}\{\Gamma_p(\tfrac{1}{2}n)\}^{-1}\, {}_0F_0^{(n)}(AS, B), \tag{25}$$

using Wishart's integral. This proves (22).

3. Distributions related to the p.d. quadratic functions.

THEOREM 1. *Let $X:p \times n$ be a real random matrix whose density function is*

$$(26)\qquad (2\pi)^{-\frac{1}{2}pn} |\Sigma|^{-\frac{1}{2}n} |B|^{-\frac{1}{2}p} \exp(-\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}XB^{-1}X')$$

where $\Sigma:p \times p$ and $B:n \times n$ are p.d. *Then the density function of $S = XLX'$, L being a $n \times n$* p.d. *matrix, is*

$$(27)\qquad 2^{-\frac{1}{2}pn}\{\Gamma_p(\tfrac{1}{2}n)\}^{-1} |LB|^{-\frac{1}{2}p} |\Sigma|^{-\frac{1}{2}n} |S|^{\frac{1}{2}(n-p-1)} \cdot\exp(-\tfrac{1}{2}q^{-1}\operatorname{tr}\Sigma^{-1}S)\, {}_0F_0^{(n)}(T, \tfrac{1}{2}q^{-1}\Sigma^{-1}S)$$

where $q > 0$ and $T = I_n - qL^{-\frac{1}{2}}B^{-1}L^{-\frac{1}{2}}$.

PROOF. Let us use the transformation $Y = XL^{\frac{1}{2}}$ in (26). Then the density of Y is

$$(2\pi)^{-\frac{1}{2}pn} |\Sigma|^{-\frac{1}{2}n} |LB|^{-\frac{1}{2}p} \exp(-\tfrac{1}{2}q^{-1}\operatorname{tr}\Sigma^{-1}YY' + \tfrac{1}{2}q^{-1}\operatorname{tr}\Sigma^{-1}YTY')$$

and $S = YY'$. Now, the use of (22) gives (27).

THEOREM 2. *The moment generating function of S defined in Theorem 1 is*

$$(28)\qquad E\{\exp(\operatorname{tr}ZS)\} = |LBq^{-1}|^{-\frac{1}{2}p} |z|^{-\frac{1}{2}n}\{{}_1F_0^{(n)}(\tfrac{1}{2}n; T, z^{-1}) = \textstyle\prod_{j=1}^{n} |I_p - \phi_j z^{-1}|^{-\frac{1}{2}}\}$$

where ϕ_j's are the ch. *roots of T, $z = I_p - 2qZ\Sigma$ and E stands for expectation.*

PROOF. The first part follows from (27) with the help of (6). For the second part, we can write after a transformation $X \to \Sigma^{\frac{1}{2}}X(Lq)^{-\frac{1}{2}}$ in $E\exp(\operatorname{tr}ZXLX')$ as

$$(29)\qquad E(\exp(\operatorname{tr}ZS)) = (2\pi)^{-\frac{1}{2}pn} |LBq^{-1}|^{-\frac{1}{2}p} \int_X \exp(\tfrac{1}{2}\operatorname{tr}XTX' - \tfrac{1}{2}\operatorname{tr}zXX')\, dX.$$

Since XX' is invariant under post multiplication of X by an orthogonal matrix, we can consider T to be a diagonal matrix with ϕ_j's as diagonal elements. Let $X = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$. Then (29) can be rewritten as

$$E\exp(\operatorname{tr}ZS) = |LBq^{-1}|^{-\frac{1}{2}p}[\textstyle\prod_{j=1}^{p}\{(2\pi)^{-\frac{1}{2}p}\int_{\mathbf{x}_j}\exp[-\tfrac{1}{2}\mathbf{x}_j'(z - \phi_j I_p)\mathbf{x}_j]\, d\mathbf{x}_j\}]$$

and this gives the second part of (28).

THEOREM 3. *Let $X:p \times n$ and $Y:p \times m$ be independently distributed, the density function of X be given by (26) and the density function of Y be given by*

$$(2\pi)^{-\frac{1}{2}pm} |\Sigma|^{-\frac{1}{2}m} \exp(-\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}YY').$$

Then, the density function of $F = Y'(XLX')^{-1}Y$ if $m \leq p \leq n$ is given by

$$(30)\qquad \Gamma_p(\tfrac{1}{2}m + \tfrac{1}{2}n)\{\Gamma_p(\tfrac{1}{2}n)\Gamma_m(\tfrac{1}{2}p)\}^{-1}q^{\frac{1}{2}p(m+n)}|BL|^{-\frac{1}{2}p} |I_m + qF|^{-\frac{1}{2}(m+n)} \cdot|F|^{\frac{1}{2}(p-m-1)}\, {}_1F_0^{(p)}(\tfrac{1}{2}m + \tfrac{1}{2}n; T, R^*)$$

where $q > 0$, $R^* = \begin{pmatrix} (I_m + qF)^{-1} & 0 \\ 0 & I_{p-m} \end{pmatrix}$ *and* $T = I_n - q(LB)^{-1}$.

PROOF. Since F is invariant under the transformation $X \to \Sigma^{\frac{1}{2}}X$ and $Y \to \Sigma^{\frac{1}{2}}Y$, we shall have $\Sigma \to I$ and hence the joint density function of $Z = S^{-\frac{1}{2}}Y : p \times m$ and $S = XLX'$ with the help of (26) is

$$(31) \quad \{2^{\frac{1}{2}p(m+n)}\Gamma_p(\tfrac{1}{2}n)|LB|^{\frac{1}{2}p}\pi^{\frac{1}{2}pm}\}^{-1} |S|^{\frac{1}{2}(m+n-p-1)} \cdot \exp(-\tfrac{1}{2}\operatorname{tr}(q^{-1}I_p + ZZ')S)\, {}_0F_0^{(n)}(T, \tfrac{1}{2}q^{-1}S).$$

Integrating S and noting

$$C_\kappa(I_p + qZZ')^{-1} = C_\kappa\begin{pmatrix}(I_m + qZ'Z)^{-1} & 0\\ 0 & I_{p-m}\end{pmatrix} = C_\kappa(R^*), \qquad \text{(say)},$$

we can write the density function of $Z : p \times m$ as

$$(32) \quad \{\Gamma_p(\tfrac{1}{2}n)|LB|^{\frac{1}{2}p}\pi^{\frac{1}{2}pm}\}^{-1} \cdot \Gamma_p(\tfrac{1}{2}m + \tfrac{1}{2}n)|I_m q^{-1} + Z'Z|^{-\frac{1}{2}(m+n)}\, {}_1F_0^{(n)}(\tfrac{1}{2}m + \tfrac{1}{2}n, T, R^*).$$

Since $p \geqq m$, we use the Wishart's integral and obtain the density function of $F = Z'Z$, which is given by (30).

THEOREM 4. *Let $X : p \times n$ and $Y : p \times m$ be independently distributed, the density function of X be given by* (26) *and the density function of Y be given by*

$$(2\pi)^{-\frac{1}{2}pm} |\Sigma_1|^{-\frac{1}{2}m} \exp(-\tfrac{1}{2}\operatorname{tr}\Sigma_1^{-1}YY').$$

Then the density function of $F_1 = X'(YY')^{-1}X$ when $n \leqq p \leqq m$ is given by

$$(33) \quad c\,|\Omega|^{-\frac{1}{2}n} |B|^{-\frac{1}{2}p} |F_1|^{\frac{1}{2}(p-n-1)} \cdot |I_n + (qB)^{-1}F_1|^{-\frac{1}{2}(m+n)}\, {}_1F_0^{(p)}(\tfrac{1}{2}m + \tfrac{1}{2}n; \Omega^*, F_1(Bq + F_1)^{-1})$$

where $\Omega^* = I_p - q\Omega^{-1}$, $\Omega = \Sigma^{\frac{1}{2}}\Sigma_1^{-1}\Sigma^{\frac{1}{2}}$

and $c = \Gamma_p(\tfrac{1}{2}m + \tfrac{1}{2}n)\{\Gamma_p(\tfrac{1}{2}m)\Gamma_n(\tfrac{1}{2}p)\}^{-1}$, $q > 0$.

PROOF. Since F_1 is invariant under $X \to \Sigma^{\frac{1}{2}}X$ and $Y \to \Sigma^{\frac{1}{2}}Y$. Then $\Sigma \to I$, $\Sigma_1^{-1} \to \Omega$. Now in the joint density function of X and Y which is given by

$$(2\pi)^{-\frac{1}{2}p(m+n)}|\Omega|^{\frac{1}{2}m} |B|^{-\frac{1}{2}p} \exp(-\tfrac{1}{2}\operatorname{tr}XB^{-1}X' - \tfrac{1}{2}\operatorname{tr}\Omega YY'),$$

we transform X to Z by $Z = (YY')^{-\frac{1}{2}}X$, and then integrating with respect to Y, we get the density function of Z as

$$(34) \qquad \pi^{-\frac{1}{2}pm} |\Omega|^{\frac{1}{2}m} |B|^{-\frac{1}{2}p}\Gamma_p(\tfrac{1}{2}m + \tfrac{1}{2}n)\{\Gamma_p(\tfrac{1}{2}m)\}^{-1} |\Omega + ZB^{-1}Z'|^{-\frac{1}{2}(m+n)}.$$

We note that $F_1 = Z'Z$ and if $\Omega^* = I_p - q\Omega^{-1}$, $q > 0$, then

$$|\Omega + ZB^{-1}Z'| = |\Omega|\, |B|^{-1} |B + Z'Zq^{-1}|\, |\mathbf{I}_p - Z(Bq + Z'Z)^{-1}Z'\Omega^*|.$$

Now, integrating Z such that $F_1 = Z'Z$ is fixed, we get the density function of F_1 as

$$(35) \quad \pi^{-\frac{1}{2}pm} |\Omega|^{-\frac{1}{2}n} |B|^{-\frac{1}{2}p} \cdot \int_{F_1 = Z'Z} \{|\mathbf{I}_n + (Bq)^{-1}Z'Z|\, |\mathbf{I}_p - Z(Bq + Z'Z)^{-1}Z'\Omega^*|\}^{-\frac{1}{2}(m+n)}\, dZ.$$

Since the integral in (35) is symmetric and homogeneous in (Ω^*), we get after using the method applied in the proof of Lemma 4 the density function of F_1 as mentioned in (33).

We make below few remarks or notes when $m \geqq p$ and $n \geqq p$ in the two theorems.

NOTE 1. In Theorem 4, when $m \geqq p$ and $n \geqq p$, the density function of $F_2 = (YY')^{-\frac{1}{2}}XX'(YY')^{-\frac{1}{2}}$ is given by

$$(36) \quad c_1 |B|^{-\frac{1}{2}p} |\Omega|^{-\frac{1}{2}n} |F_2|^{\frac{1}{2}(n-p-1)} \cdot |I_p + (q\Omega)^{-1}F_2|^{-\frac{1}{2}(m+n)} {}_1F_0^{(n)}(\tfrac{1}{2}m + \tfrac{1}{2}n; T, F_2(q\Omega + F_2)^{-1})$$

where $T = I_n - qB^{-1}$, $q > 0$, and $c_1 = \Gamma_p(\frac{1}{2}m + \frac{1}{2}n)\{\Gamma_p(\frac{1}{2}m)\Gamma_p(\frac{1}{2}n)\}^{-1}$.

NOTE 2. In Theorem 4, when $m \geqq p$ and $n \geqq p$, the density function of $F_3 = (XX')^{-\frac{1}{2}}YY'(XX')^{-\frac{1}{2}}$ can be obtained from (36) by the relation of transformation $F_2 \to F_3^{-1}$. This means that the density functions of F_3 and $F_4 = (YY')^{\frac{1}{2}}(XX')^{-1}(YY')^{\frac{1}{2}}$ are identical.

NOTE 3. When $q \to \infty$ in (33) and (36), we get the density functions of F_1 and F_2 as

$$(37) \qquad c |\Omega|^{-\frac{1}{2}n} |B|^{-\frac{1}{2}p} |F_1|^{\frac{1}{2}(p-n-1)} {}_1F_0^{(p)}(\tfrac{1}{2}m + \tfrac{1}{2}n; -\Omega^{-1}, F_1B^{-1})$$

and

$$(38) \qquad c_1 |\Omega|^{-\frac{1}{2}n} |B|^{-\frac{1}{2}p} |F_2|^{\frac{1}{2}(n-p-1)} {}_1F_0^{(n)}(\tfrac{1}{2}m + \tfrac{1}{2}n; -B^{-1}, F_2\Omega^{-1})$$

where c and c_1 are constants defined in (33) and (36) respectively. From the expressions (37) and (38), we can obtain the density functions of the ch. roots of F_2 and F_3. We may also note that when $B = I_n$ in (33), we can explicitly write down the density function of the ch. roots of F_2 when $m \geqq p$ and $n \leqq p$ which will converge rapidly by choosing q, while that of F_3 can be obtained from (36) when $n \geqq p$ and $m \geqq p$ and $\Omega = I_p$, and the density function will depend on B. Those derivable from (37) and (38) do not require such conditions.

4. Moments of certain statistics. (a) Let $X: p \times n$ be distributed as normal whose density function is given by (26). Then, for any symmetric matrix Z, and for ℓ_j's being the ch. roots of LB, we have by Theorem 2,

$$(39) \quad E \exp(\operatorname{tr} ZXLX') = \prod_{j=1}^{n} |I_p - 2\ell_j Z\Sigma|^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \sum_{\kappa} (\tfrac{1}{2}n)_\kappa C_\kappa(LB) C_\kappa(Z\Sigma) 2^k / k!\, C_\kappa(I_n).$$

Now, we note that the density function of $S = X'LX$ given in (27) can be rewritten as

$$(40) \quad 2^{-\frac{1}{2}pn}\{\Gamma_p(\tfrac{1}{2}n)\}^{-1} |\Sigma|^{-\frac{1}{2}n} |Q|^{\frac{1}{2}p} \int_{O(n)} |S|^{\frac{1}{2}(n-p-1)} \exp(-\tfrac{1}{2} \operatorname{tr} \Sigma^{-\frac{1}{2}} H_1 Q H_1' \Sigma^{-\frac{1}{2}} S)\, dH$$

where $H' = (H_1' \; H_2')$ is an $n \times n$ orthogonal matrix with H_1 and H_2 of dimension $p \times n$ and $(n - p) \times n$, respectively, and $Q^{-1} = L^{\frac{1}{2}}BL^{\frac{1}{2}}$. Using this in finding $E \exp(\operatorname{tr} ZS)$ and interchanging the integration signs, we get

$$E \exp(\mathrm{tr}\, ZXLX') = |Q|^{\frac{1}{2}p} \int_{O(n)} |H_1QH_1'|^{-\frac{1}{2}n}$$

$$(41)\quad \cdot|I_p - 2(\Sigma^{\frac{1}{2}}Z\Sigma^{\frac{1}{2}})(H_1QH_1')^{-1}|^{-\frac{1}{2}n}\, dH = \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} (\frac{1}{2}n)_{\kappa} 2^k (k!)^{-1} |Q|^{\frac{1}{2}p}$$

$$\cdot \int_{O(n)} |H_1QH_1'|^{-\frac{1}{2}n} C_{\kappa}\{(\Sigma^{\frac{1}{2}}Z\Sigma^{\frac{1}{2}})(H_1QH_1')^{-1}\}\, dH.$$

Since this is homogeneous and symmetric function in $\Sigma^{\frac{1}{2}}Z\Sigma^{\frac{1}{2}}$, we have as is the proof of Lemma 4,

$$(42)\quad E \exp(\mathrm{tr}\, ZXLX') = \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} (\frac{1}{2}n)_{\kappa} 2^k (k!)^{-1} C_{\kappa}(Z\Sigma)\{C_{\kappa}(I_p)\}^{-1} |Q|^{\frac{1}{2}p}$$

$$\int_{O(n)} |H_1QH_1'|^{-\frac{1}{2}n} C_{\kappa}(H_1QH_1')^{-1}\, dH.$$

Now (42) and (39) must be identical in Z and hence comparing the coefficients of $C_{\kappa}(Z\Sigma)$, we must have

$$(43)\quad |Q|^{\frac{1}{2}p} \int_{O(n)} |H_1QH_1'|^{-\frac{1}{2}n} C_{\kappa}(H_1QH_1')^{-1}\, dH = C_{\kappa}(Q^{-1})C_{\kappa}(I_p)/C_{\kappa}(I_n),$$

where $H' = (H_1'\ H_2')$ is as defined in (40). Now with the help of (43), using the density function of S as given in (40), we get

$$(44)\qquad EC_{\kappa}(ZS) = 2^k(\tfrac{1}{2}n)_{\kappa} C_{\kappa}(LB)C_{\kappa}(Z\Sigma)/C_{\kappa}(I_n).$$

The answer was given by Constantine [1] when $LB = I_n$. Further, when $LB = I_n$, then $S = X'LX$ is distributed as Wishart whose density function is $W(S; \frac{1}{2}n; \Sigma)$. Hence using (8), we have

$$(45)\quad EC_{\kappa}(ZS^{-1}) = \Gamma_p(\tfrac{1}{2}n, -\kappa)\{\Gamma_p(\tfrac{1}{2}n)\}^{-1} 2^{-k} C_{\kappa}(Z\Sigma^{-1}), \text{ if } n > (p-1) + k_1.$$

(b) When the distribution of $S = XLX'$ is given by (27), then it is easy to show that

$$(46)\quad E\,|S|^k = q^{p(\frac{1}{2}n+k)} 2^{pk} \Gamma_p(\tfrac{1}{2}n + k)\{\Gamma_p(\tfrac{1}{2}n)\}^{-1} |LB|^{-\frac{1}{2}p} |\Sigma|^k\, {}_1F_0^{(n)}(\tfrac{1}{2}n + k; T, I_p).$$

Now when $LB = I_n$, we get the well-known result

$$E\,|S|^k = (2^{pk})\Gamma_p(\tfrac{1}{2}n + k)|\Sigma|^k/\Gamma_p(\tfrac{1}{2}n).$$

In (36) under the condition $\Omega = I_p$ or $\Sigma_1 = \Sigma_2$, we shall choose $q = 1$ in order to obtain the jth moment of $|S|/|YY' + S|$ and that of $|YY'|/|S + YY'|$ are given by

$$E\{|S|/|S + YY'|\}^j \Gamma_p(\tfrac{1}{2}n)/\Gamma_p(\tfrac{1}{2}n + j)$$

$$(47)\qquad = E\{|YY'|/|S + YY'|\}^j \Gamma_p(\tfrac{1}{2}m)/\Gamma_p(\tfrac{1}{2}m + j)$$

$$= \Gamma_p(\tfrac{1}{2}n + \tfrac{1}{2}m)\{\Gamma_p(\tfrac{1}{2}m + \tfrac{1}{2}n + j)\}^{-1}\, {}_2F_1(\tfrac{1}{2}p, j; \tfrac{1}{2}m + \tfrac{1}{2}n + j; I_n - BL).$$

When $BL = I_n$, we get the well-known results for the moments of the likelihood ratio statistics. Hence the distribution of $|YY'|/|S + YY'|$ is obtained from that of $|S|/|S + YY'|$ by interchanging n and m.

(c) If the density function of $V{:}p \times p$ is given by

$$(48)\qquad \text{constant}\, |V|^{t-\frac{1}{2}(m+1)}\, |I_p + V|^{-t-u},$$

then from (13), (14), (7), and (17), we have

$$EC_\kappa(ZV) = (t)_\kappa \Gamma_p(u, -\kappa)C_\kappa(Z)/\Gamma_p(u) \qquad \text{if} \quad u \geqq p + k_1,$$

$$EC_\kappa(ZV^{-1}) = (u)_\kappa \Gamma_p(t, -\kappa)C_\kappa(Z)/\Gamma_p(t) \qquad \text{if} \quad t \geqq p + k_1,$$

$$(49) \quad EC_\kappa(Z(V^{-1} + I_p)^{-1}) = (t)_\kappa C_\kappa(Z)/(t+u)_\kappa,$$

$$EC_\kappa(Z(V + I_p)^{-1}) = (t)_\kappa C_\kappa(Z)/(t+u)_\kappa, \qquad \text{and}$$

$$EC_\kappa(Z(V^{-1} + I_p)) = \Gamma_p(t, -\kappa)\Gamma_p(t+u)C_\kappa(Z)/\Gamma_p(t+u, -\kappa)\Gamma_p(t) \quad \text{if} \quad t \geqq p + k_1.$$

From these, we can easily write down the moments for $(\operatorname{tr} V)$, $(\operatorname{tr} V^{-1})$, $\operatorname{tr}(V^{-1} + I_p)^{-1}$, $\operatorname{tr}(V + I_p)^{-1}$ and $(\operatorname{tr} V^{-1} + p)$ with some conditions.

5. Corresponding results for complex Gaussian variates.

(5.1) In this section, we shall state the above results for complex Gaussian distributions studied by Wooding [29], Goodman [3], [4], James [13] and Khatri [15], [17], [18]. We shall denote

$$(50) \quad \tilde{\Gamma}_m(a) = \pi^{\frac{1}{2}m(m-1)}\textstyle\prod_{i=1}^m \Gamma(a - i + 1), \qquad \tilde{\Gamma}_m(a, \kappa) = \pi^{\frac{1}{2}m(m-1)} \textstyle\prod_{i=1}^m \Gamma(a + k_i - i + 1),$$

$$\tilde{\Gamma}_m(a, -\kappa) = \pi^{\frac{1}{2}m(m-1)} \textstyle\prod_{i=1}^m \Gamma(a - m - k_i + i) \quad \text{and}$$

$$[a]_\kappa = \tilde{\Gamma}_m(a, \kappa)/\tilde{\Gamma}_m(a) = \textstyle\prod_{i=1}^m (a - i + 1)_{k_i}.$$

The corresponding hypergeometric functions are defined as

$$(51) \quad {}_p\tilde{F}_q^{(m)}(a_1, \cdots, a_p; b_1, \cdots, b_q; A, B) = \sum_{k=0}^{\infty}\sum_{\kappa} \frac{[a_1]_\kappa \cdots [a_p]_\kappa}{[b_1]_\kappa \cdots [b_q]_\kappa} \frac{\tilde{C}_\kappa(A)\tilde{C}_\kappa(B)}{\tilde{C}_\kappa(I_m)k!}.$$

When $B = I$, it is denoted as ${}_p\tilde{F}_q(a_1, \cdots, a_p; b_1, \cdots, b_q; A)$, and $\tilde{C}_\kappa(A)$ is a zonal polynomial of a Hermitian matrix A and is the symmetric functions of ch. roots of A.

(5.2) We shall denote by dU the invariant measure on the unitary group $U(n)$ normalized to make the total measure unity.

$$(52) \quad \int_{U(n)} \tilde{C}_\kappa(AUB\bar{U}')\, dU = \tilde{C}_\kappa(A)\tilde{C}_\kappa(B)/\tilde{C}_\kappa(I_n).$$

$$(53) \quad \int_{\bar{A}'=A>0} \exp(-\operatorname{tr} A)|A|^{a-m}\tilde{C}_\kappa(AB)\, dA = \tilde{\Gamma}_m(a, \kappa)C_\kappa(B),$$

and

$$\int_{\bar{A}'=A>0} \exp(-\operatorname{tr} A)|A|^{a-m}\tilde{C}_\kappa(BA^{-1})\, dA = \tilde{\Gamma}_m(a, -\kappa)\tilde{C}_\kappa(B). \tag{54}$$

(5.3) Let $F:p \times p$ be a Hermitian p.d. matrix, and its density function is

$$\tilde{\Gamma}_p(t+u)\{\tilde{\Gamma}_p(t)\tilde{\Gamma}_p(u)\}^{-1}|F|^{t-p}|I+F|^{-t-u}.$$

Then,

$$E\tilde{C}_\kappa(ZF) = [t]_\kappa\tilde{\Gamma}_p(u, -\kappa)\tilde{C}_\kappa(Z)/\tilde{\Gamma}_p(u) \qquad \text{if } u \geqq p + k_1,$$

$$E\tilde{C}_\kappa(ZF^{-1}) = [u]_\kappa\tilde{\Gamma}_p(t, -\kappa)\tilde{C}_\kappa(Z)/\tilde{\Gamma}_p(t) \qquad \text{if } t \geqq p + k_1,$$

$$E\tilde{C}_\kappa(Z(F^{-1}+I_p)^{-1}) = [t]_\kappa\tilde{C}_\kappa(Z)/[t+u]_\kappa, \tag{55}$$

$$E\tilde{C}_\kappa(Z(F+I_p)^{-1}) = [u]_\kappa\tilde{C}_\kappa(Z)/[t+u]_\kappa, \qquad \text{and}$$

$$E\tilde{C}_\kappa(ZF^{-1}+Z) = \tilde{\Gamma}_p(t, -\kappa)\tilde{\Gamma}_p(t+u)\tilde{C}_\kappa(Z)/\{\tilde{\Gamma}_p(t+u, -\kappa)\tilde{\Gamma}_p(t)\} \qquad \text{if } t \geqq p + k_1.$$

(5.4) If the complex random matrix $X:p \times n$ is distributed as Gaussian whose density function is given by

$$\pi^{-pn}|\Sigma|^{-n}|B|^{-p}\exp(-\operatorname{tr}\Sigma^{-1}XB^{-1}\bar{X}') \tag{56}$$

where Σ and B are Hermitian p.d., then the density function of $XL\bar{X}' = S$ (L being a Hermitian p.d. matrix) is given by

$$(\tilde{\Gamma}_p(n)|LB|^p|\Sigma|^n)^{-1}|S|^{n-p}\exp(-q^{-1}\operatorname{tr}\Sigma^{-1}S)\,{}_0\tilde{F}_0^{(n)}(T, q^{-1}\Sigma^{-1}S) \tag{57}$$

where $q > 0$ and $T = I_n - qL^{-\frac{1}{2}}B^{-1}L^{-\frac{1}{2}}$. Moreover

$$E\tilde{C}_\kappa(ZS) = [n]_\kappa\tilde{C}_\kappa(LB)\,\tilde{C}_\kappa(Z\Sigma)/\tilde{C}_\kappa(I_n). \tag{58}$$

When $LB = I$, then $E\tilde{C}_\kappa(ZS^{-1})$ can be written down with the help of (54).

(5.5) Let the density function of $X:p \times n$ be given by (56), the density function of $Y:p \times m$ be $\pi^{-pm}|\Sigma|^{-m}\exp(-\operatorname{tr}\Sigma^{-1}Y\bar{Y}')$ and X and Y be independent. Then for $m \leqq p \leqq n$, the density function of $F = \bar{Y}'(XL\bar{X}')^{-1}Y$ is given by

$$\tilde{\Gamma}_p(m+n)\{\tilde{\Gamma}_p(n)\tilde{\Gamma}_m(p)\}^{-1} \cdot |BL|^{-p}|F|^{p-m}|I_mq^{-1}+F|^{-m-n}\,{}_1\tilde{F}_0^{(n)}(m+n; T, R^*) \tag{59}$$

where $q > 0$, $T = I_n - q(LB)^{-1}$ and $R^* = \begin{pmatrix}(I_m+qF)^{-1} & 0\\ 0 & I_{p-m}\end{pmatrix}$.

(5.6) Let the density function of $X:p \times n$ be given by (56), X and $Y:p \times m$ be independent and the density function of Y be given by

$$\pi^{-pm}|\Sigma_1|^{-m}\exp(-\operatorname{tr}\Sigma_1^{-1}Y\bar{Y}'). \tag{60}$$

Then for $n \leqq p \leqq m$, the density function of $F_1 = \bar{X}'(Y\bar{Y}')^{-1}X$ is given by

$$\tilde{\Gamma}_p(m+n)\{\tilde{\Gamma}_p(m)\tilde{\Gamma}_n(p)\}^{-1}|B|^{-p}|\Omega|^{-n}|F_1|^{p-n} \cdot |I_n+(qB)^{-1}F_1|^{-m-n}\,{}_1\tilde{F}_0^{(p)}(m+n; \Omega^*, F_1(Bq+F_1)^{-1}) \tag{61}$$

where $q > 0$. $\Omega^* = I_p - q\Omega^{-1}$ and $\Omega = \Sigma^{\frac{1}{2}}\Sigma_1^{-1}\Sigma^{\frac{1}{2}}$.

(5.7) Let $X:p \times n$ and $Y:p \times m$ be independently distributed and their respective density functions be given by (56) and (60). Then for $n \geqq p$, $m \geqq p$, the density function of $F_2 = (Y\bar{Y}')^{-\frac{1}{2}}(X\bar{X}')(Y\bar{Y}')^{-\frac{1}{2}}$ is given by

$$(62) \quad \tilde{\Gamma}_p(m+n)\{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(m)\}^{-1} |\Omega|^{-n} |B|^{-p} |F_2|^{n-p} \cdot |I_p + (q\Omega)^{-1}F_2|^{-m-n} {}_1\tilde{F}_0^{(n)}(m+n; T, F_2(q\Omega + F_2)^{-1})$$

where $\Omega = \Sigma^{\frac{1}{2}}\Sigma_1^{-1}\Sigma^{\frac{1}{2}}$ and $T = I_n - q(LB)^{-1}$. The density functions of F_2^{-1} and $F_3 = (X\bar{X}')^{-\frac{1}{2}}(Y\bar{Y}')(X\bar{X}')^{-\frac{1}{2}}$ are identical. Further, we can obtain the density function of the ch. roots of F_2 in the form of (62) provided $\Omega = I_p$ or $\Sigma_1 = \Sigma$; otherwise, we can obtain the density function of the ch. roots of F_2 in the form obtained by taking $q \to \infty$ in (62). Further, if $S = XL\bar{X}'$, we have

$$\begin{aligned} &E(|Y\bar{Y}'|/|S + Y\bar{Y}'|)^j\tilde{\Gamma}_p(m)/\tilde{\Gamma}_p(m+j) \\ (63) \qquad &= E(|S|/|S + Y\bar{Y}'|)^j\tilde{\Gamma}_p(n)/\tilde{\Gamma}_p(n+j) \\ &= \tilde{\Gamma}_p(m+n)\{\tilde{\Gamma}_p(m+n+j)\}^{-1} {}_2\tilde{F}_1(p, j; m+n+j; I_n - BL). \end{aligned}$$

Hence the distribution of $|Y\bar{Y}'|/|S + Y\bar{Y}'|$ can be obtained from that of $|S|/|S + Y\bar{Y}'|$ by interchanging n and m.

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